Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
Prof. Dr. K.-H. Neeb
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TECHNISCHE UNIVERSITÄT DARMSTADT

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12. Problem sheet on "Lie Groups and Their Representations"

Exercise 12.1 Show that for $X = -X^* \in M_n(\mathbb{C})$ the matrix e^X is unitary and that the exponential function

$$\exp : \mathrm{Aherm}_n(\mathbb{C}) := \{X \in M_n(\mathbb{C}) \colon X^* = -X\} \to \mathrm{U}_n(\mathbb{C}), \quad X \mapsto e^X$$

is surjective.

Exercise 12.2 Show that for $X^\top = -X \in M_n(\mathbb{R})$ the matrix e^X is orthogonal and that the exponential function

$$\exp : \mathrm{Skew}_n(\mathbb{R}) := \{X \in M_n(\mathbb{R}) \colon X^\top = -X\} \to \mathrm{O}_n(\mathbb{R})$$

is not surjective. Can you determine which orthogonal matrices are contained in the image? Can you interprete the result geometrically in terns of the geometry of the flow $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, (t, v) \mapsto e^{tX}v$.

Exercise 12.3 Show that for $\mathcal{A} := C(\mathbb{S}^1)$ the exponential function

$$\exp : \mathrm{Aherm}(\mathcal{A}) := \{a \in \mathcal{A} \colon a^* = -a\} \to \mathrm{U}(\mathcal{A}) = C(\mathbb{S}^1, \mathbb{T}), \quad a \mapsto e^a$$

is not surjective. It requires some covering theory to determine which elements $f \in C(\mathbb{S}^1, \mathbb{T})$ lie in its image. Hint: Use the winding number with respect to 0.

Exercise 12.4 Show that for any measure space $(X, \mathfrak{S})$ and the C^*-algebra $\mathcal{A} := L^\infty(X, \mathfrak{S})$, the exponential function

$$\exp : \mathrm{Aherm}(\mathcal{A}) \to \mathrm{U}(\mathcal{A}), \quad a \mapsto e^a$$

is surjective.

Exercise 12.5 Show that for every von Neumann algebra $\mathcal{A}$, the exponential function

$$\exp : \mathrm{Aherm}(\mathcal{A}) \to \mathrm{U}(\mathcal{A}), \quad a \mapsto e^a$$

is surjective. This applies in particular to $\mathcal{A} = B(\mathcal{H})$, so that for every complex Hilbert space $\mathcal{H}$, the exponential function $\exp : \mathrm{Aherm}(\mathcal{H}) \to \mathrm{U}(\mathcal{H}), a \mapsto e^a$ is surjective.

Exercise 12.6 Show that every cyclic representation of an abelian involutive semigroup $(S, *)$ is multiplicity free. Hint: Since $\pi(S)' = \pi(S)'''$ one may assume that $S = \mathcal{A}$ is a commutative C^*-algebra. In this case we know the cyclic representations and the corresponding commutants (Exercise 4.2.1).

Conclude with Exercise 10.2 that a representation of an abelian involutive semigroup on a separable Hilbert space is cyclic if and only if it is multiplicity free.

Exercise 12.7 The function

$$\Phi : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} e^{-\frac{1}{t}}, & \text{for } t > 0 \\ 0, & \text{for } t \leq 0 \end{cases}$$

is smooth. Hint: The higher derivatives of $e^{-\frac{1}{t}}$ are of the form $P(t^{-1})e^{-\frac{1}{t}}$, where P is a polynomial.

(b) For $\lambda > 0$ the function $\Psi(t) := \Phi(t)\Phi(\lambda - t)$ is a non-negative smooth function with $\operatorname{supp}(\Psi) = [0, \lambda]$.

Exercise 12.8 (A smoothing procedure) Let $f \in C^1_c(\mathbb{R})$ be a C^1-function with compact support and $\gamma \in C(\mathbb{R}, E)$, where E is a Banch space. Then the convolution

$$h := f * \gamma \colon \mathbb{R} \to E, \quad t \mapsto \int_{\mathbb{R}} f(s)\gamma(t-s)\; ds = \int_{\mathbb{R}} f(t-s)\gamma(s)\; ds$$

of f and γ is continuously differentiable with $h' = f' * \gamma$. Suppose, in addition, that $\int_{\mathbb{R}} f\, dt = 1$. Can you find an estimate for $\|\gamma(t) - (f * \gamma)(t)\|$? Hint:

$$\int_{\mathbb{R}} f(t-s)\gamma(s)\; ds = \int_{t-\operatorname{supp}(f)} f(t-s)\gamma(s)\; ds.$$

Exercise 12.9 Let $X_1, \ldots, X_n$ be Banach spaces and $\beta \colon X_1 \times \ldots \times X_n \to Y$ a continuous n-linear map.

(a) Show that there exists a constant $C \geq 0$ with

$$\|\beta(x_1, \ldots, x_n)\| \leq C\|x_1\| \cdots \|x_n\| \quad \text{for} \quad x_i \in X_i.$$

(b) Show that β is differentiable with

$$\mathtt{d}\beta(x_1, \ldots, x_n)(h_1, \ldots, h_n) = \sum_{j=1}^{n} \beta(x_1, \ldots, x_{j-1}, h_j, x_{j+1}, \ldots, x_n).$$

Exercise 12.10 Let Y be a Banach space and $a_{n,m}$, $n, m \in \mathbb{N}$, elements in Y with

$$\sum_{n,m} \|a_{n,m}\| := \sup_{N \in \mathbb{N}} \sum_{n,m \leq N} \|a_{n,m}\| < \infty.$$

(a) Show that

$$A := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{n,m}$$

and that both iterated sums exist.

(b) Show that for each sequence $(S_n)_{n \in \mathbb{N}}$ of finite subsets $S_n \subseteq \mathbb{N} \times \mathbb{N}$, $n \in \mathbb{N}$, with $S_n \subseteq S_{n+1}$ and $\bigcup_n S_n = \mathbb{N} \times \mathbb{N}$ we have

$$A = \lim_{n \to \infty} \sum_{(j,k) \in S_n} a_{j,k}.$$

Exercise 12.11 (Cauchy Product Formula) Let X, Y, Z be Banach spaces and $\beta \colon X \times Y \to Z$ a continuous bilinear map. Suppose that if $x := \sum_{n=0}^{\infty} x_n$ is absolutely convergent in X and if $y := \sum_{n=0}^{\infty} y_n$ is absolutely convergent in Y, then

$$\beta(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta(x_k, y_{n-k}).$$

Hint: Use Exercise 12.10(b).