Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
Prof. Dr. K.-H. Neeb
SS 2009

TECHNISCHE UNIVERSITÄT DARMSTADT

25. Juni 2009

10. Problem sheet on "Lie Groups and Their Representations"

Exercise 10.1 Let $(\pi, \mathcal{H})$ be a representation of the involutive semigroup $(S, *)$ and $v \in \mathcal{H}$. Show that the following assertions are equivalent:

(a) v is a cyclic vector for $\pi(S)$.

(b) v is a cyclic vector for the von Neumann algebra $\pi(S)''$.

(c) v separating for the von Neumann algebra $\pi(S)'$, i.e., the map $\pi(S)' \to \mathcal{H}, A \mapsto Av$ is injective.

Hint: To see that (c) implies (a), consider the projection P onto $(\pi(S)v)^{\perp}$, which is an element of $\pi(S)'$.

Exercise 10.2 Let $(\pi, \mathcal{H})$ be a multiplicity free representation of the involutive semigroup $(S, *)$ on the separable Hilbert space $\mathcal{H}$. Show that $(\pi, \mathcal{H})$ is cyclic. Hint: Write $\mathcal{H}$ as a direct sum of at most countably many cyclic representations $(\pi_j, \mathcal{H}_j, v_j)$ with cyclic unit vectors $(v_j)_{j\in J}$ and find $c_j > 0$ such that $v := \sum_{j\in J} c_j v_j$ converges in $\mathcal{H}$. Now show that v is a separating vector for $\pi(S)'$ and use Exercise 10.1. Note that the orthogonal projections P_j onto $\mathcal{H}_j$ are contained in $\pi(S)'$.

Exercise 10.3 Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert space $\mathcal{H}$ which converges weakly to v and satisfies $\|v_n\| \to \|v\|$. Then we have $v_n \to v$.

Exercise 10.4 Let $\mathcal{A}$ be a unital C^*-algebra and $a = a^* \in \mathcal{A}$ with $\|a\| < 1$. Show that

$$b := \sqrt{1-a} := \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n a^n$$

is hermitian and satisfies $b^2 = a$. Show further that

$$u := a + i\sqrt{\mathbf{1} - a} \in \mathrm{U}(\mathcal{A})$$

and conclude that $\mathcal{A} = \operatorname{span} \mathrm{U}(\mathcal{A})$. Hint: To verify $b^2 = \mathbf{1} - a$, it suffices to consider the commutative C^*-algebra generated by a.

Exercise 10.5 Let G be a group acting in a measure preserving fashion on the σ-finite measure space $(X, \mathfrak{S}, \mu)$. The measure μ is said to be *ergodic* (with respect to this action) if any G-invariant subset $E \in \mathfrak{S}$ either

$$\mu(E) = 0 \quad \text{or} \quad \mu(E^c) = 0.$$

Show that, if μ is ergodic, then the unitary representation of $\mathcal{M}(X, \mathbb{T}) \rtimes G$ on $L^2(X, \mu)$ by

$$(\pi(\theta, g)f)(x) := \theta(x) f(g^{-1}.x)$$

(cf. Remark 2.2.5) is irreducible. We suggest the following steps:

(i) Any element of the commutant of $\pi(\mathcal{M}(X, \mathbb{T}))$ coincides with $\rho(h)f := hf$ for some $h \in L^\infty(X, \mu)$. Hint: Exercise 10.4 implies $\mathcal{M}(X, \mathbb{T})$ has the same commutant as $L^\infty(X, \mu)$; then use Exercise 4.2.1, asserting that $L^\infty(X, \mu)' = L^\infty(X, \mu)$ holds if $L^\infty(X, \mu)$ is considered as operators on $L^2(X, \mu)$.

(ii) If $\rho(h)$ commutes with $\pi(G)$, then h coincides μ-almost everywhere with a constant function.

(iii) Use Schur's Lemma to conclude that π is irreducible because

$$\pi(\mathcal{M}(X, \mathbb{T}) \rtimes G)' = \mathbb{C}\mathbf{1}.$$

Exercise 10.6 (Metric characterization of midpoints) Let $(X, \|\cdot\|)$ be a normed space and $x, y \in X$ distinct points. Let

$$M_0 := \{z \in X : \|z - x\| = \|z - y\| = \tfrac{1}{2}\|x - y\|\} \quad \text{and} \quad m := \frac{x + y}{2}.$$

For a subset $A \subseteq X$ we define its *diameter*

$$\delta(A) := \sup\{\|a - b\| : a, b \in A\}.$$

Show that:

(1) If X is a pre-Hilbert space (i.e., a vector space with a hermitian scalar product), then $M_0 = \{m\}$ is a one-element set.

(2) For $z \in M_0$ we have $\|z - m\| \leq \frac{1}{2}\delta(M_0) \leq \frac{1}{2}\|x - y\|$.

(3) For $n \in \mathbb{N}$ we define inductively:

$$M_n := \{p \in M_{n-1} : (\forall z \in M_{n-1})\ \|z - p\| \leq \tfrac{1}{2}\delta(M_{n-1})\}.$$

Then we have for each $n \in \mathbb{N}$:

(a) M_n is a convex set.

(b) M_n is invariant under the point reflection $s_m(a) := 2m - a$ in m.

(c) $m \in M_n$.

(d) $\delta(M_n) \leq \frac{1}{2}\delta(M_{n-1})$.

(4) $\bigcap_{n \in \mathbb{N}} M_n = \{m\}$.

Exercise 10.7 (Isometries of normed spaces are affine maps) Let $(X, \|\cdot\|)$ be a normed space endowed with the metric $d(x, y) := \|x - y\|$. Show that each isometry $\varphi\colon (X, d) \to (X, d)$ is an affine map by using the following steps:

(1) It suffices to assume that $\varphi(0) = 0$ and to show that this implies that φ is a linear map.

(2) $\varphi(\frac{x+y}{2}) = \frac{1}{2}(\varphi(x) + \varphi(y))$ for $x, y \in X$. Hint: Exercise 10.6.

(3) φ is continuous.

(4) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in 2^{\mathbb{Z}} \subseteq \mathbb{R}$.

(5) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for $x, y \in X$.

(6) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in \mathbb{R}$.