Fachbereich Mathematik
AG Algebra, Geometrie, Funktionalanalysis
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1. Problem sheet on "Lie Groups and Their Representations"

Exercise 1.1 Let $\mathcal{A}$ be a Banach algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\mathcal{A}$ has no unit, we cannot directly associate a "unit group" to $\mathcal{A}$. However, there is a different way to do that by considering on $\mathcal{A}$ the multiplication

$$x * y := x + y + xy.$$

Show that:

(a) The space $\mathcal{A}_+ := \mathcal{A} \times \mathbb{K}$ is a unital Banach algebra with respect to the multiplication

$$(a,t)(a',t') := (aa' + ta' + t'a, tt')$$

and

$$\|(a,t)\| := \|a\| + |t|.$$

(b) The map $\eta: \mathcal{A} \to \mathcal{A}_+, x \mapsto (x, 1)$ is injective and satisfies $\eta(x*y) = \eta(x)\eta(y)$. Conclude in particular that $(\mathcal{A}, *, 0)$ is a monoid, i.e., a semigroup with neutral element 0.

(c) An element $a \in \mathcal{A}$ is said to be *quasi-invertible* if it is an invertible element in the monoid $(\mathcal{A}, *, 0)$. Show that the set $\mathcal{A}^\times$ of quasi-invertible elements of $\mathcal{A}$ is an open subset and that $(\mathcal{A}^\times, *, 0)$ is a topological group.

Exercise 1.2 Let $\mathcal{H}$ be a Hilbert space. Show that:

(1) The involution on $B(\mathcal{H})$ is continuous with respect to the weak operator topology.

(2) On every bounded subset $K \subseteq B(\mathcal{H})$, the multiplication $(A, B) \mapsto AB$ is continuous with respect to the strong operator topology.

(3) On the unit sphere $\mathbb{S} := \{x \in \mathcal{H}: \|x\| = 1\}$ the norm topology coincides with the weak topology.

Exercise 1.3 (Antilinear Isometries) Let $\mathcal{H}$ be a complex Hilbert space. Show that:

(a) There exists an antilinear isometric involution τ on $\mathcal{H}$. Hint: Use an orthonormal basis $(e_j)_{j \in J}$ of $B(\mathcal{H})$.

(b) A map $\varphi: \mathcal{H} \to \mathcal{H}$ is an antilinear isometry if and only if

$$\langle \varphi(v), \varphi(w) \rangle = \langle w, v \rangle \quad \text{for} \quad v, w \in \mathcal{H}.$$

(c) If σ is an antilinear isometric involution of $\mathcal{H}$, then there exists an orthonormal basis $(e_j)_{j \in J}$ fixed pointwise by σ. Hint: Show that $\mathcal{H}^\sigma := \{v \in \mathcal{H}: \sigma(v) = v\}$ is a real Hilbert space with $\mathcal{H}^\sigma \oplus i\mathcal{H}^\sigma = \mathcal{H}$ and pick an ONB in $\mathcal{H}^\sigma$.

Exercise 1.4 (Antilinear Isometries) Let $\mathcal{H}$ be a complex Hilbert space. Show that:

(a) In the group $\mathrm{U}_s(\mathcal{H})$ of semilinear (=linear or antilinear) surjective isometries of $\mathcal{H}$, the unitary group $\mathrm{U}(\mathcal{H})$ is a normal subgroup of index 2.

(b) Each antilinear isometry φ of $\mathcal{H}$ induces a map $\overline{\varphi}\colon \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H}), [v] \mapsto [\varphi(v)]$ preserving $\beta([v],[w]) = \frac{|\langle v,w\rangle|^2}{\|v\|^2\|w\|^2}$, i.e.,

$$\beta(\overline{\varphi}[v], \overline{\varphi}[w]) = \frac{|\langle \varphi(v), \varphi(w)\rangle|^2}{\|\varphi(v)\|^2\|\varphi(w)\|^2} = \beta([v],[w]).$$

(c) An element $g \in \mathrm{U}(\mathcal{H})$ induces the identity on $\mathbb{P}(\mathcal{H})$ if and only if $g \in \mathbb{T}\mathbf{1}$.

(d) If there exists an antilinear isometry inducing the identity on $\mathbb{P}(\mathcal{H})$, then $\dim \mathcal{H} = 1$. Hint: Show first that $\sigma^2 = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{T}$. Find $\mu \in \mathbb{T}$ such that $\tau := \mu\sigma$ is an involution and use Exercise 1.3(c).

Exercise 1.5 Let G be a group, endowed with a topology τ. Show that (G,τ) is a topological group if (and only if) the following conditions are satisfied:

(i) The left multiplication maps $\lambda_g\colon G \to G, x \mapsto gx$ are continuous.

(ii) The right multiplication maps $\rho_g\colon G \to G, x \mapsto xg$ are continuous.

(iii) The inversion map $\eta_G\colon G \to G$ is continuous in $\mathbf{1}$.

(iv) The multiplication $m_G\colon G \times G \to G$ is continuous in $(\mathbf{1},\mathbf{1})$.

Exercise 1.6 Let G and N be topological groups and suppose that the homomorphism $\alpha\colon G \to \mathrm{Aut}(N)$ defines a continuous map

$$G \times N \to N, \quad (g,n) \mapsto \alpha(g)(n).$$

Then $N \times G$ is a group with respect to the multiplication

$$(n,g)(n',g') := (n\alpha(g)(n'), gg'),$$

called the semidirect product of N and G with respect to α. It is denoted $N \rtimes_\alpha G$. Show that it is a topological group with respect to the product topology.

A typical example is the group

$$\mathrm{Mot}(\mathcal{H}) := \mathcal{H} \rtimes_\alpha \mathrm{U}(\mathcal{H})$$

of affine isometries of a complex Hilbert space $\mathcal{H}$; also called the *motion group*. In this case $\alpha(g)(v) = gv$ and $\mathrm{Mot}(\mathcal{H})$ acts on $\mathcal{H}$ by $(b,g).v := b + gv$ (hence the name). On $\mathrm{U}(\mathcal{H})$ we may either use the norm topology or the strong topology. For both we obtain group topologies on $\mathrm{Mot}(\mathcal{H})$ (verify this!).