Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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4. Problem sheet on "Lie Groups and Their Representations"

Exercise 4.1 Show that, if $A \in M_2(\mathbb{C})$ satisfies

$$c^* A c \geq 0 \quad \text{for all} \quad c \in \mathbb{C}^2,$$

then $A^* = A$.

Exercise 4.2 Let X be a non-empty set and $T \subseteq X \times X$ be a subset containing the diagonal. Then the characteristic function χ_T of T is a positive definite kernel if and only if T is an equivalence relation.

Exercise 4.3 Let $\mathcal{H}_K \subseteq \mathbb{K}^X$ be a reproducing kernel Hilbert space and $\mathcal{H} = \widehat{\bigoplus}_{j\in J} \mathcal{H}_j$ be a direct Hilbert space sum. Show that there exist positive definite kernels $K^j \in \mathcal{P}(X)$ with $K = \sum_{j\in J} K^j$ and $\mathcal{H}_j = \mathcal{H}_{K^j}$ for $j \in J$. Hint: Consider $\mathcal{H}_j$ as a Hilbert space with continuous point evaluations and let K^j be its reproducing kernel.

Exercise 4.4 Let $G = \mathrm{Aff}_1(\mathbb{R}) \cong \mathbb{R} \ltimes \mathbb{R}^\times$ denote the affine group of $\mathbb{R}$, where (b, a) corresponds to the affine map $\varphi_{b,a}(x) := ax + b$. This group is sometimes called the *$ax + b$-group*. Show that a Haar measure on this group is obtained by

$$\int_G f(a,b)\, d\mu_G(a,b) := \int_{\mathbb{R}} \int_{\mathbb{R}^\times} f(a,b)\, \frac{da}{|a|^2} d\, b.$$

Show further that $\Delta_G(b, a) = |a|^{-1}$, which implies that G is not unimodular.

Exercise 4.5 We consider the group $G := \mathrm{GL}_2(\mathbb{R})$ and the *real projective line*

$$\mathbb{P}_1(\mathbb{R}) = \{[v] := \mathbb{R}v \colon 0 \neq v \in \mathbb{R}^2\}$$

of 1-dimensional linear subspaces of $\mathbb{R}^2$. We write $[x : y]$ for the line $\mathbb{R}\begin{pmatrix} x \\ y \end{pmatrix}$. Show that:

(a) We endow $\mathbb{P}_1(\mathbb{R})$ with the quotient topology with respect to the map $q \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{P}_1(\mathbb{R}), v \mapsto [v]$. Show that $\mathbb{P}_1(\mathbb{R})$ is homeomorphic to $\mathbb{S}^1$.

(b) The map $\mathbb{R} \to \mathbb{P}_1(\mathbb{R}), x \mapsto [x : 1]$ is injective and its complement consists of the single point $\infty := [1 : 0]$ (the horizontal line). We thus identify $\mathbb{P}_1(\mathbb{R})$ with the one-point compactification of $\mathbb{R}$. These are the so-called *homogeneous coordinates* on $\mathbb{P}_1(\mathbb{R})$.

(c) The natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}_1(\mathbb{R})$ by $g.[v] := [gv]$ is given in the coordinates of (b) by

$$g.x = \sigma_g(x) := \frac{ax + b}{cx + d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(d) There exists a unique Radon measure μ with total mass π on $\mathbb{P}_1(\mathbb{R})$ which is invariant under the group $O_2(\mathbb{R})$. Hint: Identify $\mathbb{P}_1(\mathbb{R})$ with the compact group $SO_2(\mathbb{R})/\{\pm\mathbf{1}\} \cong \mathbb{T}$.

(e) Show that, in homogeneous coordinates, we have $d\mu(x) = \frac{dx}{1+x^2}$. Hint: $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.0 = -\tan x$, and the image of Lebesgue measure on $]-\pi/2, \pi/2[$ under tan is $\frac{dx}{1+x^2}$.

(f) Show that the action of $SL_2(\mathbb{R})$ on $\mathbb{P}_1(\mathbb{R})$ preserves the measure class of μ. Hint: Show that $\sigma_g(x) := \frac{ax+b}{cx+d}$ satisfies $\sigma_g'(x) = \frac{1}{(cx+d)^2}$ and derive the formula

$$\delta(\sigma_g)(x) = \frac{d((\sigma_g)_*\mu)}{d\mu} = \frac{1+x^2}{(a-cx)^2 + (b-dx)^2}, \quad \delta(\sigma_g)(\infty) = \frac{1}{c^2+d^2}.$$

(g) The density function also has the following metric interpretation with respect to the euclidean norm on $\mathbb{R}^2$:

$$\delta(\sigma_g)([v]) = \frac{\|g^{-1}v\|^2}{\|v\|^2}.$$

The corresponding unitary representations of $SL_2(\mathbb{R})$ on $L^2(\mathbb{P}_1(\mathbb{R}), \mu)$ defined by

$$\pi_s(g)f := \delta(\sigma_g)^{\frac{1}{2}+is}(\sigma_g)_* f$$

(cf. Example 2.2.7) form the so-called *spherical prinicipal series.*

Exercise 4.6 Let $(X, \mathfrak{S}, \mu)$ be a probability space. Show that on $X = \mathfrak{S}$, the kernel

$$K(E, F) := \mu(E \cap F) - \mu(E)\mu(F)$$

is positive definite. Hint: Consider the hyperplane $\{f \in L^2(X, \mu) \colon \int_X f \, d\mu = 0\}$.

Exercise 4.7 Show that on $X := [0, 1]$, the kernel $K(x, y) := \min(x, y) - xy$ is positive definite.

Exercise 4.8 Let $\lambda = dX$ denote Lebesgue measure on the space $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of real $(n \times n)$-matrices. Show that a Haar measure on $GL_n(\mathbb{R})$ is given by

$$d\mu_{GL_n(\mathbb{R})}(g) = \frac{1}{|\det(g)|^n} d\lambda(g).$$

Hint: Calculate the determinant of the linear maps $\lambda_g \colon M_n(\mathbb{R}) \to M_n(\mathbb{R}), x \mapsto gx$.