Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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11. Problem sheet on “Lie Groups and Their Representations”

Exercise 11.1 Let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a unitary representation of the locally compact group G on $\mathcal{H}$ which is norm-continuous, i.e., continuous with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. Show that there exists an $f \in C_c(G)$ for which the operator $\pi(f)$ is invertible.

Exercise 11.2 Let $\alpha\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of commutative Banach-$*$-algebras which is *non-degenerate* in the sense that no (non-zero) character of $\mathcal{B}$ vanishes on $\alpha(\mathcal{A})$. Show that

$$\widehat{\alpha}\colon \widehat{\mathcal{B}} \to \widehat{\mathcal{A}}, \quad \chi \mapsto \chi \circ \alpha$$

is a continuous map which is proper, i.e., inverse images of compact subsets of $\widehat{\mathcal{A}}$ are compact. Hint: Extend $\widehat{\alpha}$ to a continuous map $\mathrm{Hom}(\mathcal{B}, \mathbb{C}) \to \mathrm{Hom}(\mathcal{A}, \mathbb{C})$.

Exercise 11.3 Let $f\colon X \to Y$ be a proper map between locally compact spaces. Show that

(a) f is a closed map, i.e., maps closed subsets to closed subsets.

(b) If f is injective, then it is a topological embedding onto a closed subset.

(c) There is a well-defined homomorphism $f^*\colon C_0(Y) \to C_0(X)$ of C^*-algebras, defined by $f^*h := h \circ f$. Identifying X with $C_0(X)\widehat{\ }$ and Y with $C_0(Y)\widehat{\ }$, we have $\widehat{f^*} = f$.

(d) For each regular Borel measure μ on X, the push-forward measure $f_*\mu$ on Y, defined by $(f_*\mu)(E) := \mu(f^{-1}(E))$ is regular. Hint: To verify outer regularity, pick an open $O \supseteq f^{-1}(E)$ with $\mu(O \setminus f^{-1}(E)) < \varepsilon$. Then $U := f(O^c)^c$ is an open subset of Y containing E and $\widetilde{O} := f^{-1}(U)$ satisfies $f^{-1}(E) \subseteq \widetilde{O} \subseteq O$, which leads to $(f_*\mu)(U \setminus E) < \varepsilon$.

Exercise 11.4 (Cyclic spectral measures) Let $P\colon \mathfrak{S} \to P_{\mathcal{H}}$ be a spectral measure on $(X, \mathfrak{S})$ with a cyclic vector v. Find a unitary isomorphism $\Phi\colon L^2(X, P^v) \to \mathcal{H}$ with $\Phi(\chi_E f) = P(E)\Phi(f)$ for $f \in L^2(X, P^v)$.

Exercise 11.5 (Unitary one-parameter groups) Let $P\colon \mathfrak{S} \to P_{\mathcal{H}}$ be a spectral measure on $(X, \mathfrak{S})$ and $f\colon X \to \mathbb{R}$ be a measurable function. Show that

(a) $\pi\colon \mathbb{R} \to \mathrm{U}(\mathcal{H}), \pi(t) := P(e^{itf})$ defines a continuous unitary representation of $\mathbb{R}$ on $\mathcal{H}$.

(b) If f is bounded, then π is norm continuous.

(c) If f is norm-continuous, then f is essentially bounded.

Exercise 11.6 Let G be a locally compact group. Show that the convolution product on $C_c(G)$ satisfies

$$\|f * h\|_\infty \leq \|f\|_1 \cdot \|h\|_\infty.$$

Conclude that convolution extends to a continuous bilinear map

$$L^1(G, \mu_G) \times C_0(G) \to C_0(G).$$

Conclude that for $f \in L^1(G, \mu_G)$ and $h \in C_c(G)$, the convolution product $f * h$ can be represented by a continuous function in $C_0(G)$.

Exercise 11.7 Let G be a compact group. Show that every left or right invariant closed subspace of $L^2(G)$ consists of continuous functions. Hint: Use Exercise 11.6 and express the integrated representation of $L^1(G)$ on $L^2(G)$ in terms of the convolution product.

Exercise 11.8 (Strongly ergodic measures) Let $(X, \mathfrak{S})$ be a measurable space and $\sigma\colon G \times X \to X$ an action of a group G on X by measurable maps. Show that for a finite G-invariant measure μ on $(X, \mathfrak{S})$, the following are equivalent

(a) $L^2(X, \mu)^G = \mathbb{C}\mathbf{1}$, i.e., the only elements of $L^2(X, \mu)$ invariant under the representation $(\pi(g)f)(x) := f(g^{-1}x)$ are the constants.

(b) $L^\infty(X, \mu)^G = \mathbb{C}\mathbf{1}$.

Then the measure μ is called *strongly* G-ergodic.

Exercise 11.9 Let G be a compact group and μ_G be normalized Haar measure of G. Show μ_G is strongly ergodic for the multiplication action $\sigma(g, h) := gh$ of G on itself is strongly ergodic. Hint: Exercise 11.7.

Exercise 11.10 Let G be a compact group, μ_G be normalized Haar measure of G, H a closed subgroup of G, $q\colon G \to G/H$ the quotient map and $\mu := q_*\mu_G$. Show that μ is strongly ergodic with respect to the left translation action $\sigma(g, xH) := gxH$ of G on the quotient space G/H of left cosets of H. Hint: Exercise 11.9

Exercise 11.11 Let H be a compact group, $G \subseteq H$ be a dense subgroup and μ_H normalized Haar measure of H. Show that μ_H is strongly ergodic with respect to the multiplication action $\sigma(g, h) := gh$ of G on H.

Exercise 11.12 Show that the Haar measure on $\mathbb{T}$ is ergodic for the action of $\mathbb{Z}$ on $\mathbb{T}$ by $n.e^{it} := e^{i(t+n\theta)}$, where θ is an irrational number.