Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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SS 2009

TECHNISCHE UNIVERSITÄT DARMSTADT

30. April 2009

3. Problem sheet on "Lie Groups and Their Representations"

Exercise 3.1 Show that each σ-finite measure μ on a measurable space $(X, \mathfrak{S})$ is equivalent to a finite measure.

Exercise 3.2 Let μ and λ be equivalent σ-finite measures on $(X, \mathfrak{S})$ and $h := \frac{d\mu}{d\lambda}$. Show that

$$\Phi\colon L^2(X, \mu) \to L^2(X, \nu), \quad f \mapsto \sqrt{h} f$$

defines a unitary map.

Exercise 3.3 (Affine actions) Let V be a vector space and $\rho\colon G \to \mathrm{GL}(V)$ be a representation of the group G on V. Show that:

(1) For each 1-cocycle $f \in Z^1(G, V)_\rho$ we obtain on V an action of G by affine maps via $\sigma_f(g)(v) := \rho(g)v + f(g)$.

(2) The affine action σ_f has a fixed point if and only if f is a coboundary, i.e., of the form $f(g) = \rho(g)v - v$ for some $v \in V$.

(3) If G is finite and V is defined over a field of characteristic zero, then the group $H^1(G, V)_\rho$ is trivial.

Exercise 3.4 (Affine isometric actions) Let $\mathcal{H}$ be a Hilbert space and $(\pi, \mathcal{H})$ be a unitary representation of G. Show that:

(1) For each 1-cocycle $f \in Z^1(G, \mathcal{H})_\pi$, we obtain an action of G on $\mathcal{H}$ by affine isometries $\sigma_f(g)(v) := \pi(g)v + f(g)$.

(2) If this action has a fixed point, then f is bounded. [1]

(3) Consider the real unitary (=orthogonal) representation of $G = \mathbb{R}$ on $\mathcal{H} = \mathbb{R}^3$ by

$$\pi(t) := \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that $f\colon \mathbb{R} \to \mathbb{R}^3, f(t) := (0,0,t)^\top = te_3$ is a 1-cocycle with non-trivial cohomology class. Find a set of representatives for the corresponding affine action and describe the orbits geometrically.

[1] The converse is also true. It is a consequence of the Bruhat–Tits Fixed Point Theorem.

Exercise 3.5 Let $(S, *)$ be an involutive semigroup. Show that:

(a) Every cyclic representation is non-degenerate.

(b) If $(\pi, \mathcal{H})$ is a direct sum of cyclic representations, then it is is non-degenerate.

(c) Every non-degenerate representation $(\pi, \mathcal{H})$ of S is a direct sum of cyclic representations. Hint: One can follows the argument in Proposition 1.3.10, but one step requires additional care, namely that for $0 \neq v \in \mathcal{H}$, the representation of S on the closed subspace $\mathcal{K} := \overline{\operatorname{span} \pi(S)v}$ is cyclic. One has to argue that $v \in \mathcal{K}$ to see that this is the case, and this is where it is needed that $(\pi, \mathcal{H})$ is non-degenerate.

Exercise 3.6 Let G be a group and $\mathcal{H}$ be a complex Hilbert space. We consider the trivial homomorphism $\alpha = \mathbf{1}\colon G \to \operatorname{Aut}(\operatorname{U}(\mathcal{H}))$, i.e., $\alpha(G) = \{\operatorname{id}_{\operatorname{U}(\mathcal{H})}\}$ for each $g \in G$. Show that there is a bijection between $H^1(G, \operatorname{U}(\mathcal{H}))$ and the of equivalence classes of unitary representations of the group G on $\mathcal{H}$.

Exercise 3.7 (From measure classes to cohomology classes) Let $\sigma\colon G \times X \to X$ be an action of G by measurable maps on $(X, \mathfrak{S})$ and $[\mu]$ be a G-invariant σ-finite measure class. Verify the following assertions:

(a) Let $\mathcal{R}$ denote the group of all measurable functions $f\colon X \to \mathbb{R}_+^\times$ with respect to pointwise multiplication and write $\mathcal{R}_0$ for the subgroup of those functions which are constant 1 on the complement of a μ-zero set. Then $\widetilde{\mathcal{R}} := \mathcal{R}/\mathcal{R}_0$ is a group, whose elements are denoted $[f]$, and

$$g.[f] := [(\sigma_g)_* f]$$

defines an action of G on $\widetilde{\mathcal{R}}$ by automorphisms.

(b) $\delta_\mu(g) := \frac{d(\sigma_g)_*\mu}{d\mu}$ defines a $\widetilde{\mathcal{R}}$-valued 1-cocycle on G.

(c) If $[h] \in \widetilde{\mathcal{R}}$, then $\delta_{h\mu}(g) = \frac{(\sigma_g)_* h}{h}\delta_\mu(g)$. Conclude that the cohomology class $[\delta_\mu] \in H^1(G, \widetilde{\mathcal{R}}]$ does not depend on the representative μ of the measure class $[\mu]$.

(d) Show that the measure class $[\mu]$ contains a G-invariant element if and only if the cohomology class $[\delta_\mu]$ vanishes.

Exercise 3.8 Consider the Gaußian measure

$$d\gamma(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\|x\|^2} dx$$

on $\mathbb{R}^n$. Its measure class is invariant under the action of the affine group $\operatorname{Aff}_n(\mathbb{R})$. Find a formula for the action of this group on $L^2(\mathbb{R}^n, \gamma)$.

Exercise 3.9 (Left- and right cocycles) Let $\widehat{G} = N \rtimes_\alpha G$ be a semidirect product defined by the homomorphism $\alpha\colon G \to \operatorname{Aut}(N)$. Show that:

(1) The map $G \times N \to \widehat{G}, (g, n) \mapsto (\mathbf{1}, g)(n, \mathbf{1})$ is a group isomorphism if we define the multiplication on $G \times N$ by

$$(g, n)(g', n') := (gg', \alpha(g')^{-1}(n)n').$$

We write $G \ltimes_\alpha N$ for this group.

(2) A map $s\colon G \to G \ltimes_\alpha N, s(g, n) = (g, f(g))$ is a group homomorphism if and only if it is a *right cocycle*, i.e.,

$$f(gh) = \alpha(h)^{-1}(f(g))f(h) \quad \text{for} \quad g, h \in G.$$