Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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5. Problem sheet on "Lie Groups and Their Representations"

Exercise 5.1 Let C be a convex cone in a real vector space. Show that for any family $(F_i)_{i\in I}$ of faces of C, the intersection $\bigcap_{i\in I} F_i$ also is a face.

Exercise 5.2 Let C be a convex cone in a real vector space and $f\colon V \to \mathbb{R}$ a linear functional with $f(C) \subseteq \mathbb{R}_+$. Show that $\ker f \cap C$ is a face of C.

Exercise 5.3 Let C be a convex cone in a topological vector space V. Show that every proper face F of C is contained in the boundary ∂C. Hint: Show that the face generated by any $x \in C^0$ is all of C by showing that $C \subseteq \bigcup_{\lambda>0}(\lambda x - C)$.

Exercise 5.4 On the interval $[0,1] \subseteq \mathbb{R}$, we consider $\mathcal{H} = L^2([0,1], dx)$ and the map

$$\gamma\colon [0,1] \to \mathcal{H}, \quad \gamma(x) := \chi_{[0,x]}.$$

Show that:

(a) $K(x,y) := \langle \gamma(y), \gamma(x)\rangle = \min(x,y)$.

(b) $\mathrm{im}(\gamma)$ is total in $\mathcal{H}$. Hint: The subspace spanned by $\mathrm{im}(\gamma)$ contains all Riemannian step functions (those corresponding to finite partitions of $[0,1]$ into subintervals). From this one derives that its closure contains all continuous functions which span a dense subspace.

(c) The reproducing kernel space $\mathcal{H}_K$ consists of continuous functions and the corresponding realization map is

$$\varphi_\gamma\colon L^2([0,1]) \to \mathcal{H}_K, \quad \varphi_\gamma(f)(x) := \int_0^x f(t)\,dt.$$

The space $\mathcal{H}_K$ is the Sobolev space of all continuous functions on $[0,1]$, vanishing in 0 whose derivatives are L^2-functions.

Exercise 5.5 Let $\mathcal{A}$ be a Banach algebra and $\chi\colon \mathcal{A} \to \mathbb{C}$ be an algebra homomorphism. Show that:

(a) χ extends to the unital Banach algebra $\mathcal{A}_+ := \mathcal{A} \times \mathbb{C}$ with the multiplication

$$(a,t)(a',t') := (aa' + ta' + t'a, tt')$$

(cf. Exercise 1.1.11).

(b) If $\mathcal{A}$ is unital and $\chi \neq 0$, then

$$\chi(\mathbf{1}) = 1 \quad \text{and} \quad \chi(\mathcal{A}^\times) \subseteq \mathbb{C}^\times.$$

Conclude further that $\chi(B_1(\mathbf{1})) \subseteq \mathbb{C}^\times$ and derive that χ is continuous with $\|\chi\| \leq 1$.

Exercise 5.6 (Concrete commutants) Let $(X, \mathfrak{S}, \mu)$ be a finite measure space, $\mathcal{H} := L^2(X, \mu)$ the corresponding Hilbert space and

$$\pi\colon L^\infty(X, \mu) \to B(L^2(X, \mu)), \quad \pi(f)g := fg$$

be the homomorphism from Lemma 2.1.5. Show that:

(1) $1 \in L^2(X, \mu)$ is a cyclic vector for π, i.e., not contained in a proper closed subspace invariant under $L^\infty(X, \mu)$.

(2) If $B \in \pi(L^\infty(X, \mu))'$, then

(a) $B(f) = B(1)f$ for $f \in L^2(X, \mu)$. Hint: Verify this relation first for bounded functions f.

(b) $B(1)$ is bounded. Hint: Apply B to the characteristic function of the set $E_n := \{x \in X : n \leq |B(1)|(x) \leq n + 1\}$.

(c) $B = \pi(B(1))$.

(5) $\pi(L^\infty(X, \mu)) = \pi(L^\infty(X, \mu))'$ is its own commutant, hence in particular a von Neumann algebra.

Exercise 5.7 Suppose that Y is a compact space $y_0 \in Y$ and $X := Y \setminus \{y_0\}$. Show that the restriction map yields an isometric isomorphism of C^*-algebras:

$$r\colon C_*(Y, \mathbb{C}) := \{f \in C(Y, \mathbb{C}) : f(y_0) = 0\} \to C_0(X, \mathbb{C}).$$

Exercise 5.8 Let $\mathcal{A}$ be a C^*-algebra. Show that:

(i) If $a = a^* \in \mathcal{A}$ is a hermitian element, then $\|a^n\| = \|a\|^n$ holds for each $n \in \mathbb{N}$. Hint: Consider the commutative C^*-subalgebra generated by a.

(ii) If $\mathcal{B}$ is a Banach-$*$-algebra and $\alpha\colon \mathcal{B} \to \mathcal{A}$ a continuous morphism of Banach-$*$-algebras, then $\|\alpha\| \leq 1$. Hint: Let $C := \|\alpha\|$ and derive with (i) for $b \in \mathcal{B}$ the relation

$$\|\alpha(b)\|^{2n} = \|\alpha(bb^*)\|^n = \|\alpha((bb^*)^n)\| \leq C\|(bb^*)^n\| \leq C\|b\|^{2n}.$$

Finally, use that $C^{1/n} \to 1$.

Exercise 5.9 Show that on $X :=]0, \infty[$ the kernel $K(x, y) := \frac{1}{x+y}$ is positive definite. Hint: Consider the elements $e_\lambda(x) := e^{-\lambda x}$ in $L^2(\mathbb{R}_+, dx)$.