Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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2. Problem sheet on “Lie Groups and Their Representations”

Exercise 2.1 Let $\mathcal{H}$ be a Hilbert space and $\mathrm{U}(\mathcal{H})_s$ its unitary group, endowed with the strong (=weak) operator topology. Show that the action map

$$\sigma\colon \mathrm{U}(\mathcal{H})_s \times \mathcal{H} \to \mathcal{H}, \quad (g,v) \mapsto gv$$

is continuous. Conclude that each continuous unitary representation $(\pi, \mathcal{H})$ of a topological group G defines a continuous action of G on $\mathcal{H}$ by $g.v := \pi(g)v$.

Exercise 2.2 Let $(a_n)_{n\in\mathbb{N}}$ be asequence of real numbers. Show that we obtain a continuous unitary representation of $G = (\mathbb{R}, +)$ on $\mathcal{H} = \ell^2(\mathbb{N}, \mathbb{C})$ by

$$\pi(t)x = (e^{ita_1}x_1, e^{ita_2}x_2, \ldots).$$

Show further that, if the sequence (a_n) is unbounded, then π is not norm continuous. Is it norm continuous if the sequence (a_n) is bounded?

Exercise 2.3 Let $(\pi, \mathcal{H})$ be a representation of an involutive semigroup $(S, *)$. Show that:

(a) $(\pi, \mathcal{H})$ is non-degenerate if and only if $\pi(S)v \subseteq \{0\}$ implies $v = 0$.

(b) Show that $(\pi, \mathcal{H})$ is an orthogonal direct sum of a non-degenerate representation and a zero representation $(\zeta, \mathcal{K})$, i.e., $\zeta(S) = \{0\}$.

Exercise 2.4 Let $(\pi, \mathcal{H})$ be a representation of the involutive semigroup (G, η_G), where G is a group. Show that:

(a) $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_j = \ker(\eta(\mathbf{1}) - j\mathbf{1})$, is an orthogonal direct sum.

(b) $(\pi, \mathcal{H})$ is non-degenerate if and only if $\pi(\mathbf{1}) = \mathbf{1}$.

Exercise 2.5 Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be unitary representations of G. Show that the space $B_G(\mathcal{K}, \mathcal{H})$ of all intertwining operators is a closed subspace of the Banach space $B(\mathcal{K}, \mathcal{H})$

Exercise 2.6 Let $b\colon V \times V \to \mathbb{C}$ be a hermitian form on the complex Vector space V, i.e., b is linear in the first argument and satisfies $b(y,x) = \overline{b(x,y)}$. Show that b satisfies the *polarization identity* which permits the recover all values of b from those on the diagonal:

$$b(x,y) = \frac{1}{4}\sum_{k=0}^{3} i^k b(x + i^k y, x + i^k y).$$

Exercise 2.7 Show that for each summable family $(x_j)_{j\in J}$ in the Banach space X, the set

$$J^\times = \{j \in J \colon x_j \neq 0\}$$

is countable, and that, if $J^\times = \{j_n \colon n \in \mathbb{N}\}$ is an enumeration of $J^\times$, then $\sum_{j\in J} x_j = \sum_{n=1}^\infty x_{j_n}$. Hint: Show that each set $J_n := \{j \in J \colon \|x\|_n > \frac{1}{n}\}$ is finite.

Exercise 2.8 Show that for an orthogonal family $(x_j)_{j\in J}$ in the Hilbert space $\mathcal{H}$, the following are equivalent:

(i) $(x_j)_{j\in J}$ is summable.

(ii) $(\|x_j\|^2)_{j\in J}$ is summable in $\mathbb{R}$.

Show further that, if this is the case, then $\|\sum_{j\in J} x_j\|^2 = \sum_{j\in J}\|x_j\|^2$ and the set $\{j \in J : x_j \neq 0\}$ is countable.

Exercise 2.9 Let $(\mathcal{H}_j)_{j\in J}$ be an orthogonal family of closed subspaces of the Hilbert space $\mathcal{H}$. Show that for each $x = (x_j)_{j\in J} \in \widehat{\bigoplus}\mathcal{H}_j$, the sum $\Phi(x) := \sum_{j\in J} x_j$ converges in $\mathcal{H}$ and that $\Phi\colon \widehat{\bigoplus}_{j\in J}\mathcal{H}_j \to \mathcal{H}, (x_j)_{j\in J} \mapsto \sum_{j\in J} x_j$ defines an isometric embedding (cf. Exercise 2.8).

Exercise 2.10 (Endomorphisms as matrices) Let V be a vector space which is the direct sum

$$V = V_1 \oplus \cdots \oplus V_n$$

of the subspaces V_i, $i = 1,\ldots,n$. Accordingly, we write $v \in V$ as a sum $v = v_1 + \cdots + v_n$ with $v_i \in V$. To each $\varphi \in \mathrm{End}(V)$ we associate the map $\varphi_{ij} \in \mathrm{Hom}(V_j, V_i)$, defined by $\varphi_{ij}(v) = \varphi(v)_i$ for $v \in V_j$. Show that

(a) $\varphi(v)_i = \sum_{j=1}^n \varphi_{ij}(v_j)$ for $v = \sum_{j=1}^n v_j \in V$.

(b) The map

$$\Gamma\colon \bigoplus_{i,j=1}^n \mathrm{Hom}(V_j, V_i) \to \mathrm{End}(V), \quad \Gamma((\psi_{ij}))(v) := \sum_{i,j=1}^n \psi_{ij}(v_j)$$

is a linear isomorphism. In this sense we may identify endomorphisms of V with $(n \times n)$-matrices with entries in $\mathrm{Hom}(V_j, V_i)$ in position (i,j).

(c) If V is a Banach space and each V_i is a closed subspace, then the map

$$S\colon V_1 \times \cdots \times V_n \to V, \quad (v_1,\ldots,v_n) \mapsto \sum_{i=1}^n v_i$$

is a homeomorphism. Moreover, a linear endomorphism $\varphi\colon V \to V$ is continuous if and only if each φ_{ij} is continuous. Hint: For the first assertion use the Open Mapping Theorem. Conclude that if $\iota_i\colon V_i \to V$ denotes the inclusion map and $p_j\colon V \to V_j$ the projection map, then both are continuous. Then use that $\varphi_{ij} = p_i \circ \varphi \circ \eta_j$.

Exercise 2.11 Let G be a group. Show that:

(a) Each unitary representation $(\pi,\mathcal{H})$ of G is equivalent to a representation $(\rho, \ell^2(J,\mathbb{C}))$ for some set J. Therefore it makes sense to speak of the **set** of equivalence classes of representations with a fixed Hilbert dimension $|J|$.

(b) Two unitary representations $\pi_j\colon G \to \mathrm{U}(\mathcal{H})$, $j = 1,2$, are equivalent if and only if there exists a unitary operator $U \in \mathrm{U}(\mathcal{H})$ with

$$\pi_2(g) = U\pi_1(g)U^{-1} \quad \text{for each} \quad g \in G.$$

Therefore the set of equivalence classes of unitary representations of G on $\mathcal{H}$ is the set of orbits of the action of $\mathrm{U}(\mathcal{H})$ on the set $\mathrm{Hom}(G, \mathrm{U}(\mathcal{H}))$ for the action $(U * \pi)(g) := U\pi(g)U^{-1}$.