An Introduction to Unitary Representations of Lie Groups

Karl-Hermann Neeb

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Notation and Conventions

If $\mathcal{H}$ is a complex Hilbert space, then its scalar product is written $\langle\cdot,\cdot\rangle$. It is linear in the first and antilinear in the second argument

$$\lambda\langle v,w\rangle = \langle\lambda v,w\rangle = \langle v,\overline{\lambda}w\rangle,$$

and $\|v\| := \sqrt{\langle v,v\rangle}$ is the corresponding norm.

- $\mathbb{N} := \{1,2,3,\ldots\}$
- $\mathbb{R}_+ := \{x\in\mathbb{R}\colon x\geq 0\} = [0,\infty[$.
- $\mathbb{R}^\times := \mathbb{R}\setminus\{0\}$, $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$, $\mathbb{T} := \{z\in\mathbb{C}\colon |z|=1\}$.

For Banach spaces X and Y we write

$$B(X,Y) := \{A\colon X\to Y\colon A \text{ linear}, \ \|A\|<\infty\}$$

for the Banach space of bounded linear operators from X to Y. For $X=Y$ we abbreviate $B(X) := B(X,X)$ and write $\mathrm{GL}(X)$ for the group of invertible elements in $B(X)$. If $\mathcal{H}$ is a complex Hilbert space, then we have an antilinear isometric map $B(\mathcal{H})\to B(\mathcal{H}), A\mapsto A^*$, determined uniquely by

$$\langle Av,w\rangle = \langle v,A^*w\rangle \quad \text{for } v,w\in\mathcal{H}.$$

We write

$$\mathrm{U}(\mathcal{H}) := \{g\in\mathrm{GL}(\mathcal{H})\colon g^{-1}=g^*\}$$

for the unitary group. For $\mathcal{H}=\mathbb{C}^n$, the corresponding matrix group is denoted

$$\mathrm{U}_n(\mathbb{C}) := \{g\in\mathrm{GL}_n(\mathbb{C})\colon g^{-1}=g^*\}.$$

If G is a group, we write $\mathbf{1}$ for its neutral element and

$$\lambda_g(x)=gx, \quad \rho_g(x)=xg \quad \text{and} \quad c_g(x)=gxg^{-1}$$

for left multiplications, right multiplications, resp., conjugations.

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Introduction

A *unitary representation* of a group G is a homomorphism $\pi\colon G \to \mathrm{U}(\mathcal{H})$ of G to the unitary group

$$\mathrm{U}(\mathcal{H}) = \{g \in \mathrm{GL}(\mathcal{H})\colon g^* = g^{-1}\}$$

of a complex Hilbert space $\mathcal{H}$. Such a representation is said to be *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only $\pi(G)$-invariant closed subspaces of $\mathcal{H}$. The two fundamental problems in representation theory are:

(FP1) To classify, resp., parameterize the irreducible representations of G, and

(FP2) to explain how a general unitary representation can be decomposed into irreducible ones. This is called the problem of *harmonic analysis* because it contains in particular the expansion of a periodic L^2-function as a Fourier series.

As formulated above, both problems are not well-posed. First, one has to specify the class of representations one is interested in, and this class may depend on the group G, resp., additional structure on this group. Only in very rare situations, one studies arbitrary unitary representations. If, f.i., G is a *topological group*, i.e., if G carries a topology for which the group operations are continuous, one is only interested in unitary representations which are continuous in the sense that for each $v \in \mathcal{H}$, the orbit map

$$\pi^v\colon G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

is continuous. If G is a *Lie group*, a concept refining that of a topological group, so that it makes sense to talk about smooth functions on G, then we consider only representations for which the subspace

$$\mathcal{H}^\infty := \{v \in \mathcal{H}\colon \pi^v\colon G \to \mathcal{H} \ \text{ is smooth}\}$$

of smooth vectors is dense in $\mathcal{H}$.

This means that there are three basic contexts for representation theory

- the discrete context (G is considered as a discrete group, no restrictions)
- the topological context (G is a topological group; continuity required)
- the Lie context (G is a Lie group; smoothness required).

In each of these contexts, the two fundamental problems mentioned above are of a completely different nature because they concern different classes of representations. For example one can show that the harmonic analysis problem has a good solution for the topological group $\mathrm{GL}_2(\mathbb{R})$, but not for the same group, considered as a discrete one. To make statements like this more precise is one of the fundamental tasks of representation theory.

To give a first impression of the major difficulties involved in this program, we discuss some examples.

Remark 0.0.1. If the group G is abelian, then one can show that all irreducible representations $(\pi, \mathcal{H})$ are one-dimensional, so that $\pi(a) = \chi(a)\mathbf{1}$ holds for a group homomorphism

$$\chi\colon A \to \mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$$

into the circle group. Such homomorphisms are called *characters.* For a topological group G we write

$$\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$$

for the set of continuous characters. They form a group under pointwise multiplication, called the *character group.* Since all irreducible representations are one-dimensional, the group $\widehat{G}$ parameterizes the irreducible representations and the solution of (FP1) therefore consists in a description of the group $\widehat{G}$.

The second fundamental problem (FP2) is much harder to deal with. If $(\pi, \mathcal{H})$ is a unitary representation, then each irreducible subrepresentation is one-dimensional, hence given by a G-eigenvector $v \in \mathcal{H}$ satisfying

$$\pi(g)v = \chi(g)v \quad \text{for} \quad g \in G$$

and some character $\chi \in \widehat{G}$. Now one would like to "decompose" $\mathcal{H}$ into the G-eigenspaces

$$\mathcal{H}_\chi := \{v \in \mathcal{H} \colon (\forall g \in G)\, \pi(g)v = \chi(g)v\}.$$

As the following two examples show, there are situations where this is possible, but this is not always the case.

Example 0.0.2. To solve (FP1) for the group $G := \mathbb{T}$, we first note that for each $n \in \mathbb{Z}$, $\chi_n(z) := z^n$ defines a continuous character of $\mathbb{T}$, and one can show that these are all continuous characters. Therefore $\chi_n\chi_m = \chi_{n+m}$ leads to

$$\widehat{\mathbb{T}} = \mathrm{Hom}(\mathbb{T}, \mathbb{T}) \cong \mathbb{Z}.$$

The group $\mathbb{T}$ has a continuous representation on the space $\mathcal{H} = L^2(\mathbb{T}, \mu)$, where μ is the probability measure on $\mathbb{T}$ specified by

$$\int_{\mathbb{T}} f(z)\, d\mu(z) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, dt$$

and

$$(\pi(t)f)(z) := f(tz).$$

Then the $\mathbb{T}$-eigenfunctions in $\mathcal{H}$ corresponding to χ_n are the functions χ_n themselves, and it is a basic result in the theory of Fourier series that any function $f \in \mathcal{H}$ can be expanded as a Fourier series

$$f = \sum_{n \in \mathbb{Z}} a_n \chi_n$$

converging in $\mathcal{H}$. In this sense $\mathcal{H}$ is a (topological) direct sum of the subspaces $\mathbb{C}\chi_n$, which means that the representation π decomposes nicely into irreducible pieces.

Example 0.0.3. For the group $G := \mathbb{R}$, the solution of (FP1) asserts that each continuous character is of the form

$$\chi_\lambda(x) := e^{i\lambda x}, \quad \lambda \in \mathbb{R},$$

so that $\chi_\lambda \chi_\mu = \chi_{\lambda+\mu}$ leads to

$$\widehat{\mathbb{R}} \cong \mathbb{R}.$$

The group $\mathbb{R}$ has a continuous representation on the space $\mathcal{H} = L^2(\mathbb{R}, dx)$ given by

$$(\pi(x)f)(y) := f(x+y).$$

Then the $\mathbb{R}$-eigenfunctions in $\mathcal{H}$ corresponding to χ_λ solve the equation

$$f(x+y) = e^{i\lambda x} f(y)$$

almost everywhere on $\mathbb{R}$, which leads to $f = c\chi_\lambda$ for some $c \in \mathbb{C}$. Unfortunately, the functions χ_λ are not square integrable because $|\chi_\lambda| = 1$ and $\mathbb{R}$ has infinite measure. Therefore the representation $(\pi, \mathcal{H})$ contains no irreducible subspaces and we need refined methods to say what it means to decompose it into irreducible ones.

The problem of decomposing functions into simpler pieces with respect to the transformation behavior under a certain symmetry group arises in many situations, not only in mathematics, but also in the natural sciences. In mathematics, unitary representation theory has many applications in areas ranging from number theory, geometry, real and complex analysis to partial differential equations.

However, one of the strongest motivation for the systematic development of the theory of unitary group representations, started in the 1940s, was its close connection to Quantum Mechanics. This connection is due to the fact that the states of a quantum mechanical system are modeled by the set

$$\mathbb{P}(\mathcal{H}) := \{[v] = \mathbb{C}v \colon 0 \neq v \in \mathcal{H}\}$$

of one-dimensional subspaces of a complex Hilbert space, its *projective space*. This spaces carries several interesting structures. The most important one for physics is the function

$$\beta \colon \mathbb{P}(\mathcal{H}) \times \mathbb{P}(\mathcal{H}) \to [0,1], \quad \beta([v],[w]) := \frac{|\langle v, w\rangle|^2}{\|v\|^2 \|w\|^2},$$

which is interpreted as a *transition probability* between the two states $[v]$ and $[w]$. A central feature of quantum physical models is that systems are often specified by their symmetries. This means that each system has a symmetry group G. This group acts on the corresponding set $\mathbb{P}(\mathcal{H})$ of states in such a way that it preserves the transition probabilities, i.e., we have a group action $G \times \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H}), (g,[v]) \mapsto g[v]$, satisfying

$$\beta(g[v], g[w]) = \beta([v],[w]) \quad \text{for} \quad g \in G, 0 \neq v, w \in \mathcal{H}.$$

To link these structures to unitary representations, we have to quote Wigner's fundamental theorem that for each bijection φ of $\mathbb{P}(\mathcal{H})$ preserving β, there exists either a linear or an antilinear surjective isometry $\psi\colon \mathcal{H} \to \mathcal{H}$ such that $\varphi[v] = [\psi(v)]$ holds for each $0 \neq v \in \mathcal{H}$. We thus obtain a surjective homomorphism

$$\Gamma\colon\ \mathrm{U}_s(\mathcal{H}) \to \mathrm{Aut}(\mathbb{P}(\mathcal{H}), \beta), \quad \psi \mapsto \varphi,$$

where $\mathrm{U}_s(\mathcal{H})$ denotes the set of *semilinear unitary operators*, where semilinear means either linear or antilinear. If $G \subseteq \mathrm{Aut}(\mathbb{P}(\mathcal{H}), \beta)$ is a quantum mechanical symmetry group, we thus obtain a subgroup $G^\sharp := \Gamma^{-1}(G) \subseteq \mathrm{U}_s(\mathcal{H})$ with a semilinear unitary representation on $\mathcal{H}$, and the subgroup $G^\sharp_u := G^\sharp \cap \mathrm{U}(\mathcal{H})$ of index two is a unitary group. One subtlety that we observe here is that the homomorphism Γ is not injective, if $\dim \mathcal{H} > 1$, its kernel consists of the circle group $\mathbb{T}\mathbf{1} = \{z\mathbf{1} \colon |z| = 1, z \in \mathbb{C}\}$, so that

$$G \cong G^\sharp/\mathbb{T}$$

and $G^\sharp$ is an *extension* of the group G by the circle group $\mathbb{T}$.

It is this line of reasoning that leads us from quantum mechanical modeling to the problem of classifying irreducible unitary representation of a group G, resp., its extensions $G^\sharp$, because these representations correspond to systems with the same kind of symmetry. Similar questions lead in particular to the problem of classifying elementary particles in terms of representations of certain compact Lie groups.

Chapter 1

Continuous Unitary Representations

Throughout these notes we shall mainly be concerned with continuous representations of topological groups. Therefore Section 1.1 introduces topological groups and some important examples. In Section 1.2 we discuss continuity of unitary representations and provide some methods that can be used to verify continuity easily in many situations. We also introduce the strong topology on the unitary group $\mathrm{U}(\mathcal{H})$ for which a continuous unitary representation of G is the same as a continuous group homomorphism $\pi\colon G \to \mathrm{U}(\mathcal{H})$. As a first step in the decomposition theory of representations, we discuss in Section 1.3 direct sums of unitary representations and show that every representation is a direct sum of cyclic ones. Later we shall study cyclic representations in terms of positive definite functions on G.

1.1 Topological Groups

Definition 1.1.1. A *topological group* is a pair (G, τ) of a group G and a Hausdorff topology τ for which the group operations

$$m_G\colon G \times G \to G, \quad (x, y) \mapsto xy \quad \text{and} \quad \eta_G\colon G \to G, \quad x \mapsto x^{-1}$$

are continuous if $G \times G$ carries the product topology. Then we call τ a *group topology* on the group G.

Remark 1.1.2. The continuity of the group operations can also be translated into the following conditions which are more direct than referring to the product topology on G. The continuity of the multiplication m_G in $(x, y) \in G \times G$ means that for each neighborhood V of xy there exist neighborhoods U_x of x and U_y of y with $U_xU_y \subseteq V$. Similarly, the continuity of the inversion map η_G in x means that for each neighborhood V of x^{-1}, there exist neighborhoods U_x of x with $U_x^{-1} \subseteq V$.

Remark 1.1.3. For a group G with a topology τ, the continuity of m_G and η_G already follows from the continuity of the single map

$$\varphi \colon G \times G \to G, \quad (g,h) \mapsto gh^{-1}.$$

In fact, if φ is continuous, then the inversion $\eta_G(g) = g^{-1} = \varphi(\mathbf{1}, g)$ is the composition of φ and the continuous map $G \to G \times G, g \mapsto (\mathbf{1}, g)$. The continuity of η_G further implies that the product map

$$\mathrm{id}_G \times \eta_G \colon G \times G \to G \times G, \quad (g,h) \mapsto (g, h^{-1})$$

is continuous, and therefore $m_G = \varphi \circ (\mathrm{id}_G \times \eta_G)$ is continuous.

Remark 1.1.4. Every subgroup of a topological group is a topological group.

Example 1.1.5. (1) $G = (\mathbb{R}^n, +)$ is an abelian topological group with respect to any metric defined by a norm.

More generally, the additive group $(X, +)$ of every Banach space is a topological group.

(2) $(\mathbb{C}^\times, \cdot)$ is a topological group and the circle group $\mathbb{T} := \{z \in \mathbb{C}^\times \colon |z| = 1\}$ is a compact subgroup.

(3) The group $\mathrm{GL}_n(\mathbb{R})$ of invertible $(n \times n)$-matrices is a topological group with respect to matrix multiplication. The continuity of the inversion follows from Cramer's Rule, which provides an explicit formula for the inverse in terms of determinants: For $g \in \mathrm{GL}_n(\mathbb{R})$, we define $b_{ij}(g) := \det(g_{mk})_{m \neq j, k \neq i}$. Then the inverse of g is given by

$$(g^{-1})_{ij} = \frac{(-1)^{i+j}}{\det g} b_{ij}(g)$$

(see Proposition 1.1.10 for a different argument).

(4) Any group G is a topological group with respect to the discrete topology.

We have already argued above that the group $\mathrm{GL}_n(\mathbb{R})$ carries a natural group topology. This group is the unit group of the algebra $M_n(\mathbb{R})$ of real $(n \times n)$-matrices. As we shall see now, there is a vast generalization of this construction.

Definition 1.1.6. A *Banach algebra* is a triple $(\mathcal{A}, m_{\mathcal{A}}, \|\cdot\|)$ of a Banach space $(\mathcal{A}, \|\cdot\|)$, together with an associative bilinear multiplication

$$m_{\mathcal{A}} \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}, (a,b) \mapsto ab$$

for which the norm $\|\cdot\|$ is *submultiplicative*, i.e.,

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \text{for} \quad a, b \in \mathcal{A}.$$

By abuse of notation, we shall mostly call $\mathcal{A}$ a Banach algebra, if the norm and the multiplication are clear from the context.

A *unital Banach algebra* is a pair $(\mathcal{A}, \mathbf{1})$ of a Banach algebra $\mathcal{A}$ and an element $\mathbf{1} \in \mathcal{A}$ satisfying $\mathbf{1}a = a\mathbf{1} = a$ for each $a \in \mathcal{A}$.

The subset

$$\mathcal{A}^\times := \{a \in \mathcal{A} \colon (\exists b \in \mathcal{A})\ ab = ba = \mathbf{1}\}$$

is called the *unit group of* $\mathcal{A}$ (cf. Exercise 1.1.11).

Example 1.1.7. (a) If $(X, \|\cdot\|)$ is a Banach space, then the space $B(X)$ of continuous linear operators $A \colon X \to X$ is a unital Banach algebra with respect to the *operator norm*

$$\|A\| := \sup\{\|Ax\| \colon x \in X, \|x\| \leq 1\}$$

and composition of maps. Note that the submultiplicativity of the operator norm, i.e.,

$$\|AB\| \leq \|A\| \cdot \|B\|,$$

is an immediate consequence of the estimate

$$\|ABx\| \leq \|A\| \cdot \|Bx\| \leq \|A\| \cdot \|B\| \cdot \|x\| \quad \text{for} \quad x \in X.$$

In this case the unit group is also denoted $\mathrm{GL}(X) := B(X)^\times$.

(b) If X is a compact space and $\mathcal{A}$ a Banach algebra, then the space $C(X, \mathcal{A})$ of $\mathcal{A}$-valued continuous functions on X is a Banach algebra with respect to pointwise multiplication $(fg)(x) := f(x)g(x)$ and the norm $\|f\| := \sup_{x \in X} \|f(x)\|$ (Exercise 1.1.9)

(c) An important special case of (b) arises for $\mathcal{A} = M_n(\mathbb{C})$, where we obtain $C(X, M_n(\mathbb{C}))^\times = C(X, \mathrm{GL}_n(\mathbb{C})) = \mathrm{GL}_n(C(X, \mathbb{C}))$.

Example 1.1.8. For any norm $\|\cdot\|$ on $\mathbb{R}^n$, the choice of a basis yields an isomorphism of algebras $M_n(\mathbb{R}) \cong B(\mathbb{R}^n)$, so that $\mathrm{GL}_n(\mathbb{R}) \cong B(\mathbb{R}^n)^\times = \mathrm{GL}(\mathbb{R}^n)$.

Remark 1.1.9. In a Banach algebra $\mathcal{A}$, the multiplication is continuous because $a_n \to a$ and $b_n \to b$ implies $\|b_n\| \to \|b\|$ and therefore

$$\|a_n b_n - ab\| = \|a_n b_n - ab_n + ab_n - ab\| \leq \|a_n - a\| \cdot \|b_n\| + \|a\| \cdot \|b_n - b\| \to 0.$$

In particular, left and right multiplications

$$\lambda_a \colon \mathcal{A} \to \mathcal{A}, x \mapsto ax, \quad \text{and} \quad \rho_a \colon \mathcal{A} \to \mathcal{A}, x \mapsto xa,$$

are continuous with

$$\|\lambda_a\| \leq \|a\| \quad \text{and} \quad \|\rho_a\| \leq \|a\|.$$

Proposition 1.1.10. *The unit group $\mathcal{A}^\times$ of a unital Banach algebra is an open subset and a topological group with respect to the topology defined by the metric $d(a, b) := \|a - b\|$.*

Proof. The proof is based on the convergence of the Neumann series $\sum_{n=0}^\infty x^n$ for $\|x\| < 1$. For any such x we have

$$(\mathbf{1} - x)\sum_{n=0}^\infty x^n = \Big(\sum_{n=0}^\infty x^n\Big)(\mathbf{1} - x) = \mathbf{1},$$

so that $\mathbf{1} - x \in \mathcal{A}^\times$. We conclude that the open unit ball $B_1(\mathbf{1})$ is contained in $\mathcal{A}^\times$.

Next we note that left multiplications $\lambda_g \colon \mathcal{A} \to \mathcal{A}$ with elements $g \in \mathcal{A}^\times$ are continuous (Remark 1.1.9), hence homeomorphisms because $\lambda_g^{-1} = \lambda_{g^{-1}}$ is also continuous. Therefore $gB_1(\mathbf{1}) = \lambda_g B_1(\mathbf{1}) \subseteq \mathcal{A}^\times$ is an open subset, showing that g is an interior point of $\mathcal{A}^\times$. Therefore $\mathcal{A}^\times$ is open.

The continuity of the multiplication of $\mathcal{A}^\times$ follows from the continuity of the multiplication on $\mathcal{A}$ by restriction and corestriction (Remark 1.1.9). The continuity of the inversion in $\mathbf{1}$ follows from the estimate

$$\|(\mathbf{1} - x)^{-1} - \mathbf{1}\| = \|\sum_{n=1}^\infty x^n\| \leq \sum_{n=1}^\infty \|x\|^n = \frac{1}{1 - \|x\|} - 1 = \frac{\|x\|}{1 - \|x\|},$$

which tends to 0 for $x \to 0$. The continuity of the inversion in $g_0 \in \mathcal{A}^\times$ now follows from the continuity in $\mathbf{1}$ via

$$g^{-1} - g_0^{-1} = g_0^{-1}(g_0 g^{-1} - \mathbf{1}) = g_0^{-1}((g g_0^{-1})^{-1} - \mathbf{1})$$

because left and right multiplication with g_0^{-1} is continuous. This shows that $\mathcal{A}^\times$ is a topological group. □

As we shall see throughout these notes, dealing with unitary representations often leads us to Banach algebras with an extra structure given by an involution.

Definition 1.1.11. (a) An *involutive algebra* $\mathcal{A}$ is a pair $(\mathcal{A}, *)$ of a complex algebra $\mathcal{A}$ and a map $\mathcal{A} \to \mathcal{A}, a \mapsto a^*$, satisfying

(1) $(a^*)^* = a$ (Involutivity)

(2) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ (Antilinearity).

(3) $(ab)^* = b^* a^*$ ($*$ is an antiautomorphism of $\mathcal{A}$).

Then $*$ is called an *involution* on $\mathcal{A}$. A *Banach-$*$-algebra* is an involutive algebra $(\mathcal{A}, *)$, where $\mathcal{A}$ is a Banach algebra and $\|a^*\| = \|a\|$ holds for each $a \in \mathcal{A}$. If, in addition,

$$\|a^* a\| = \|a\|^2 \quad \text{for} \quad a \in \mathcal{A},$$

then $(\mathcal{A}, *)$ is called a *C^*-algebra*.

Example 1.1.12. (a) The algebra $B(\mathcal{H})$ of bounded operators on a complex Hilbert space $\mathcal{H}$ is a C^*-algebra. Here the main point is that for each $A \in B(\mathcal{H})$ we have

$$\|A\| = \sup\{|\langle Av, w\rangle| \colon \|v\|, \|w\| \leq 1\},$$

which immediately implies that $\|A^*\| = \|A\|$. It also implies that

$$\|A^*A\| = \sup\{|\langle Av, Aw\rangle|\colon \|v\|, \|w\| \leq 1\} \geq \sup\{\|Av\|^2\colon \|v\| \leq 1\} = \|A\|^2,$$

and since $\|A^*A\| \leq \|A^*\| \cdot \|A\| = \|A\|^2$ is also true, we see that $B(\mathcal{H})$ is a C^*-algebra.

(b) From (a) it immediately follows that every closed $*$-invariant subalgebra of $\mathcal{A} \subseteq B(\mathcal{H})$ also is a C^*-algebra.

(c) If X is a compact space, then the Banach space $C(X, \mathbb{C})$, endowed with $\|f\| := \sup_{x\in X} |f(x)|$ is a C^*-algebra with respect to $f^*(x) := \overline{f(x)}$. In this case $\|f^*f\| = \||f|^2\| = \|f\|^2$ is trivial.

(d) If X is a locally compact space, then we say that a continuous function $f\colon X \to \mathbb{C}$ *vanishes at infinity* if for each $\varepsilon > 0$ there exists a compact subset $K \subseteq X$ with $|f(x)| \leq \varepsilon$ for $x \notin K$. We write $C_0(X, \mathbb{C})$ for the set of all continuous functions vanishing at infinity and endow it with the norm $\|f\| := \sup_{x\in X} |f(x)|$. (cf. Exercise 1.1.10). Then $C_0(X, \mathbb{C})$ is a C^*-algebra with respect the involution $f^*(x) := \overline{f(x)}$.

Example 1.1.13. (a) If $\mathcal{H}$ is a (complex) Hilbert space, then its unitary group

$$\mathrm{U}(\mathcal{H}) := \{g \in \mathrm{GL}(\mathcal{H})\colon g^* = g^{-1}\}$$

is a topological group with respect to the metric $d(g, h) := \|g-h\|$. It is a closed subgroup of $\mathrm{GL}(\mathcal{H}) = B(\mathcal{H})^\times$.

For $\mathcal{H} = \mathbb{C}^n$, endowed with the standard scalar product, we also write

$$\mathrm{U}_n(\mathbb{C}) := \{g \in \mathrm{GL}_n(\mathbb{C})\colon g^* = g^{-1}\} \cong \mathrm{U}(\mathbb{C}^n),$$

and note that

$$\mathrm{U}_1(\mathbb{C}) = \{z \in \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})\colon |z| = 1\} \cong \mathbb{T}$$

is the circle group.

(b) If $\mathcal{A}$ is a unital C^*-algebra, then its *unitary group*

$$\mathrm{U}(\mathcal{A}) := \{g \in \mathcal{A}\colon gg^* = g^*g = \mathbf{1}\}$$

also is a topological group with respect to the norm topology.

Exercises for Section 1.1

Exercise 1.1.1. (Antilinear Isometries) Let $\mathcal{H}$ be a complex Hilbert space. Show that:

(a) There exists an antilinear isometric involution τ on $\mathcal{H}$. Hint: Use an orthonormal basis $(e_j)_{j\in J}$ of $\mathcal{H}$.

(b) A map $\varphi\colon \mathcal{H} \to \mathcal{H}$ is an antilinear isometry if and only if

$$\langle\varphi(v), \varphi(w)\rangle = \langle w, v\rangle \quad \text{for} \quad v, w \in \mathcal{H}.$$

(c) If σ is an antilinear isometric involution of $\mathcal{H}$, then there exists an orthonormal basis $(e_j)_{j\in J}$ fixed pointwise by σ. Hint: Show that $\mathcal{H}^\sigma := \{v \in \mathcal{H} \colon \sigma(v) = v\}$ is a real Hilbert space with $\mathcal{H}^\sigma \oplus i\mathcal{H}^\sigma = \mathcal{H}$ and pick an ONB in $\mathcal{H}^\sigma$.

Exercise 1.1.2. (Antilinear Isometries) Let $\mathcal{H}$ be a complex Hilbert space. Show that:

(a) In the group $\mathrm{U}_s(\mathcal{H})$ of semilinear (=linear or antilinear) surjective isometries of $\mathcal{H}$, the unitary group $\mathrm{U}(\mathcal{H})$ is a normal subgroup of index 2.

(b) Each antilinear isometry φ of $\mathcal{H}$ induces a map $\overline{\varphi} \colon \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H}), [v] \mapsto [\varphi(v)]$ preserving $\beta([v],[w]) = \frac{|\langle v,w\rangle|^2}{\|v\|^2\|w\|^2}$, i.e.,

$$\beta(\overline{\varphi}[v], \overline{\varphi}[w]) = \frac{|\langle \varphi(v), \varphi(w)\rangle|^2}{\|\varphi(v)\|^2\|\varphi(w)\|^2} = \beta([v],[w]).$$

(c) An element $g \in \mathrm{U}(\mathcal{H})$ induces the identity on $\mathbb{P}(\mathcal{H})$ if and only if $g \in \mathbb{T}\mathbf{1}$.

(d) If there exists an antilinear isometry inducing the identity on $\mathbb{P}(\mathcal{H})$, then $\dim \mathcal{H} = 1$. Hint: Show first that $\sigma^2 = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{T}$. Find $\mu \in \mathbb{T}$ such that $\tau := \mu\sigma$ is an involution and use Exercise 1.1.1(c).

Exercise 1.1.3. Let G be a topological group. Show that the following assertions hold:

(i) The left multiplication maps $\lambda_g \colon G \to G, x \mapsto gx$ are homeomorphisms.

(ii) The right multiplication maps $\rho_g \colon G \to G, x \mapsto xg$ are homeomorphisms.

(iii) The conjugation maps $c_g \colon G \to G, x \mapsto gxg^{-1}$ are homeomorphisms.

(iv) The inversion map $\eta_G \colon G \to G, x \mapsto x^{-1}$ is a homeomorphism.

Exercise 1.1.4. Let G be a group, endowed with a topology τ. Show that (G,τ) is a topological group if the following conditions are satisfied:

(i) The left multiplication maps $\lambda_g \colon G \to G, x \mapsto gx$ are continuous.

(ii) The inversion map $\eta_G \colon G \to G, x \mapsto x^{-1}$ is continuous.

(iii) The multiplication $m_G \colon G \times G \to G$ is continuous in $(\mathbf{1},\mathbf{1})$.

Hint: Use (i) and (ii) to derive that all right multiplications and hence all conjugations are continuous.

Exercise 1.1.5. Let G be a group, endowed with a topology τ. Show that (G,τ) is a topological group if the following conditions are satisfied:

(i) The left multiplication maps $\lambda_g \colon G \to G, x \mapsto gx$ are continuous.

(ii) The right multiplication maps $\rho_g \colon G \to G, x \mapsto xg$ are continuous.

(iii) The inversion map $\eta_G \colon G \to G$ is continuous in $\mathbf{1}$.

(iv) The multiplication $m_G \colon G \times G \to G$ is continuous in $(\mathbf{1}, \mathbf{1})$.

Exercise 1.1.6. Show that if $(G_i)_{i \in I}$ is a family of topological groups, then the product group $G := \prod_{i \in I} G_i$ is a topological group with respect to the product topology.

Exercise 1.1.7. Let G and N be topological groups and suppose that the homomorphism $\alpha \colon G \to \mathrm{Aut}(N)$ defines a continuous map

$$G \times N \to N, \quad (g, n) \mapsto \alpha(g)(n).$$

Then $N \times G$ is a group with respect to the multiplication

$$(n, g)(n', g') := (n\alpha(g)(n'), gg'),$$

called the semidirect product of N and G with respect to α. It is denoted $N \rtimes_\alpha G$. Show that it is a topological group with respect to the product topology.

A typical example is the group

$$\mathrm{Mot}(\mathcal{H}) := \mathcal{H} \rtimes_\alpha \mathrm{U}(\mathcal{H})$$

of affine isometries of a complex Hilbert space $\mathcal{H}$; also called the *motion group*. In this case $\alpha(g)(v) = gv$ and $\mathrm{Mot}(\mathcal{H})$ acts on $\mathcal{H}$ by $(b, g).v := b + gv$ (hence the name). On $\mathrm{U}(\mathcal{H})$ we may either use the norm topology or the strong topology. For both we obtain group topologies on $\mathrm{Mot}(\mathcal{H})$ (verify this!).

Exercise 1.1.8. Let X be a topological space and G be a topological group. We want to define a topology on the group $C(X, G)$, endowed with the pointwise product $(fg)(x) := f(x)g(x)$. We specify a set τ of subsets of $C(X, G)$ by $O \in \tau$ if for each $f \in O$ there exists a compact subset $K \subseteq X$ and an open $\mathbf{1}$-neighborhood $U \subseteq G$ such that

$$W(K, U) := \{f \in C(X, G) \colon f(K) \subseteq U\}$$

satisfies $gW(K, U) \subseteq O$. Show that τ defines a group topology on $C(X, G)$. It is called the *compact open topology*, or the *topology of uniform convergence on compact subsets of* X. Hint: You may cut the problem into the following steps:

(i) For compact subsets $K_1, \ldots, K_n$ of X and open $\mathbf{1}$-neighborhoods $U_1, \ldots, U_n$ in G, we have

$$W\Big(\bigcup_{i=1}^n K_i, \bigcap_{i=1}^n U_i\Big) \subseteq \bigcap_{i=1}^n W(K_i, U_i).$$

(ii) $W(K, U) \in \tau$ for $K \subseteq X$ compact and $U \subseteq G$ an open $\mathbf{1}$-neighborhood. Hint: If $f(K) \subseteq U$, there exists a $\mathbf{1}$-neighborhood V in G with $f(K)V \subseteq U$, and then $fW(K, V) \subseteq W(K, U)$.

(iii) τ is a topology on $C(X, G)$.

(iv) Use Exercise 1.1.4 to show that $C(X,G)$ is a topological group. For the continuity of the multiplication in $\mathbf{1}$, use that $W(K,V)W(K,V) \subseteq W(K,U)$ whenever $VV \subseteq U$.

Exercise 1.1.9. Let X be a compact space and $\mathcal{A}$ be a Banach algebra. Show that:

(a) The space $C(X,\mathcal{A})$ of $\mathcal{A}$-valued continuous functions on X is a complex associative algebra with respect to pointwise multiplication $(fg)(x) := f(x)g(x)$.

(b) $\|f\| := \sup_{x\in X} \|f(x)\|$ is a submultiplicative norm on $C(X,\mathcal{A})$ for which $C(X,\mathcal{A})$ is complete, hence a Banach algebra. Hint: Continuous functions on compact spaces are bounded and uniform limits of sequences of continuous functions are continuous.

(c) $C(X,\mathcal{A})^\times = C(X,\mathcal{A}^\times)$.

(d) If $\mathcal{A}$ is a C^*-algebra, then $C(X,\mathcal{A})$ is also a C^*-algebra with respect to the involution $f^*(x) := f(x)^*$, $x \in X$.

Exercise 1.1.10. Let X be a locally compact space and $\mathcal{A}$ be a Banach algebra. We say that a continuous function $f\colon X \to \mathcal{A}$ *vanishes at infinity* if for each $\varepsilon > 0$ there exists a compact subset $K \subseteq X$ with $\|f(x)\| \leq \varepsilon$ for $x \notin K$. We write $C_0(X,\mathcal{A})$ for the set of all continuous $\mathcal{A}$-valued functions vanishing at infinity. Show that all assertions of Exercise 1.1.9 remain true in this more general context.

Exercise 1.1.11. Let $\mathcal{A}$ be a complex Banach algebra over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. If $\mathcal{A}$ has no unit, we cannot directly associate a "unit group" to $\mathcal{A}$. However, there is a different way to do that by considering on $\mathcal{A}$ the multiplication

$$x * y := x + y + xy.$$

Show that:

(a) The space $\mathcal{A}_+ := \mathcal{A} \times \mathbb{K}$ is a unital Banach algebra with respect to the multiplication

$$(a,t)(a',t') := (aa' + ta' + t'a, tt').$$

(b) The map $\eta\colon \mathcal{A} \to \mathcal{A}_+, x \mapsto (x,1)$ is injective and satisfies $\eta(x * y) = \eta(x)\eta(y)$. Conclude in particular that $(\mathcal{A},*,0)$ is a monoid, i.e., a semigroup with neutral element 0.

(c) An element $a \in \mathcal{A}$ is said to be *quasi-invertible* if it is an invertible element in the monoid $(\mathcal{A},*,0)$. Show that the set $\mathcal{A}^\times$ of quasi-invertible elements of $\mathcal{A}$ is an open subset and that $(\mathcal{A}^\times,*,0)$ is a topological group.

1.2 Continuous Unitary Representations

For a topological group G, we only want to consider unitary representations which are continuous in some sense. Since we have already seen above that the unitary group $\mathrm{U}(\mathcal{H})$ of a Hilbert space is a topological group with respect to the metric induced by the operator norm, it seems natural to call a unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ continuous if it is continuous with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. However, the norm topology on $\mathrm{U}(\mathcal{H})$ is very fine, so that continuity with respect to this topology is a condition which is much too strong for many applications. We therefore need a suitable weaker topology on the unitary group.

We start by defining some topologies on the space $B(\mathcal{H})$ of all continuous operators which are weaker than the norm topology.

Definition 1.2.1. Let $\mathcal{H}$ be a Hilbert space. On $B(\mathcal{H})$ we define the *weak operator topology* τ_w as the coarsest topology for which all functions

$$f_{v,w}\colon B(\mathcal{H}) \to \mathbb{C}, \quad A \mapsto \langle Av, w\rangle, \quad v, w \in \mathcal{H},$$

are continuous. We define the *strong operator topology* τ_s as the coarsest topology for which all maps

$$B(\mathcal{H}) \to \mathcal{H}, \quad A \mapsto Av, \quad v \in \mathcal{H},$$

are continuous. This topology is also called the *topology of pointwise convergence.*

Remark 1.2.2. (a) Since

$$f_{v,w}(A) - f_{v,w}(B) = \langle (A-B)v, w\rangle \leq \|(A-B)v\| \cdot \|w\|$$

by the Cauchy–Schwarz Inequality, the functions $f_{v,w}$ are continuous on $B(\mathcal{H})$ with respect to the strong operator topology. Therefore the weak operator topology is weaker (=coarser) than the strong one.

(b) In the weak operator topology all left and right multiplications

$$\lambda_A\colon B(\mathcal{H}) \to B(\mathcal{H}), \quad X \mapsto AX \quad \text{ and } \rho_A\colon B(\mathcal{H}) \to B(\mathcal{H}), \quad X \mapsto XA$$

are continuous. Indeed, for $v, w \in \mathcal{H}$, we have

$$f_{v,w}(\lambda_A(X)) = \langle AXv, w\rangle = f_{v,A^*w}(X),$$

so that $f_{v,w} \circ \lambda_A$ is continuous, and this implies that λ_A is continuous. Similarly, we obtain $f_{v,w} \circ \rho_A = f_{Av,w}$, and hence the continuity of ρ_A.

Proposition 1.2.3. *On the unitary group* $\mathrm{U}(\mathcal{H})$ *the weak and the strong operator topology coincide and turn it into a topological group.*

We write $\mathrm{U}(\mathcal{H})_s$ for the topological group $(\mathrm{U}(\mathcal{H}), \tau_s)$.

Proof. For $v \in \mathcal{H}$ and $g_i \to g$ in $\mathrm{U}(\mathcal{H})$ in the weak operator topology, we have

$$\begin{aligned}\|g_i v - gv\|^2 &= \|g_i v\|^2 + \|gv\|^2 - 2\,\mathrm{Re}\langle g_i v, gv\rangle = 2\|v\|^2 - 2\,\mathrm{Re}\langle g_i v, gv\rangle \\ &\to 2\|v\|^2 - 2\,\mathrm{Re}\langle gv, gv\rangle = 0.\end{aligned}$$

Therefore the orbit maps $\mathrm{U}(\mathcal{H}) \to \mathcal{H}, g \mapsto gv$ are continuous with respect to the weak operator topology, so that the weak operator topology on $\mathrm{U}(\mathcal{H})$ is finer than the strong one. Since it is also coarser by Remark 1.2.2, both topologies coincide on $\mathrm{U}(\mathcal{H})$.

The continuity of the multiplication in $\mathrm{U}(\mathcal{H})$ is most easily verified in the strong operator topology, where it follows from the estimate

$$\begin{aligned}\|g_i h_i v - ghv\| &= \|g_i(h_i - h)v + (g_i - g)hv\| \leq \|g_i(h_i - h)v\| + \|(g_i - g)hv\| \\ &= \|(h_i - h)v\| + \|(g_i - g)hv\|.\end{aligned}$$

This expression tends to zero for $g_i \to g$ and $h_i \to h$ in the strong operator topology.

The continuity of the inversion follows in the weak topology from the continuity of the functions

$$f_{v,w}(g^{-1}) = \langle g^{-1}v, w\rangle = \langle v, gw\rangle = \overline{\langle gw, v\rangle} = \overline{f_{w,v}(g)}$$

for $v, w \in \mathcal{H}$ and $g \in \mathrm{U}(\mathcal{H})$. □

Remark 1.2.4. (a) If $\dim \mathcal{H} < \infty$, then the norm topology and the strong operator topology coincide on $B(\mathcal{H})$, hence in particular on $\mathrm{U}(\mathcal{H})$. In fact, choosing an orthonormal basis $(e_1, \ldots, e_n)$ in $\mathcal{H}$, we represent $A \in B(\mathcal{H})$ by the matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, where $a_{ij} = \langle Ae_j, e_i\rangle = f_{e_j,e_i}(A)$. If $E_{ij} \in M_n(\mathbb{C})$ denote the matrix units, we then have $A = \sum_{i,j=1}^n a_{ij}E_{ij}$, so that

$$\|A\| \leq \sum_{i,j=1}^n |a_{ij}|\|E_{ij}\| = \sum_{i,j=1}^n |f_{e_j,e_i}(A)|\|E_{ij}\|,$$

which shows that convergence in the weak topology implies convergence in the norm topology.

(b) If $\dim \mathcal{H} = \infty$, then the strong operator topology on $\mathrm{U}(\mathcal{H})$ is strictly weaker than the norm topology. In fact, let $(e_i)_{i \in I}$ be an orthonormal basis of $\mathcal{H}$. Then I is infinite, so that we may w.l.o.g. assume that $\mathbb{N} \subseteq I$. For each n we then define the unitary operator $g_n \in \mathrm{U}(\mathcal{H})$ by $g_n e_i := (-1)^{\delta_{in}} e_i$. For $n \neq m$, we then have

$$\|g_n - g_m\| \geq \|(g_n - g_m)e_n\| = \| - 2e_n\| = 2,$$

and

$$\langle g_n v, w\rangle - \langle v, w\rangle = \langle g_n v - v, w\rangle = \langle -2\langle v, e_n\rangle e_n, w\rangle = -2\langle v, e_n\rangle\langle e_n, w\rangle \to 0$$

implies that $\lim_{n\to\infty} g_n = \mathbf{1}$ in the weak operator topology.

Definition 1.2.5. Let $\mathcal{H}$ be a complex Hilbert space and G a topological group. A continuous homomorphism

$$\pi \colon G \to \mathrm{U}(\mathcal{H})_s$$

is called a *(continuous) unitary representation* of G. We often denote unitary representations as pairs $(\pi, \mathcal{H})$. In view of Proposition 1.2.3, the continuity of π is equivalent to the continuity of all the *representative functions*

$$\pi_{v,w} \colon G \to \mathbb{C}, \quad \pi_{v,w}(g) := \langle \pi(g)v, w \rangle.$$

A representation $(\pi, \mathcal{H})$ is called *norm continuous*, if it is continuous with respect to the operator norm on $\mathrm{U}(\mathcal{H})$. Clearly, this condition is stronger

Here is a convenient criterion for the continuity of a unitary representation:

Lemma 1.2.6. *A unitary representation $(\pi, \mathcal{H})$ of the topological group G is continuous if and only if there exists a subset $E \subseteq \mathcal{H}$ for which* $\operatorname{span} E$ *is dense and the functions $\pi_{v,w}$ are continuous for $v, w \in E$.*

Proof. The condition is clearly necessary because we may take $E = \mathcal{H}$.

To see that it is also sufficient, we show that all functions $\pi_{v,w}$, $v, w \in \mathcal{H}$, are continuous. If $F := \operatorname{span} E$, then all functions $\pi_{v,w}$, $v, w \in F$, are continuous because the space $C(G, \mathbb{C})$ of continuous functions on G is a vector space.

Let $v, w \in \mathcal{H}$ and $v_n \to v$, $w_n \to w$ with $v_n, w_n \in F$. We claim that the sequence π_{v_n,w_n} converges uniformly to $\pi_{v,w}$, which then implies its continuity. In fact, for each $g \in G$ we have

$$\begin{aligned}
|\pi_{v_n,w_n}(g) - \pi_{v,w}(g)| &= |\langle \pi(g)v_n, w_n \rangle - \langle \pi(g)v, w \rangle| \\
&= |\langle \pi(g)(v_n - v), w_n \rangle - \langle \pi(g)v, w - w_n \rangle| \\
&\le \|\pi(g)(v_n - v)\| \|w_n\| + \|\pi(g)v\| \|w - w_n\| \\
&= \|v_n - v\| \|w_n\| + \|v\| \|w - w_n\| \to 0. \qquad \square
\end{aligned}$$

Example 1.2.7. If $(e_j)_{j \in J}$ is an orthonormal basis of $\mathcal{H}$, then $E := \{e_j : j \in J\}$ is a total subset. We associate to $A \in B(\mathcal{H})$ the matrix $(a_{jk})_{j,k \in J}$, defined by

$$a_{jk} := \langle Ae_k, e_j \rangle,$$

so that

$$A \sum_{k \in J} x_k e_k = \sum_{j \in J} \Big(\sum_{k \in J} a_{jk} x_k \Big) e_j.$$

Now Lemma 1.2.6 asserts that a unitary representation $(\pi, \mathcal{H})$ of G is continuous if and only if all functions

$$\pi_{jk}(g) := \langle \pi(g)e_k, e_j \rangle = \pi_{e_k, e_j}(g)$$

are continuous. These functions are the entries of $\pi(g)$, considered as a $(J \times J)$-matrix.

To deal with unitary group representations, we shall frequently have to deal with representations of more general structures, called involutive semigroups.

Definition 1.2.8. A pair $(S, *)$ of a semigroup S and an involutive antiautomorphism $s \mapsto s^*$ is called an *involutive semigroup*. Then we have $(st)^* = t^* s^*$ for $s, t \in S$ and $(s^*)^* = s$.

Example 1.2.9. (a) Any abelian semigroup S becomes an involutive semigroup with respect to $s^* := s$.

(b) If G is a group and $g^* := g^{-1}$, then $(G, *)$ is an involutive semigroup.

(c) An example of particular interest is the multiplicative semigroup $S = (B(\mathcal{H}), \cdot)$ of bounded operators on a complex Hilbert space $\mathcal{H}$ (Example 1.1.12(a)).

Definition 1.2.10. (a) A *representation* $(\pi, \mathcal{H})$ *of the involutive semigroup* $(S, *)$ is a homomorphism $\pi \colon S \to B(\mathcal{H})$ of semigroups satisfying $\pi(s^*) = \pi(s)^*$ for each $s \in S$.

(b) A representation $(\pi, \mathcal{H})$ of $(S, *)$ is called *non-degenerate*, if $\pi(S)\mathcal{H}$ spans a dense subspace of $\mathcal{H}$. This is in particular the case if $\mathbf{1} \in \pi(S)$.

(c) A representation $(\pi, \mathcal{H})$ is called *cyclic* if there exists a $v \in \mathcal{H}$ for which $\pi(S)v$ spans a dense subspace of $\mathcal{H}$.

(d) A representation $(\pi, \mathcal{H})$ is called *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed $\pi(S)$-invariant subspaces of $\mathcal{H}$.

Example 1.2.11. If G is a group with $g^* = g^{-1}$, then the representations of the involutive semigroup $(G, *)$ mapping $\mathbf{1} \in G$ to $\mathbf{1} \in B(\mathcal{H})$, are precisely the unitary representations of G. All unitary representations of groups are non-degenerate since $\pi(\mathbf{1}) = \mathbf{1}$.

Exercises for Section 1.2

Exercise 1.2.1. Let $\mathcal{H}$ be a Hilbert space. Show that:

(1) The involution on $B(\mathcal{H})$ is continuous with respect to the weak operator topology.

(2) On every bounded subset $K \subseteq B(\mathcal{H})$ the multiplication $(A, B) \mapsto AB$ is continuous with respect to the strong operator topology.

(3) On the unit sphere $\mathbb{S} := \{x \in H \colon \|x\| = 1\}$ the norm topology coincides with the weak topology.

Exercise 1.2.2. Let $\mathcal{H}$ be a Hilbert space and $\mathrm{U}(\mathcal{H})_s$ its unitary group, endowed with the strong (=weak) operator topology. Show that the action map

$$\sigma \colon \mathrm{U}(\mathcal{H})_s \times \mathcal{H} \to \mathcal{H}, \quad (g, v) \mapsto gv$$

is continuous. Conclude that each continuous unitary representation $(\pi, \mathcal{H})$ of a topological group G defines a continuous action of G on $\mathcal{H}$ by $g.v := \pi(g)v$.

Exercise 1.2.3. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Show that we obtain a continuous unitary representation of $G = (\mathbb{R}, +)$ on $\mathcal{H} = \ell^2(\mathbb{N}, \mathbb{C})$ by

$$\pi(t)x = (e^{ita_1}x_1, e^{ita_2}x_2, \ldots).$$

Show further that, if the sequence (a_n) is unbounded, then π is not norm continuous. Is it norm continuous if the sequence (a_n) is bounded?

Exercise 1.2.4. Let $(\pi, \mathcal{H})$ be a representation of an involutive semigroup $(S, *)$. Show that:

(a) $(\pi, \mathcal{H})$ is non-degenerate if and only if $\pi(S)v \subseteq \{0\}$ implies $v = 0$.

(b) Show that $(\pi, \mathcal{H})$ is an orthogonal direct sum of a non-degenerate representation and a zero representation $(\zeta, \mathcal{K})$, i.e., $\zeta(S) = \{0\}$.

Exercise 1.2.5. Let $(\pi, \mathcal{H})$ be a representation of the involutive semigroup (G, η_G), where G is a group. Show that:

(a) $(\pi, \mathcal{H})$ is non-degenerate if and only if $\pi(\mathbf{1}) = \mathbf{1}$.

(b) $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_j = \ker(\eta(\mathbf{1}) - j\mathbf{1})$, is an orthogonal direct sum.

Exercise 1.2.6. Let (X, d) be a metric space and $G := \mathrm{Aut}(X, d)$ be the group of automorphisms of (X, d), i.e., the group of bijective isometries. Show that the coarsest topology on G for which all functions

$$f_x \colon G \to \mathbb{R}, \quad f_x(g) := d(g.x, x)$$

are continuous turns G into a topological group and that the action $\sigma \colon G \times X \to X, (g, x) \mapsto g.x$ is continuous.

1.3 Discrete Decomposition of Unitary Representations

One major goal of the theory of unitary representations is to decompose a unitary representation into simpler pieces. The first basic observation is that for any closed invariant subspace $\mathcal{K} \subseteq \mathcal{H}$, its orthogonal complement is also invariant, so that we obtain a decomposition into the two subrepresentations on $\mathcal{K}$ and $\mathcal{K}^\perp$. The next step is to iterate this process whenever either $\mathcal{K}$ of $\mathcal{K}^\perp$ is not irreducible. This method works well if $\mathcal{H}$ is finite dimensional, but in general it may not lead to a decomposition into irreducible pieces. However, we shall apply this strategy to show at least that every unitary representation is a direct sum of cyclic ones.

We start with the discussion of invariant subspaces.

Lemma 1.3.1. *Let $\mathcal{K} \subseteq \mathcal{H}$ be a closed subspace, $P \in B(\mathcal{H})$ be the orthogonal projection on $\mathcal{K}$ and $S \subseteq B(\mathcal{H})$ be a $*$-invariant subset. Then the following are equivalent*

(i) *$\mathcal{K}$ is S-invariant.*

(ii) *$\mathcal{K}^\perp$ is S-invariant.*

(iii) *P commutes with S.*

Proof. (i) $\Rightarrow$ (ii): If $w \in \mathcal{K}^\perp$ and $v \in \mathcal{K}$, we have for any $s \in S$ the relation $\langle sw, v\rangle = \langle w, s^*v\rangle = 0$ because $s^*v \in S\mathcal{K} \subseteq \mathcal{K}$.

(ii) $\Rightarrow$ (iii): First we observe that the same argument as above implies that the invariance of $\mathcal{K}^\perp$ entails the invariance of $\mathcal{K} = (\mathcal{K}^\perp)^\perp$.

We write $v = v_0 + v_1$, according to the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. Then we have for any $s \in S$:

$$sPv = sv_0 = Psv_0 = P(sv_0 + sv_1) = Psv,$$

so that P commutes with S.

(iii) $\Rightarrow$ (i) follows from the fact that $\mathcal{K} = \ker(P - \mathbf{1})$ is an eigenspace of P, hence invariant under every operator commuting with P. □

We record an important consequence for unitary representations:

Proposition 1.3.2. *If $(\pi, \mathcal{H})$ is a continuous unitary representation of the group G and $\mathcal{H}_1 \subseteq \mathcal{H}$ a closed invariant subspace, then $\mathcal{H}_2 := \mathcal{H}_1^\perp$ is also invariant.*

Writing elements of $B(\mathcal{H})$ according to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ as matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in B(\mathcal{H}_1)$, $b \in B(\mathcal{H}_2, \mathcal{H}_1)$, $c \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $d \in B(\mathcal{H}_2)$ (Exercise 1.3.6), we then have

$$\pi(g) = \begin{pmatrix} \pi_1(g) & \mathbf{0} \\ \mathbf{0} & \pi_2(g) \end{pmatrix},$$

where $(\pi_i, \mathcal{H}_i)$, $i = 1, 2$, are continuous unitary representations.

Proof. The invariance of $\mathcal{H}_2$ follows from Lemma 1.3.1 because $\pi(G) \subseteq B(\mathcal{H})$ is $*$-invariant. Therefore $\pi_j(g) := \pi(g)|_{\mathcal{H}_j}^{\mathcal{H}_j}$ is a unitary operator for each $g \in G$ and we obtain two unitary representations $(\pi_j, \mathcal{H}_j)$. Their continuity follows immediately from the continuity of $(\pi, \mathcal{H})$. □

Definition 1.3.3. (a) If $(\pi, \mathcal{H})$ is a unitary representation of G and $\mathcal{K} \subseteq \mathcal{H}$ a closed G-invariant subspace, then $\rho(g) := \pi(g)|_{\mathcal{K}}^{\mathcal{K}}$ defines a unitary representation $(\rho, \mathcal{K})$ which is called a *subrepresentation of* $(\pi, \mathcal{H})$.

(b) If $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ are unitary representations, then a bounded operator $A\colon \mathcal{K} \to \mathcal{H}$ satisfying

$$A \circ \rho(g) = \pi(g) \circ A \quad \text{for all} \quad g \in G$$

is called an *intertwining operator*. We write $B_G(\mathcal{K}, \mathcal{H})$ for the set of all intertwining operators. It is a closed subspace of the Banach space $B(\mathcal{K}, \mathcal{H})$ (Exercise 1.3.7).

Two unitary representations $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ are said to be *equivalent* if there exists a unitary intertwining operator $A\colon \mathcal{K} \to \mathcal{H}$. It is easy to see that this defines indeed an equivalence relation on the class of all unitary representations. We write $[\pi]$ for the equivalence class of the representation $(\pi, \mathcal{H})$.

To understand the decomposition of representations into smaller pieces, we also need infinite "direct sums" of representations, hence the concept of a direct sum of Hilbert spaces which in turn requires the somewhat subtle concept of summability in Banach spaces.

Definition 1.3.4. Let I be a set and X a Banach space. Then a family $(x_i)_{i\in I}$ is called *summable to* $x \in X$ if for each $\varepsilon > 0$ there exists a finite subset $I_\varepsilon \subseteq I$ with the property that for every finite subset $F \supseteq I_\varepsilon$ we have

$$\|\sum_{i\in F} x_i - x\| < \varepsilon.$$

If $(x_i)_{i\in I}$ is summable to x, we write $x = \sum_{i\in I} x_i$.[1]

Remark 1.3.5. (a) Note that for $I = \mathbb{N}$ the summability of a family $(x_n)_{n\in\mathbb{N}}$ in a Banach space X is stronger than the convergence of the series $\sum_{n=1}^\infty x_n$. In fact, if $x = \sum_{n\in\mathbb{N}} x_n$ holds in the sense of summability and $\mathbb{N}_\varepsilon \subseteq \mathbb{N}$ is a finite subset with the property that for every finite subset $F \supseteq \mathbb{N}_\varepsilon$ we have $\|\sum_{n\in F} x_n - x\| < \varepsilon$, then we have for $N > \max \mathbb{N}_\varepsilon$ in particular

$$\|\sum_{n=1}^{N} x_n - x\| < \varepsilon,$$

showing that the series $\sum_{n=1}^\infty x_n$ converges to x.

(b) If, conversely, the series $\sum_{n=1}^\infty x_n$ converges absolutely to some $x \in X$ and $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$ with $\sum_{n=N}^\infty \|x_n\| < \varepsilon$. With $\mathbb{N}_\varepsilon := \{1, \ldots, N\}$ we then find for every finite superset $F \supseteq \mathbb{N}_\varepsilon$ that

$$\|x - \sum_{n\in F} x_n\| \leq \sum_{n\in\mathbb{N}\setminus F} \|x_n\| \leq \sum_{n>N} \|x_n\| < \varepsilon.$$

Therefore we also have $x = \sum_{n\in\mathbb{N}} x_n$ in the sense of summability.

(c) For $X = \mathbb{R}$ and $I = \mathbb{N}$ summability of $(x_n)_{n\in\mathbb{N}}$ implies in particular convergence of all reordered series $\sum_{n=1}^\infty x_{\sigma(n)}$, where $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a bijection. Therefore Riemann's Reordering Theorem shows that summability implies absolute convergence.

(d) If $(x_i)_{i\in I}$ is a family in $\mathbb{R}_+ = [0, \infty[$, then the situation is much simpler. Here summability is easily seen to be equivalent to the existence of the supremum of the set $\mathcal{F} := \{\sum_{i\in F} x_i \colon F \subseteq I, |F| < \infty\}$ of all partial sums, and in this case $\sum_{i\in I} x_i = \sup \mathcal{F}$.

[1] This can also be formulated in terms of convergence of nets. First we order the set $\mathcal{I} := \{F \subseteq I \colon |F| < \infty\}$ of finite subsets of I by set inclusion, so that $F \mapsto \sum_{i\in F} x_i$ is a net in X, called the *net of partial sums*. Then the summability of $(x_i)_{i\in I}$ in X is equivalent to the convergence of this net in X.

Lemma 1.3.6. *Let $(\mathcal{H}_j)_{j\in J}$ be a family of Hilbert spaces and*

$$\mathcal{H} := \Big\{(x_j)_{j\in J} \in \prod_{j\in J}\mathcal{H}_j \colon \sum_{j\in J}\|x_j\|^2 < \infty\Big\}.$$

Then $\mathcal{H}$ is a Hilbert space with respect to the scalar product

$$\langle (x_j)_{j\in J}, (y_j)_{j\in J}\rangle = \sum_{j\in J}\langle x_j, y_j\rangle.$$

Proof. First we show that $\mathcal{H}$ is a linear subspace of the complex vector space $\prod_{j\in J}\mathcal{H}_j$, in which we define addition and scalar multiplication componentwise. Clearly, $\mathcal{H}$ is invariant under multiplication with complex scalars. For $a, b \in \mathcal{H}_j$ the parallelogram identity

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2$$

(Exercise) implies in particular that

$$\|a+b\|^2 \leq 2(\|a\|^2 + \|b\|^2).$$

For $x = (x_j)_{j\in J}, y = (y_j)_{j\in J} \in \mathcal{H}$, we therefore obtain

$$\sum_{j\in J}\|x_j + y_j\|^2 \leq 2\sum_{j\in J}\|x_j\|^2 + 2\sum_{j\in J}\|y_j\|^2 < \infty.$$

This shows that $x + y \in \mathcal{H}$, so that $\mathcal{H}$ is indeed a linear subspace.

For $x, y \in \mathcal{H}$, the polarization identity

$$\langle x, y\rangle = \frac{1}{4}\big(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle\big)$$

(Exercise 1.3.1(i)) and $x \pm y, x \pm iy \in \mathcal{H}$ imply that the sum

$$\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle$$

exists. For $0 \neq x$, some x_i is non-zero, so that $\langle x, x\rangle \geq \langle x_i, x_i\rangle > 0$. It is a trivial verification that $\langle\cdot,\cdot\rangle$ is a hermitian form. Therefore $\mathcal{H}$, endowed with $\langle\cdot,\cdot\rangle$, is a pre-Hilbert space.

It remains to show that it is complete. This is proved in the same way as the completeness of the space ℓ^2 of square-summable sequences, which is the special case $J = \mathbb{N}$ and $\mathcal{H}_j = \mathbb{C}$ for each $j \in J$. Let $(x^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}$. Then $\|x_j^n - x_j^m\| \leq \|x^n - x^m\|$ holds for each $j \in J$, so that $(x_j^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_j$. Now the completeness of the spaces $\mathcal{H}_j$ imply the existence of elements $x_j \in \mathcal{H}_j$ with $x_j^n \to x_j$. For every finite subset $F \subseteq J$, we then have

$$\sum_{j\in F}\|x_j\|^2 = \lim_{n\to\infty}\sum_{j\in F}\|x_j^n\|^2 \leq \lim_{n\to\infty}\sum_{j\in J}\|x_j^n\|^2 = \lim_{n\to\infty}\|x^n\|^2,$$

which exists because $(x^n)_{n\in\mathbb{N}}$ is a Cauchy sequence. This implies that $x := (x_j)_{j\in J} \in \mathcal{H}$ with $\|x\|^2 \leq \lim_{n\to\infty} \|x^n\|^2$.

Finally, we show that $x^n \to x$ holds in $\mathcal{H}$. So let $\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ with $\|x^n - x^m\| \leq \varepsilon$ for $n, m \geq N_\varepsilon$. For a finite subset $F \subseteq J$, we then have

$$\sum_{j\in F} \|x_j - x_j^n\|^2 = \lim_{m\to\infty} \sum_{j\in F} \|x_j^m - x_j^n\|^2 \leq \lim_{m\to\infty} \|x^m - x^n\|^2 \leq \varepsilon^2$$

for $n \geq N_\varepsilon$. We therefore obtain

$$\|x - x^n\|^2 = \sup_{F\subseteq J, |F|<\infty} \sum_{j\in F} \|x_j - x_j^n\|^2 \leq \varepsilon^2.$$

This implies that $x^n \to x$ in $\mathcal{H}$, and thus $\mathcal{H}$ is complete. $\square$

Definition 1.3.7. For a family of $(\mathcal{H}_j)_{j\in J}$ of Hilbert spaces, we define

$$\widehat{\bigoplus_{j\in J}}\mathcal{H}_j := \Big\{(x_j)_{j\in J} \in \prod_{j\in J} \mathcal{H}_j \colon \sum_{j\in J} \|x_j\|^2 < \infty\Big\}$$

with the scalar product from Lemma 1.3.6. We call this space the *Hilbert space direct sum* of the spaces $(\mathcal{H}_j)_{j\in J}$. This space is larger than the direct vector space sum of the $\mathcal{H}_j$, which is a dense subspace of $\widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ (Exercise). In the following we always identify $\mathcal{H}_i$ with the subspace

$$\mathcal{H}_i \cong \{(x_j)_{j\in J} \colon (\forall j \neq i)\ x_j = 0\}.$$

Note that the requirement that $(\|x_j\|^2)_{j\in J}$ is summable implies in particular that, for each $x \in \mathcal{H}$, only countably many x_j are non-zero, even if J is uncountable (Exercise 1.3.2).

Example 1.3.8. (a) If $\mathcal{H}_j = \mathbb{C}$ for each $j \in J$, we also write

$$\ell^2(J, \mathbb{C}) := \widehat{\bigoplus}_{j\in J}\mathbb{C} = \Big\{(x_j)_{j\in J} \in \mathbb{C}^J \colon \sum_{j\in J} |x_j|^2 < \infty\Big\}.$$

On this space we have

$$\langle x, y\rangle = \sum_{j\in J} \langle x_j, y_j\rangle \quad \text{ and } \quad \|x\|^2 = \sum_{j\in J} |x_j|^2.$$

For $J = \{1, \ldots, n\}$, we obtain in particular the Hilbert space

$$\mathbb{C}^n \cong \ell^2(\{1, \ldots, n\}, \mathbb{C}).$$

(b) If all Hilbert spaces $\mathcal{H}_j = \mathcal{K}$ are equal, we put

$$\ell^2(J, \mathcal{K}) := \widehat{\bigoplus}_{j\in J}\mathcal{K} = \Big\{(x_j)_{j\in J} \in \mathcal{K}^J \colon \sum_{j\in J} |x_j|^2 < \infty\Big\}.$$

On this space we also have

$$\langle x, y\rangle = \sum_{j\in J} \langle x_j, y_j\rangle \quad \text{ and } \quad \|x\|^2 = \sum_{j\in J} |x_j|^2.$$

Proposition 1.3.9. *Let $(\pi_j, \mathcal{H}_j)_{j \in J}$ be a family of continuous unitary representation of G. Then*

$$\pi(g)(v_j)_{j\in J} := \big(\pi_j(g)v_j\big)_{j\in J}$$

defines on $\mathcal{H} := \widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ a continuous unitary representation.

The representation $(\pi, \mathcal{H})$ is called the *direct sum of the representations π_j*, $j \in J$. It is also denoted $\pi = \sum_{j\in J} \pi_j$.

Proof. Since all operators $\pi_j(g)$ are unitary, we have

$$\sum_{j\in J} \|\pi_j(g)v_j\|^2 = \sum_{j\in J} \|v_j\|^2 < \infty \quad \text{for} \quad v = (v_j)_{j\in J} \in \mathcal{H}.$$

Therefore each $\pi(g)$ defines a unitary operator on $\mathcal{H}$ and we thus obtain a unitary representation $(\pi, \mathcal{H})$ of G because each π_j is a unitary representation.

To see that it is continuous, we use Lemma 1.2.6, according to which it suffices to show that, for $v \in \mathcal{H}_i$ and $w \in \mathcal{H}_j$, the function

$$\pi_{v,w}(g) = \langle \pi(g)v, w\rangle = \delta_{ij}\langle \pi_j(g)v, w\rangle$$

is continuous, which immediately follows from the continuity of the representation π_j. □

As we shall see soon, we cannot expect in general that a unitary representation decomposes into irreducible ones, but the following proposition is often a useful replacement.

Proposition 1.3.10. *Each continuous unitary representation $(\pi, \mathcal{H})$ of G is (equivalent to) a direct sum of cyclic subrepresentations $(\pi_j, \mathcal{H}_j)_{j\in J}$.*

Proof. The proof is a typical application of Zorn's Lemma. We order the set $\mathcal{M}$ of all sets $(\mathcal{H}_j)_{j\in J}$ of mutually orthogonal closed G-invariant subspaces on which the representation is cyclic by set inclusion. Clearly, each chain $\mathcal{K}$ in this ordered space has an upper bound given by the union $\bigcup \mathcal{K} \in \mathcal{M}$. Now Zorn's Lemma yields a maximal element $(\mathcal{H}_j)_{j\in J}$ in $\mathcal{M}$.

Let $\mathcal{K} := \overline{\sum_{j\in J} \mathcal{H}_j}$. Since each $\mathcal{H}_j$ is G-invariant and each $\pi(g)$ is continuous, $\mathcal{K}$ is also G-invariant. In view of Proposition 1.3.2, the orthogonal complement $\mathcal{K}^\perp$ is also G-invariant. If $\mathcal{K}^\perp$ is non-zero, we pick $0 \neq v \in \mathcal{K}^\perp$. Then $\mathcal{H}' := \overline{\operatorname{span} \pi(G)v}$ is a closed G-invariant subspace on which the representation is cyclic. Therefore $\mathcal{H}'$, together with $(\mathcal{H}_j)_{j\in J}$ is an orthogonal family of G-cyclic subspaces. This contradicts the maximality of $(\mathcal{H}_j)_{j\in J}$, and therefore $\mathcal{K}^\perp = \{0\}$, which proves that $\mathcal{K} = \mathcal{H}$.

Finally, we note that the mutual orthogonality of the spaces $\mathcal{H}_j$ implies the existence of a map

$$\Phi\colon \widehat{\bigoplus}_{j\in J}\mathcal{H}_j \to \mathcal{H}, \quad \Phi(x) := \sum_{j\in J} x_j$$

which is easily seen to be isometric (Exercise 1.3.5). Since its range is dense and complete, it is also surjective. Moreover, $\pi(g)\Phi((x_j)) = \Phi((\pi_j(g)x_j))$ implies that Φ is an equivalence of unitary representations. $\square$

Proposition 1.3.11. *Each finite dimensional continuous unitary representation $(\pi, \mathcal{H})$ of a group G is a direct sum of irreducible representations.*

Proof. This is proved easily by induction on $\dim \mathcal{H}$. If $\dim \mathcal{H} \leq 1$, there is nothing to show. Suppose that $\dim \mathcal{H} = d > 0$ and that the assertion is true for representations of dimension $< d$. Let $\mathcal{K} \subseteq \mathcal{H}$ be a minimal G-invariant subspace. Then the representation $\pi_{\mathcal{K}}$ of G on $\mathcal{K}$ is irreducible and $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$ is a G-invariant decomposition (Proposition 1.3.2). Using the induction hypothesis on the representation on $\mathcal{K}^\perp$, the assertion follows. $\square$

Corollary 1.3.12. *If G is a finite group, then each unitary representation $(\pi, \mathcal{H})$ of G is a direct sum of irreducible representations $(\pi_j, \mathcal{H}_j)_{j\in J}$.*

Proof. First we use Proposition 1.3.10 to decompose π as a direct sum of cyclic representations π_j. Hence it suffices to show that each cyclic representation is a direct sum of irreducible ones. Since G is finite, each cyclic representation is finite dimensional, so that the assertion follows from Proposition 1.3.11. $\square$

Remark 1.3.13. The preceding corollary remains true for representations of compact groups. Here the main point is to show that every cyclic representation contains a finite dimensional invariant subspace, which can be derived from the existence of Haar measure and the theory of compact operators.

Exercises for Section 1.3

Exercise 1.3.1. Let $b \colon V \times V \to \mathbb{C}$ be a sesquilinear form on the complex Vector space V, i.e., b is linear in the first argument and antilinear in the second.

(i) Show that b satisfies the *polarization identity* which permits the recover all values of b from those on the diagonal:

$$b(x, y) = \frac{1}{4} \sum_{k=0}^{3} i^k b(x + i^k y, x + i^k y).$$

(ii) Show also that, if b is positive semidefinite, then it satisfies the *Cauchy–Schwarz inequality*:

$$|b(x, y)|^2 \leq b(x, x) b(y, y) \quad \text{for} \quad v, w \in V.$$

Exercise 1.3.2. Show that for each summable family $(x_j)_{j\in J}$ in the Banach space X, the set

$$J^\times = \{j \in J \colon x_j \neq 0\}$$

is countable, and that, if $J^\times = \{j_n \colon n \in \mathbb{N}\}$ is an enumeration of $J^\times$, then $\sum_{j\in J} x_j = \sum_{n=1}^\infty x_{j_n}$. Hint: Show that each set $J_n := \{j \in J \colon \|x\|_n > \frac{1}{n}\}$ is finite.

Exercise 1.3.3. Show that for an orthogonal family $(x_j)_{j\in J}$ in the Hilbert space $\mathcal{H}$, the following are equivalent:

(i) $(x_j)_{j\in J}$ is summable.

(ii) $(\|x_j\|^2)_{j\in J}$ is summable in $\mathbb{R}$.

Show further that, if this is the case, then $\left\|\sum_{j\in J} x_j\right\|^2 = \sum_{j\in J}\|x_j\|^2$ and the set $\{j\in J\colon x_j\neq 0\}$ is countable.

Exercise 1.3.4. Show that for an orthonormal family $(x_j)_{j\in J}$ in the Hilbert space $\mathcal{H}$, the following assertions hold:

(i) $(\forall x\in\mathcal{H})\sum_{j\in J}|\langle x_j,x\rangle|^2\leq\|x\|^2$ (Bessel inequality).

(ii) $x=\sum_{j\in J}\langle x,x_j\rangle x_j$ holds if and only if $\sum_{j\in J}|\langle x,x_j\rangle|^2=\|x\|^2$ (Parseval equality).

Exercise 1.3.5. Let $(\mathcal{H}_j)_{j\in J}$ be an orthogonal family of closed subspaces of the Hilbert space $\mathcal{H}$. Show that for each $x=(x_j)_{j\in J}\in\widehat{\bigoplus}\mathcal{H}_j$, the sum $\Phi(x):=\sum_{j\in J}x_j$ converges in $\mathcal{H}$ and that $\Phi\colon\widehat{\bigoplus}_{j\in J}\mathcal{H}_j\to\mathcal{H}, (x_j)_{j\in J}\mapsto\sum_{j\in J}x_j$ defines an isometric embedding (cf. Exercise 1.3.3).

Exercise 1.3.6. Let V be a vector space which is the direct sum

$$V=V_1\oplus\cdots\oplus V_n$$

of the subspaces V_i, $i=1,\ldots,n$. Accordingly, we write $v\in V$ as a sum $v=v_1+\cdots+v_n$ with $v_i\in V$. To each $\varphi\in\operatorname{End}(V)$ we associate the map $\varphi_{ij}\in\operatorname{Hom}(V_j,V_i)$, defined by $\varphi_{ij}(v)=\varphi(v)_i$ for $v\in V_j$. Show that

(a) $\varphi(v)_i=\sum_{j=1}^n\varphi_{ij}(v_j)$ for $v=\sum_{j=1}^n v_j\in V$.

(b) The map

$$\Gamma\colon\bigoplus_{i,j=1}^n\operatorname{Hom}(V_j,V_i)\to\operatorname{End}(V),\quad\Gamma((\psi_{ij}))(v):=\sum_{i,j=1}^n\psi_{ij}(v_j)$$

is a linear isomorphism. In this sense we may identify endomorphisms of V with $(n\times n)$-matrices with entries in $\operatorname{Hom}(V_j,V_i)$ in position (i,j).

(c) If V is a Banach space and each V_i is a closed subspace, then the map

$$S\colon V_1\times\cdots\times V_n\to V,\quad(v_1,\ldots,v_n)\mapsto\sum_{i=1}^n v_i$$

is a homeomorphism. Moreover, a linear endomorphism $\varphi\colon V\to V$ is continuous if and only if each φ_{ij} is continuous. Hint: For the first assertion use the Open Mapping Theorem. Conclude that if $\iota_i\colon V_i\to V$ denotes the inclusion map and $p_j\colon V\to V_j$ the projection map, then both are continuous. Then use that $\varphi_{ij}=p_i\circ\varphi\circ\eta_j$.

Exercise 1.3.7. Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be unitary representations of G. Show that the space $B_G(\mathcal{K}, \mathcal{H})$ of all intertwining operators is a closed subspace of the Banach space $B(\mathcal{K}, \mathcal{H})$

Exercise 1.3.8. Let G be a group. Show that:

(a) Each unitary representation $(\pi, \mathcal{H})$ of G is equivalent to a representation $(\rho, \ell^2(J, \mathbb{C}))$ for some set J. Therefore it makes sense to speak of the **set** of equivalence classes of representations with a fixed Hilbert dimension.

(b) Two unitary representations $\pi_j \colon G \to \mathrm{U}(\mathcal{H})$, $j = 1, 2$, are equivalent if and only if there exists a unitary operator $U \in \mathrm{U}(\mathcal{H})$ with

$$\pi_2(g) = U\pi_1(g)U^{-1} \quad \text{for each} \quad g \in G.$$

Therefore the set of equivalence classes of unitary representations of G on $\mathcal{H}$ is the set of orbits of the action of $\mathrm{U}(\mathcal{H})$ on the set $\mathrm{Hom}(G, \mathrm{U}(\mathcal{H}))$ for the action $(U * \pi)(g) := U\pi(g)U^{-1}$.

Exercise 1.3.9. Let V be a $\mathbb{K}$-vector space and $A \in \mathrm{End}(V)$. We write $V_\lambda(A) := \ker(A - \lambda\mathbf{1})$ for the *eigenspace of* A corresponding to the eigenvalue λ and $V^\lambda(A) := \bigcup_{n\in\mathbb{N}} \ker(A - \lambda\mathbf{1})^n$ for the *generalized eigenspace of* A corresponding to λ. Show that if $A, B \in \mathrm{End}(V)$ commute, then

$$BV^\lambda(A) \subseteq V^\lambda(A) \quad \text{and} \quad BV_\lambda(A) \subseteq V_\lambda(A)$$

holds for each $\lambda \in \mathbb{K}$.

Exercise 1.3.10. Let $(S, *)$ be an involutive semigroup. Show that:

(a) Every cyclic representation is non-degenerate.

(b) If $(\pi, \mathcal{H})$ is a direct sum of cyclic representations, then it is is non-degenerate.

(c) Every non-degenerate representation $(\pi, \mathcal{H})$ of S is a direct sum of cyclic representations. Hint: One can follows the argument in Proposition 1.3.10, but one step requires additional care, namely that for $0 \neq v \in \mathcal{H}$, the representation of S on the closed subspace $\mathcal{K} := \overline{\mathrm{span}\, \pi(S)v}$ is cyclic. One has to argue that $v \in \mathcal{K}$ to see that this is the case, and this is where it is needed that $(\pi, \mathcal{H})$ is non-degenerate.

Chapter 2

Representations on L^2-spaces

In the first chapter we have seen how to deal with discrete decompositions of Hilbert spaces and unitary representations. We now turn to the continuous side. Here the simplest situations arise for Hilbert spaces of the type $L^2(X, \mu)$, where $(X, \mathfrak{S}, \mu)$ is a measure space. We start with recalling the construction of the L^2-space in Section 2.1. Then we turn to unitary operators on this space coming from measurable transformations of X and multiplication with functions (Section 2.2), and how this leads to unitary representations. In Section 2.3 we discuss the Riesz Representation Theorem and how it connects locally compact spaces with measure theory. It provides a natural source of L^2-spaces on which topological groups act unitarily. In Section 2.4 we briefly discuss Haar measure on a locally compact group G and how it leads to faithful continuous unitary representations of G.

2.1 Measures and L^2-spaces

Definition 2.1.1. Let X be a set.

(a) A subset $\mathfrak{S} \subseteq \mathbb{P}(X)$ is called a *σ-algebra* if

(SA1) $\emptyset \in \mathfrak{S}$.

(SA2) $A^c := X \setminus A \in \mathfrak{S}$ for $A \in \mathfrak{S}$.

(SA3) $\bigcup_{j=1}^\infty A_j \in \mathfrak{S}$ for every sequence $(A_j)_{j\in\mathbb{N}}$ in $\mathfrak{S}$.

Note that (SA2) and (SA3) imply that $\mathfrak{S}$ is also closed under countable intersections. If $\mathfrak{S}$ is a σ-algebra of subsets of X, then the pair $(X, \mathfrak{S})$ is called a *measurable space* or a *Borel space* and the elements of $\mathfrak{S}$ are called the *measurable subsets*.

(b) If $(X, \mathfrak{S})$ and $(X', \mathfrak{S}')$ are measurable spaces, then a map $f \colon X \to X'$ is said to be measurable if $f^{-1}(\mathfrak{S}') \subseteq \mathfrak{S}$, i.e., inverse images of measurable sets are measurable. It is easy to see that compositions of measurable maps are measurable.

A map $f: X \to X'$ is called a *Borel isomorphism* if there exists a measurable map $g: X' \to X$ with $f \circ g = \mathrm{id}_{X'}$ and $g \circ f = \mathrm{id}_X$. This is equivalent to f being bijective, measurable and f^{-1} also measurable (Exercise). We write $\mathrm{Aut}(X, \mathfrak{S})$ for the group of Borel automorphisms of $(X, \mathcal{S})$.

Definition 2.1.2. Let $(X, \mathfrak{S})$ be a measurable space.

(a) A *positive measure on* $(X, \mathfrak{S})$ is a function $\mu: \mathfrak{S} \to [0, \infty]$ with $\mu(\emptyset) = 0$ which is *countably additive*, i.e., for each disjoint sequence $(E_n)_{n \in \mathbb{N}}$ in $\mathfrak{S}$ we have

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n). \tag{2.1}$$

A positive measure is called a *probability measure* if $\mu(X) = 1$.

(b) A *measure space* is a triple $(X, \mathfrak{S}, \mu)$, where μ is a positive measure on $(X, \mathfrak{S})$.

(c) A *complex measure* is a function $\mu: \mathfrak{S} \to \mathbb{C}$ which is countably additive. In particular, ∞ is excluded as a possible value.

We shall now freely use basic results and constructions from Lebesgue measure theory, such as the integral of measurable functions and the convergence theorems.

Definition 2.1.3. For a measurable function $f: X \to \mathbb{C}$, we define

$$\|f\|_2 := \Big(\int_X |f(x)|^2 \, d\mu(x)\Big)^{\frac{1}{2}}$$

and put

$$\mathcal{L}^2(X, \mu) := \{f : \|f\|_2 < \infty\}.$$

If $f, g \in \mathcal{L}^2(X, \mu)$, then $|f|^2, |g|^2$ are integrable, and so is $(|f| + |g|)^2$ because $(|f| + |g|)^2 \leq 2|f|^2 + 2|g|^2$. Therefore $|f||g| = \frac{1}{2}\big((|f| + |g|)^2 - |f|^2 - |g|^2\big)$ in $\mathcal{L}^1(X, \mu)$, i.e., this function is integrable. For $f, g \in \mathcal{L}^2(X, \mu)$ we may thus define

$$\langle f, g \rangle := \int_X f\overline{g} \, d\mu.$$

This expression makes sense because $f\overline{g}$ is integrable. One easily verifies that we thus obtain a positive semidefinite hermitian form on $\mathcal{L}^2(X, \mu)$ whose corresponding seminorm is given by $\|f\|_2 = \sqrt{\langle f, f \rangle}$. With

$$N = \{f \in \mathcal{L}^2(X, \mu) : \|f\|_2 = 0\}$$

we now derive that

$$L^2(X, \mu) := \mathcal{L}^2(X, \mu)/N$$

is a pre-Hilbert space w.r.t. $\langle [f], [g] \rangle := \langle f, g \rangle$, and it is a standard result of integration theory that it is complete, i.e., a Hilbert space and that the step functions, i.e., the integrable measurable functions with finitely many values form a dense subspace.

Definition 2.1.4. There are two more Banach spaces associated with a measure space $(X, \mathfrak{S}, \mu)$ that we shall need in the following.

(a) For a measurable function $f\colon X \to \mathbb{C}$, we define

$$\|f\|_1 := \Big(\int_X |f(x)|\, d\mu(x)\Big) \quad \text{and} \quad \mathcal{L}^1(X,\mu) := \{f\colon \|f\|_1 < \infty\},$$

and write $L^1(X,\mu) := \mathcal{L}^1(X,\mu)/N$, where N is the subspace of functions vanishing μ-almost everywhere. In measure theory one shows that this space is a Banach space with respect to $\|\cdot\|_1$.

(b) We consider the algebra $\mathcal{L}^\infty(X,\mathbb{C})$ of bounded measurable functions $f\colon X \to \mathbb{C}$ and define

$$\|f\|_\infty := \inf\{C \geq 0\colon \mu(\{|f| > C\}) = 0\},$$

where $\{|f| > C\} := \{x \in X\colon |f(x)| > C\}$. It is easy to verify that

$$\|\lambda f\|_\infty = |\lambda|\|f\|_\infty \quad \text{for} \quad \lambda \in \mathbb{C} \quad \text{and} \quad \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Therefore $N := \{f \in \mathcal{L}^\infty(X,\mathbb{C})\colon \|f\|_\infty = 0\}$ (the set of all functions vanishing μ-almost everywhere) is a subspace, and

$$L^\infty(X,\mu) := \mathcal{L}^\infty(X)/N$$

is a normed space with respect to $\|[f]\|_\infty := \|f\|_\infty$, which is easily seen to be complete. In fact, for every Cauchy sequence $[f_n]$ in this space, there exists an $E \in \mathfrak{S}$ of measure zero such that the restrictions $f_n|_{X\setminus E}$ form a Cauchy sequence with respect to the sup norm. If f is its limit, then we extend f by 0 on E to obtain an element $[f] \in L^\infty(X,\mu)$ with $[f_n] \to [f]$.

For $(fg)(x) := f(x)g(x)$ and $f^*(x) := \overline{f(x)}$, we have

$$\|fg\|_\infty \leq \|f\|_\infty\|g\|_\infty \quad \text{and} \quad \|f^*\|_\infty = \|f\|_\infty, \quad \|f^*f\|_\infty = \|f\|_\infty^2,$$

so that $L^\infty(X,\mu)$ carries the structure of a C^*-algebra.

For the following lemma, we recall that a positive measure μ on $(X,\mathfrak{S})$ is said to be *σ-finite* if $X = \bigcup_{n\in\mathbb{N}} E_n$ with $E_n \in \mathfrak{S}$ and $\mu(E_n) < \infty$. This is an important assumption for many results in measure theory, such as Fubini's Theorem and the Radon–Nikodym Theorem.

Lemma 2.1.5. *For each $f \in L^\infty(X,\mu)$, we obtain a bounded operator $\lambda_f \in B(L^2(X,\mu))$ by $\lambda_f(g) := fg$. It satisfies $\|\lambda_f\| \leq \|f\|_\infty$, and the corresponding map*

$$\lambda\colon L^\infty(X,\mu) \to B(L^2(X,\mu)), \quad f \mapsto \lambda_f$$

is a homomorphism of C^*-algebras. If, in addition, μ is σ-finite, then λ is isometric, i.e., $\|\lambda_f\| = \|f\|_\infty$ for each f.

Proof. Since $|f(x)g(x)| \leq \|f\|_\infty |g(x)|$ holds μ-almost everywhere, we see that λ_f defines a bounded operator on $L^2(X, \mu)$ with $\|\lambda_f\| \leq \|f\|_\infty$. We clearly have $\lambda_{fg} = \lambda_f \lambda_g$ and $\lambda_f^* = \lambda_{f^*}$, so that λ defines a homomorphism of C^*-algebras.

Now assume that $\|f\|_\infty > c \geq 0$. Then $F := \{|f| \geq c\}$ has positive measure, and since μ is σ-finite, it contains a subset E of positive and finite μ-measure. Then $h := \chi_E \in L^2(X, \mu)$ and

$$c\|h\|_2 \leq \|fh\|_2 \leq \|\lambda_f\| \|h\|_2$$

leads to $\|\lambda_f\| \geq c$, and since c was arbitrary, we obtain $\|f\|_\infty \leq \|\lambda_f\|$. $\square$

Exercises for Section 2.1

Exercise 2.1.1. We say that a measure μ on $(X, \mathfrak{S})$ *semifinite* if for each $E \in \mathfrak{S}$ with $\mu(E) = \infty$, there exists a measurable subset $F \subseteq E$ satisfying $0 < \mu(F) < \infty$. Show that any σ-finite measure is semifinite and that the conclusion of Lemma 2.1.5 remains valid for semifinite measure spaces. Show also that the counting measure on an uncountable set is semifinite but not σ-finite.

2.2 Unitary Representations on L^2-spaces

2.2.1 Measure Classes and Their Invariance

Definition 2.2.1. Let $(X, \mathfrak{S})$ be a measurable space and λ, μ be positive measures on $(X, \mathfrak{S})$. We say that λ is *absolutely continuous with respect to* μ if

$$\mu(E) = 0, \quad E \in \mathfrak{S} \quad \Rightarrow \quad \lambda(E) = 0.$$

We then write $\lambda \ll \mu$. We call λ and μ *equivalent* if they have the same zero sets, i.e., if $\lambda \ll \mu$ and $\mu \ll \lambda$. We then write $\lambda \sim \mu$. It is easy to see that $\sim$ defines an equivalence relation on the set of positive measures on $(X, \mathfrak{S})$. The corresponding equivalence classes $[\lambda]$ are called *measure classes.*

The Theorem of Radon–Nikodym ([Ru86, Thm. 6.10]) is a central result in abstract measure theory. It makes the relation $\lambda \ll \mu$ more concrete.

Theorem 2.2.2. (Radon–Nikodym Theorem) *If μ is a σ-finite measure on $(X, \mathfrak{S})$ and $\lambda \ll \mu$ is a finite positive measure, then there exists a unique non-negative $f \in L^1(X, \mu)$ with*

$$\lambda(E) = \int_E f(x)\, d\mu(x) \quad \textit{for} \quad E \in \mathfrak{S}.$$

We then write

$$\lambda = f \cdot \mu, \quad \text{and} \quad \frac{d\lambda}{d\mu} := f$$

is called the *Radon–Nikodym derivative of λ with respect to μ.*

Remark 2.2.3. (a) If λ is not finite, but only σ-finite, then the relation $\lambda \ll \mu$ still implies the existence of a measurable non-negative function $f\colon X \to \mathbb{R}_+$ with

$$\lambda(E) = \int_E f(x)\,d\mu(x) \quad \text{for} \quad E \in \mathfrak{S}.$$

In fact, the σ-finiteness of λ implies the existence of a disjoint sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathfrak{S}$ with $X = \bigcup_n E_n$ and $\lambda(E_n) < \infty$. Then $\lambda|_{E_n} \ll \mu|_{E_n}$ implies the existence of $f_n \in L^1(E, \mu_n)$ with $\lambda_{E_n} = f_n\mu|_{E_n}$. Then $f(x) := f_n(x)$ for $x \in E_n$ defines a measurable function on X with $\lambda = f\mu$.

From the uniqueness of f_n on each E_n, it follows that f is unique up to a function vanishing on the complement of a μ-zero set.

(b) The set

$$\{f = 0\} := \{x \in X\colon f(x) = 0\}$$

is a λ-zero set because

$$\lambda(\{f = 0\}) = \int_{\{f=0\}} f(x)\,d\mu(x) = 0.$$

If $\lambda \sim \mu$, then this implies that $\{f = 0\}$ is also a μ-zero set, and we obtain

$$\mu = f^{-1}\lambda,$$

because we may w.l.o.g. assume that $f(X) \subseteq \mathbb{R}^\times$.

We have already seen that each σ-finite positive measure μ on $(X, \mathfrak{S})$ defines a natural Hilbert space $L^2(X, \mu)$. We are interested in unitary group representations on this space. Clearly, a natural group to be considered in this context is the group

$$\mathrm{Aut}(X, \mathfrak{S}) := \{\varphi\colon X \to X\colon \varphi \text{ meas.},\ \exists\psi\colon X \to X \text{ meas.}, \psi\circ\varphi = \varphi\circ\psi = \mathrm{id}_X\}$$

of automorphism of the measurable space $(X, \mathfrak{S})$.

Recall that for a measurable map $\varphi\colon (X, \mathfrak{S}) \to (X', \mathfrak{S}')$ and a measure μ on $(X, \mathfrak{S})$, we obtain a measure $\varphi_*\mu$ on $(X, \mathfrak{S}')$ by

$$(\varphi_*\mu)(E) := \mu(\varphi^{-1}(E)) \quad \text{for} \quad E \in \mathfrak{S}'.$$

It is called the *push-forward of μ by φ*. The corresponding transformation formula for integrals reads

$$\int_{X'} f(x)\,d(\varphi_*\mu)(x) = \int_X f(\varphi(x))\,d\mu(x). \tag{2.2}$$

For a given positive measure μ on $(X, \mathfrak{S})$, we thus obtain two subgroups of $\mathrm{Aut}(X, \mathfrak{S})$:

$$\mathrm{Aut}(X, \mu) := \{\varphi \in \mathrm{Aut}(X, \mathfrak{S})\colon \varphi_*\mu = \mu\}$$

and the larger group

$$\mathrm{Aut}(X, [\mu]) := \{\varphi \in \mathrm{Aut}(X, \mathfrak{S})\colon \varphi_*\mu \sim \mu\}.$$

If $\varphi_*\mu \sim \mu$, then μ is said to be *quasi-invariant under* φ. Clearly,

$$\mathrm{Aut}(X, \mu) \subseteq \mathrm{Aut}(X, [\mu]).$$

For $\varphi \in \mathrm{Aut}(X, [\mu])$, we define

$$\delta(\varphi) := \delta_\mu(\varphi) := \frac{d(\varphi_*\mu)}{d\mu}$$

and note that, with the notation $\varphi_* f := f \circ \varphi^{-1}$, we have

$$\delta(\varphi\psi) = \delta(\varphi) \cdot \varphi_*(\delta(\psi)), \tag{2.3}$$

because

$$(\varphi\psi)_*\mu = \varphi_*\psi_*\mu = \varphi_*(\delta(\psi)\mu) = \varphi_*(\delta(\psi)) \cdot \varphi_*\mu = \varphi_*(\delta(\psi))\delta(\varphi)\mu.$$

Proposition 2.2.4. *For $f \in L^2(X, \mu)$ and $\varphi \in \mathrm{Aut}(X, [\mu])$, we put*

$$(\pi(\varphi)f)(x) := \sqrt{\delta(\varphi)(x)}\, f(\varphi^{-1}(x)).$$

Then $(\pi, L^2(X, \mu))$ is a unitary representation of the group $\mathrm{Aut}(X, [\mu])$.

Proof. Clearly, $\pi(\varphi)f$ is measurable, and we also find

$$\begin{aligned}\|\pi(\varphi)f\|_2^2 &= \int_X \delta(\varphi)(x)|f(\varphi^{-1}(x))|^2\, d\mu(x) = \int_X |f(\varphi^{-1}(x))|^2\, d(\varphi_*\mu)(x) \\ &= \int_X |f(x)|^2\, d\mu(x) = \|f\|^2,\end{aligned}$$

so that $\pi(\varphi)$ defines an isometry of $L^2(X, \mu)$. We also observe that for $\varphi, \psi \in \mathrm{Aut}(X, [\mu])$ we obtain with (2.3):

$$\begin{aligned}\pi(\varphi\psi)f &= \sqrt{\delta(\varphi\psi)}\varphi_*(\psi_* f) = \sqrt{\delta(\varphi)}\varphi_*\sqrt{\delta(\psi)}\varphi_*(\psi_* f) \\ &= \sqrt{\delta(\varphi)}\varphi_*\big(\sqrt{\delta(\psi)}(\psi_* f)\big) = \pi(\varphi)\pi(\psi)f.\end{aligned}$$

In particular, we see that each isometry $\pi(\varphi)$ is surjective with $\pi(\varphi^{-1}) = \pi(\varphi)^{-1}$. ☐

Remark 2.2.5. Let $\mathcal{M}(X, \mathbb{T})$ be the group of measurable functions $\theta \colon X \to \mathbb{T}$, where the group structure is defined by pointwise multiplication. We also associate to each $\theta \in \mathcal{M}(X, \mathbb{T})$ a unitary operator on $L^2(X, \mu)$ by

$$\rho(\theta)f := \theta f.$$

Clearly, this defines a unitary representation

$$\rho \colon \mathcal{M}(X, \mathbb{T}) \to \mathrm{U}(L^2(X, \mu)).$$

For $\varphi \in \mathrm{Aut}(X, [\mu])$ we have

$$\pi(\varphi)\rho(\theta)f = \sqrt{\delta(\varphi)}(\varphi_*\theta)(\varphi_* f) = \rho(\varphi_*\theta)\pi(\varphi)f,$$

i.e.,

$$\pi(\varphi)\rho(\theta)\pi(\varphi)^{-1} = \rho(\varphi_*\theta). \tag{2.4}$$

Next we observe that the transformation of measurable functions under $\mathrm{Aut}(X, \mathfrak{S})$ defines a homomorphism

$$\beta\colon \mathrm{Aut}(X, [\mu]) \to \mathrm{Aut}(\mathcal{M}(X, \mathbb{T})), \quad \beta(\varphi)\theta := \varphi_*\theta,$$

so that we can form the semidirect product group

$$\mathcal{M}(X, \mathbb{T}) \rtimes \mathrm{Aut}(X, [\mu]),$$

and the relation (2.4) implies that

$$\widehat{\pi}\colon \mathcal{M}(X, \mathbb{T}) \rtimes \mathrm{Aut}(X, [\mu]) \to \mathrm{U}(L^2(X, \mu)), \quad (\theta, \varphi) \mapsto \rho(\theta)\pi(\varphi) \tag{2.5}$$

defines a unitary representation of this semidirect product.

2.2.2 Intermezzo on Group Cocycles

Definition 2.2.6. Let N and G be groups and $\alpha\colon G \to \mathrm{Aut}(N)$ be a group homomorphism, so that we can form the semidirect product group

$$\widehat{G} := N \rtimes_\alpha G,$$

i.e., the product set $N \times G$, endowed with the multiplication

$$(n, g)(n', g') := (n\alpha(g)(n'), gg').$$

A map $f\colon G \to N$ is called a *(left)* 1*-cocycle*, resp., a *(left) crossed homomorphism* if

$$f(gh) = f(g) \cdot \alpha(g)(f(h)) \quad \text{for} \quad g, h \in G. \tag{2.6}$$

We write $Z^1(G, N)_\alpha$ for the set of 1-cocycles $f\colon G \to N$.

The cocycle relation (2.6) is equivalent to the map $s := (f, \mathrm{id}_G)\colon G \to N \rtimes_\alpha G$ being a group homomorphism. If $q\colon \widehat{G} \to G, (n, g) \mapsto g$ denote the corresponding quotient homomorphism, then $q \circ s = \mathrm{id}_G$. Conversely, every group homomorphism $s\colon G \to N \rtimes_\alpha G$ with $q \circ s = \mathrm{id}_G$ is of the form (f, id_G) for some $f \in Z^1(G, N)_\alpha$.

Identifying N with the subgroup $N \rtimes \{\mathbf{1}\}$ of $N \rtimes_\alpha G$, we see that N acts on the set

$$S := \{s \in \mathrm{Hom}(G, N \rtimes_\alpha G)\colon q \circ s = \mathrm{id}_g\}$$

by composition with the conjugation map

$$(n * s)(g) := (n, \mathbf{1})s(g)(n, \mathbf{1})^{-1}.$$

For $s(g) = (f(g), g)$, this leads to

$$(n * s)(g) = (nf(g)\alpha(g)(n^{-1}), g),$$

and from that we derive that

$$(n * f)(g) := nf(g)\alpha(g)(n^{-1})$$

defines an action of the group N on the set $Z^1(G, N)_\alpha$, satisfying $n * (f, \mathrm{id}_G) = (n * f, \mathrm{id}_G)$. We write

$$H^1(G, N)_\alpha := Z^1(G, N)_\alpha / N$$

for the set of N-orbits in this set. It is called the *first cohomology set for G with values in N with respect to α*. In general, this cohomology set does not carry a group structure, but it has a natural base point given by the orbit of the constant cocycle $f = \mathbf{1}$. We write $[f] \in H^1(G, N)_\alpha$ for the N-orbit of f, which is also called its *cohomology class*.

If N is abelian and written additively, the cocycle relation can be written as

$$f(gh) = f(g) + \alpha(g)(f(h)) \quad \text{for} \quad g, h \in G. \tag{2.7}$$

Since this relation is "additive in f", the the set $Z^1(G, N)_\alpha$ is a group with respect to pointwise addition. The N-action on this group is given by

$$(n * f)(g) = f(g) + n - \alpha(g)(n).$$

Cocycles of the form

$$\mathtt{d}_G(n)(g) := \alpha(g)n - n$$

are called 1-coboundaries. They form a subgroup $B^1(G, N)_\alpha$ of $Z^1(G, N)_\alpha$, so that

$$H^1(G, N)_\alpha \cong Z^1(G, N)_\alpha / B_1(G, N)_\alpha$$

can be identified with the quotient group, hence carries a natural group structure.

2.2.3 Applications to Representation Theory

Let $(X, \mathfrak{S}, \mu)$ be a σ-finite measure space and

$$\sigma \colon G \times X \to X, \quad (g, x) \mapsto g.x$$

a group action by measurable maps such that each map $\sigma_g(x) := \sigma(g, x)$ preserves the measure class $[\mu]$.

From Proposition 2.2.4, we obtain immediately that

$$(\pi(g)f)(x) := \sqrt{\delta(\sigma_g)(x)} f(\sigma_g^{-1}(x))$$

defines a unitary representation $(\pi, L^2(X,\mu))$ of G, but there are many other representations. Indeed, let $\gamma\colon G \to \mathcal{M}(X,\mathbb{T})$ be a 1-cocycle with respect to the action of G on $\mathcal{M}(X,\mathbb{T})$ by $g.\theta := \theta \circ \sigma_g^{-1}$. Then

$$(\gamma,\sigma)\colon G \to \mathcal{M}(X,\mathbb{T}) \rtimes \mathrm{Aut}(X,[\mu]), \quad g \mapsto (\gamma(g),\sigma_g)$$

is a group homomorphism, and (2.5) in Remark 2.2.5 implies that also

$$(\pi_\gamma(g)f)(x) := \gamma(g)(x)\sqrt{\delta(\sigma_g)(x)}f(\sigma_g^{-1}(x)), \quad \pi_\gamma(g)f := \gamma(g)\cdot\sqrt{\delta(\sigma_g)}(\sigma_g)_*f$$

defines a unitary representation of G on $L^2(X,\mu)$. The following lemma already shows the close connection between the problem to determine cohomology sets and the classification problem for unitary representations.

Example 2.2.7. In the situation from above, we obtain in particular a natural class of 1-cocycles by

$$\gamma(g) := \delta(\sigma_g)^{is} = e^{is\log\delta(\sigma_g)}, \quad s \in \mathbb{R}.$$

This leads to a family of unitary representations on $L^2(X,\mu)$, parameterized by $s \in \mathbb{R}$:

$$(\pi_s(g)f) := \delta(\sigma_g)^{\frac{1}{2}+is}(\sigma_g)_*f.$$

Lemma 2.2.8. *If the cocycles $\gamma_1,\gamma_2\colon G \to \mathcal{M}(X,\mathbb{T})$ are equivalent, then the representations π_{γ_1} and π_{γ_2} are equivalent.*

Proof. If $\gamma_1 \sim \gamma_2$, there exists a $\theta \in \mathcal{M}(X,\mathbb{T})$ with

$$(\gamma_2(g),g) = (\theta,\mathbf{1})(\gamma_1(g),g)(\theta,\mathbf{1})^{-1}$$

holds in $\mathcal{M}(X,\mathbb{T}) \rtimes G$. This implies that

$$\pi_{\gamma_2}(g) = \rho(\theta)\pi_{\gamma_1}(g)\rho(\theta)^{-1} \quad \text{for} \quad g \in G.$$ □

Example 2.2.9. Let $U \subseteq \mathbb{R}^n$ be an open subset and λ be Lebesgue measure, restricted to U. For each C^1-diffeomorphism $\varphi\colon U \to U$ we then have the transformation formulas

$$\int_U f(x)\,d(\varphi_*\lambda)(x) = \int_U f(\varphi(x))\,d\lambda(x)$$

and

$$\int_U f(\varphi(x))|\det(\mathtt{d}\varphi(x))|\,dx = \int_U f(x)\,dx.$$

Comparing these two implies that

$$\begin{aligned}\int_U f(x)\,d(\varphi_*\lambda)(x) &= \int_U f(\varphi(x))|\det(\mathtt{d}\varphi(x))||\det(\mathtt{d}\varphi(x))|^{-1}\,d\lambda(x)\\ &= \int_U f(x)|\det(\mathtt{d}\varphi(\varphi^{-1}(x)))|^{-1}\,d\lambda(x)\\ &= \int_U f(x)|\det(\mathtt{d}\varphi^{-1}(x))|\,d\lambda(x),\end{aligned}$$

and therefore

$$\delta(\varphi)(x) := \frac{d\varphi_*\lambda}{d\lambda} = |\det(\mathtt{d}\varphi^{-1}(x))|.$$

Therefore

$$(\pi(\varphi)f)(x) := \sqrt{|\det(\mathtt{d}\varphi^{-1}(x))|}f(\varphi^{-1}(x))$$

defines a unitary representation of the group $\mathrm{Diff}^1(U)$ of C^1-diffeomorphisms of U on $L^2(U,\lambda)$.

For $U = \mathbb{R}^n$ and the subgroup

$$\mathrm{Aff}_n(\mathbb{R}) := \{\varphi_{A,b}(x) = Ax + b \colon A \in \mathrm{GL}_n(\mathbb{R}), b \in \mathbb{R}^n\}$$

we obtain in particular

$$\delta(\varphi_{A,b})(x) = |\det A|^{-1},$$

so that

$$(\pi(\varphi_{A,b})f)(x) := \sqrt{|\det A|}^{-1} f(\varphi_{A,b}^{-1}(x))$$

defines a unitary representation of the affine group $\mathrm{Aff}_n(\mathbb{R})$ on $L^2(\mathbb{R}^n,\lambda)$.

Exercises for Section 2.2

Exercise 2.2.1. (Affine actions) Let V be a vector space and $\rho\colon G \to \mathrm{GL}(V)$ be a representation of the group G on V. Show that:

(1) For each 1-cocycle $f \in Z^1(G,V)_\rho$ we obtain on V an action of G by affine maps via $\sigma_f(g)(v) := \rho(g)v + f(g)$.

(2) The affine action σ_f has a fixed point if and only if f is a coboundary, i.e., of the form $f(g) = \rho(g)v - v$ for some $v \in V$.

(3) If G is finite and V is defined over a field of characteristic zero, then the group $H^1(G,V)_\rho$ is trivial.

Exercise 2.2.2. (Affine isometric actions) Let $\mathcal{H}$ be a Hilbert space and $(\pi,\mathcal{H})$ be a unitary representation of G. Show that:

(1) For each 1-cocycle $f \in Z^1(G,\mathcal{H})_\pi$, we obtain an action of G on $\mathcal{H}$ by affine isometries $\sigma_f(g)(v) := \pi(g)v + f(g)$.

(2) If this action has a fixed point, then f is bounded. [1]

(3) Consider the real unitary (=orthogonal) representation of $G = \mathbb{R}$ on $\mathcal{H} = \mathbb{R}^3$ by

$$\pi(t) := \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that $f\colon \mathbb{R} \to \mathbb{R}^3, f(t) := (0,0,t)^\top = te_3$ is a 1-cocycle with non-trivial cohomology class. Find a set of representatives for the corresponding affine action and describe the orbits geometrically.

[1] The converse is also true. It is a consequence of the Bruhat–Tits Fixed Point Theorem.

Exercise 2.2.3. (Left- and right cocycles) Let $\widehat{G} = N \rtimes_\alpha G$ be a semidirect product defined by the homomorphism $\alpha\colon G \to \mathrm{Aut}(N)$. Show that:

(1) The map $G \times N \to \widehat{G}, (g, n) \mapsto (\mathbf{1}, g)(n, \mathbf{1})$ is a group isomorphism if we define the multiplication on $G \times N$ by

$$(g, n)(g', n') := (gg', \alpha(g')^{-1}(n)n').$$

We write $G \ltimes_\alpha N$ for this group.

(2) A map $s\colon G \to G \ltimes_\alpha N, s(g, n) = (g, f(g))$ is a group homomorphism if and only if it is a *right cocycle*, i.e.,

$$f(gh) = \alpha(h)^{-1}(f(g))f(h) \quad \text{ for } \quad g, h \in G.$$

Exercise 2.2.4. Let $U \subseteq \mathbb{R}^n$ be an open subset. We consider the group $G := \mathrm{Diff}^1(U)$ of C^1-diffeomorphisms of U, and the group $N := C(U, \mathrm{GL}_n(\mathbb{R}))$, where the group structure on N is given by pointwise multiplication. Then $\alpha(\varphi)(f) := f \circ \varphi^{-1}$ defines a homomorphism $\alpha\colon G \to \mathrm{Aut}(N)$. Show that the Jacobi matrix

$$J\colon \mathrm{Diff}^1(U) \to C(\mathbb{R}^n, \mathrm{GL}_n(\mathbb{R})), \quad J(\varphi) := \Big(\frac{\partial \varphi_i}{\partial x_j}\Big)_{i,j=1,\ldots,n}$$

is a right cocycle with respect to α.

Exercise 2.2.5. Show that each σ-finite measure μ on a measurable space $(X, \mathfrak{S})$ is equivalent to a finite measure.

Exercise 2.2.6. Let μ and λ be equivalent σ-finite measures on $(X, \mathfrak{S})$ and $h := \frac{d\mu}{d\lambda}$. Show that

$$\Phi\colon L^2(X, \mu) \to L^2(X, \lambda), \quad f \mapsto \sqrt{h}f$$

defines a unitary map.

Exercise 2.2.7. (From measure classes to cohomology classes) Let $\sigma\colon G \times X \to X$ be an action of G by measurable maps on $(X, \mathfrak{S})$ and $[\mu]$ be a G-invariant σ-finite measure class. Verify the following assertions:

(a) Let $\mathcal{R}$ denote the group of all measurable functions $f\colon X \to \mathbb{R}_+^\times$ with respect to pointwise multiplication and write $\mathcal{R}_0$ for the subgroup of those functions which are constant 1 on the complement of a μ-zero set. Then $\widetilde{\mathcal{R}} := \mathcal{R}/\mathcal{R}_0$ is a group, whose elements are denoted $[f]$, and

$$g.[f] := [(\sigma_g)_* f]$$

defines an action of G on $\widetilde{\mathcal{R}}$ by automorphisms.

(b) $\delta_\mu(g) := \frac{d(\sigma_g)_*\mu}{d\mu}$ defines a $\widetilde{\mathcal{R}}$-valued 1-cocycle on G.

(c) If $[h] \in \widetilde{\mathcal{R}}$, then $\delta_{h\mu}(g) = \frac{(\sigma_g)_* h}{h}\delta_\mu(g)$. Conclude that the cohomology class $[\delta_\mu] \in H^1(G, \widetilde{\mathcal{R}}]$ does not depend on the representative μ of the measure class $[\mu]$.

(d) Show that the measure class $[\mu]$ contains a G-invariant element if and only if the cohomology class $[\delta_\mu]$ vanishes.

Exercise 2.2.8. Consider the Gaußian measure

$$d\gamma(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\|x\|^2} dx$$

on $\mathbb{R}^n$. Its measure class is invariant under the action of the affine group $\mathrm{Aff}_n(\mathbb{R})$. Find a formula for the action of this group on $L^2(\mathbb{R}^n, \gamma)$.

Exercise 2.2.9. Let G be a group and $\mathcal{H}$ be a complex Hilbert space. We consider the trivial homomorphism $\alpha = \mathbf{1} \colon G \to \mathrm{Aut}(\mathrm{U}(\mathcal{H}))$, i.e., $\alpha(G) = \{\mathrm{id}_{\mathrm{U}(\mathcal{H})}\}$ for each $g \in G$. Show that there is a bijection between $H^1(G, \mathrm{U}(\mathcal{H}))$ and the of equivalence classes of unitary representations of the group G on $\mathcal{H}$.

Exercise 2.2.10. Let $(X, \mathfrak{S}, \mu)$ be a measure space. Show that:

(a) If $f \in L^1(X, \mu)$, then the measurable subset $\{|f| \neq 0\}$ of X is σ-finite.

(b) If $\mathcal{H} \subseteq L^2(X, \mu)$ is a separable subspace, then there exists a σ-finite measurable subset $X_0 \subseteq X$ with the property that each $f \in \mathcal{H}$ vanishes μ-almost everywhere on $X_0^c = X \setminus X_0$.

Exercise 2.2.11. (Limitations of the Radon–Nikodym Theorem) Let $(X, \mathfrak{S}, \mu)$ be a finite measure space. Define $\nu \colon \mathfrak{S} \to \mathbb{R}_+$ by

$$\nu(E) := \begin{cases} 0 & \text{for } \mu(E) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Show that ν is a measure and that $\nu \sim \mu$, but there exists no $\mathbb{R}$-valued measurable function $f \colon X \to \mathbb{R}$ with $\nu = f\mu$. However, the constant function $f = \infty$ satisfies $\nu(E) = \int_E f(x)\, d\mu(x)$ for each $E \in \mathfrak{S}$.

Exercise 2.2.12. (Modifying homomorphisms by cocycles) Let G and N be groups and $\beta \colon G \to N$ a homomorphism. Then $\alpha(g)(n) := \beta(g)n\beta(g)^{-1}$ defines a homomorphism $\alpha \colon G \to \mathrm{Aut}(N)$. Show that:

(a) A function $f \colon G \to N$ is a 1-cocycle with respect to α if and only if the pointwise product

$$f\beta \colon G \to N, \quad g \mapsto f(g)\beta(g)$$

is a homomorphism.

(b) If $\gamma \colon G \to N$ is another homomorphism, then $f(g) := \gamma(g)\beta(g)^{-1}$ defines a 1-cocycle with respect to α.

Exercise 2.2.13. Let $(X, \mathfrak{S}, \mu)$ be a finite measure space and $G := \mathrm{Aut}(X, \mu)$ be its automorphism group. Verify the following assertions:

(a) On $\mathfrak{S}$ we obtain by $d(E, F) := \mu(E \Delta F)$ a semimetric. Let

$$[E] := \{F \in \mathfrak{S} \colon d(E, F) = 0\}$$

denote the corresponding equivalence class of E and $\overline{\mathfrak{S}} := \mathfrak{S}/\sim$ denote the set of equivalence classes. Then $d([E], [F]) := \mu(E \Delta F)$ defines a metric on $\overline{\mathfrak{S}}$.

(b) The map $\gamma \colon \mathfrak{S} \to L^2(X, \mu), E \mapsto \chi_E$ satisfies

$$d(E, F) = \mu(E \Delta F) = \|\chi_E - \chi_F\|^2.$$

(c) $N := \{g \in G \colon (\forall E \in \mathfrak{S})\, d(gE, E) = 0\}$ is a normal subgroup of G, and if $\pi \colon G \to \mathrm{U}(L^2(X, \mu)), \pi(g)f := f \circ g^{-1}$ is the canonical unitary representation of G on $L^2(X, \mu)$, then $\ker \mu = N$.

(d) On the quotient group $\overline{G} := G/N$, we consider the coarsest topology for which all functions

$$f_E \colon \overline{G} \to \mathbb{R}, \quad g \mapsto \mu(gE \Delta E)$$

are continuous. Show that $\overline{G}$ is a topological group and that π factors through a topological embedding $\overline{\pi} \colon \overline{G} \to \mathrm{U}(L^2(X, \mu))_s$. Hint: Exercise 1.2.6 and Lemma 1.2.6.

2.3 L^2-spaces on Locally Compact Spaces

In this section we discuss a central bridge between topology and measure theory, the Riesz Representation Theorem.

Definition 2.3.1. Let X be a locally compact space. A *positive Radon integral on X* is a linear functional

$$I \colon C_c(X, \mathbb{C}) \to \mathbb{R}$$

on the algebra $C_c(X) := C_c(X, \mathbb{C})$ of compactly supported continuous functions on X which is positive in the sense that

$$f \geq 0 \quad \Rightarrow \quad I(f) \geq 0.$$

For the following we recall that for a topological space X, the *sigma algebra $\mathfrak{B}(X)$ of Borel sets* is the σ-algebra generated by the open subsets. Accordingly, a *Borel measure on X* is a measure defined on $\mathfrak{B}(X)$.

Theorem 2.3.2. (Riesz Representation Theorem) *Let X be a locally compact space.*

(a) *If μ is a positive Borel measure on X with $\mu(K) < \infty$ for each compact subset K, then the integral*

$$I_\mu \colon C_c(X) \to \mathbb{C}, \quad f \mapsto \int_X f(x)\, d\mu(x)$$

is a positive Radon integral on X.

(b) *For each positive Radon integral I on X there exists a positive Borel measure μ on X, which is uniquely determined by the following properties:*

(i) $I_\mu = I$.

(ii) $\mu(K) < \infty$ *for each compact subset K of X.*

(iii) (Outer regularity) *For each Borel subset $E \subseteq X$ we have*

$$\mu(E) = \inf\{\mu(U) \colon E \subseteq U, U \mathit{open}\}.$$

(iv) *If $E \subseteq X$ is open or E is a Borel set with $\mu(E) < \infty$, then*

$$\mu(E) = \sup\{\mu(K) \colon K \subseteq E, K \ \mathit{compact}\}.$$

Then μ is called the *Radon measure* defined by I.

Proof. (Sketch) We only given the rough outline of the proof.

(a) is the simple part. First we observe that every continuous function $f \colon X \to \mathbb{R}$ is Borel measurable. If, in addition, f vanishes outside some compact set K, then $|f| \leq \|f\|_\infty \chi_K$ implies the integrability of f with respect to μ. Clearly, the so obtained linear functional I_μ is positive.

(b) The simple part of the proof is the verification of uniqueness. In view of (iii) and (iv), μ is determined by its values on compact subsets of X. So let μ_1 and μ_2 be two Borel measures satisfying (i)-(iv) and $K \subseteq X$ be compact. We have to show that $\mu_1(K) = \mu_2(K)$. So let $\varepsilon > 0$. In view of (iii), there exists an open subset $U \supseteq K$ with $\mu_2(U) \leq \mu_2(K) + \varepsilon$. Using Urysohn's Theorem, we find a function $f \in C_c(X, \mathbb{R})$ with $0 \leq f \leq 1$, $f|_K = 1$ and $f|_{X \setminus U} = 0$. Then

$$\begin{aligned} \mu_1(K) &= \int_X \chi_K(x)\, d\mu_1(x) \leq \int_X f(x)\, d\mu_1(x) = I(f) = \int_X f(x)\, d\mu_2(x) \\ &\leq \int_X \chi_U(x)\, d\mu_2(x) = \mu_2(U) \leq \mu_2(K) + \varepsilon. \end{aligned}$$

Since ε was arbitrary, we obtain $\mu_1(K) \leq \mu_2(K)$. Exchanging the roles of μ_1 and μ_2 now shows equality. This proves uniqueness and we have also seen in (a) that (i) implies (ii).

For the existence, we put for an open subset $U \subseteq X$:

$$\mu(U) := \sup\{I(f) \colon \operatorname{supp}(f) \subseteq U, 0 \leq f \leq 1\}$$

and note that $\mu(U_1) \leq \mu(U_2)$ for $U_1 \subseteq U_2$ follows immediately. This shows in particular that

$$\mu(E) = \inf\{\mu(U) \colon E \subseteq U, U \text{open}\} \tag{2.8}$$

for each open subset $E \subseteq X$. We may therefore define $\mu(E)$ for an arbitrary subset $E \subseteq X$ by (2.8). For the somewhat technical and lengthy proof, we refer to [Ru86, p.40ff]. □

Definition 2.3.3. Property (iv) in the Riesz Representation Theorem is expressed by saying that all open subsets and sets with finite measure are *inner regular*. We call μ a *regular Borel measure* if all Borel sets are *outer regular* and all open subsets are inner regular. Then the Riesz representation theorem yields a bijection between Radon integrals and regular Borel measures. Therefore regular Borel measures are often called *Radon measures*.

If μ is regular, then the proof of the following proposition shows that every Borel set of finite measure is inner regular. In sufficiently nice spaces, regularity is automatic (cf. [Ru86, Thm. 2.18]):

Proposition 2.3.4. *If X is a locally compact space in which every open subset is a countable union of compact subsets, then every Borel measure μ with $\mu(K) < \infty$ for every compact subset is regular.*

Remark 2.3.5. (a) If (X, d) is a metric, σ-compact locally compact space, then every open subset of X is σ-compact. To see this, let $O \subseteq X$ be a proper open subset and $X = \bigcup_{n\in\mathbb{N}} K_n$, where each K_n is compact, and w.l.o.g. $K_n \subseteq K_{n+1}$ for $n \in \mathbb{N}$. Then

$$f \colon X \to \mathbb{R}, \quad f(x) := \sup\{r > 0 \colon B_r(x) \subseteq O\} = \operatorname{dist}(x, X \setminus O)$$

is a continuous function (Exercise), measuring the distance to $X \setminus O$. Then each set

$$Q_n := \Big\{x \in K_n \colon f(x) \geq \frac{1}{n}\Big\}$$

is compact and $O = \bigcup_{n\in\mathbb{N}} Q_n$.

(b) Applying Proposition 2.3.4 to the Riemann integral on $C_c(\mathbb{R}^n)$, the Riesz Representation Theorem yields the Lebesgue measure on $\mathfrak{B}(\mathbb{R}^n)$.

An important property of Radon measures is that many properties of L^2-functions can be derived from corresponding properties of continuous functions. The key is the following approximation result:

Proposition 2.3.6. *If μ is a Radon measure on a locally compact space X, then $C_c(X)$ is dense in $\mathcal{L}^2(X, \mu)$, resp., $L^2(X, \mu)$.*

Proof. Since the step functions form a dense subspace of $L^2(X, \mu)$, it suffices to show that any characteristic function χ_E with $\mu(E) < \infty$ can be approximated by elements of $C_c(X)$ in the L^2-norm. Since every Borel set is inner regular, we may w.l.o.g. assume that E is compact. Then the outer regularity implies for each $\varepsilon > 0$ the existence of an open subset $U \subseteq X$ with $\mu(U \setminus E) < \varepsilon$. Next we use Urysohn's Theorem to find a continuous function $f \in C_c(X)$ with $0 \leq f \leq 1$, $f|_E = 1$, and $\mathrm{supp}(f) \subseteq U$. Then

$$\|f - \chi_E\|_2^2 = \int_X |f(x) - \chi_E(x)|^2 \, d\mu(x) = \int_{U \setminus E} |f(x)|^2 \, d\mu(x) \leq \mu(U \setminus E) < \varepsilon,$$

and this completes the proof. $\square$

Lemma 2.3.7. *Let X be a topological space and Y be a locally compact space and ν be a Radon measure on Y. Let $f \colon X \times Y \to \mathbb{C}$ be continuous and suppose that there exists a compact subset $K \subseteq Y$ with* $\mathrm{supp}(f) \subseteq X \times K$. *Then the functions*

$$f^\vee \colon X \to (C(K), \|\cdot\|_\infty), \quad x \mapsto f_x, \quad f_x(y) := f(x,y)$$

and

$$F \colon X \to \mathbb{C}, \quad x \mapsto \int_Y f(x,y) \, d\nu(y)$$

are continuous.

Proof. Let $x_0 \in X$ and $\varepsilon > 0$. Then

$$M := \{(x,y) \in X \times Y \colon |f(x,y) - f(x_0,y)| < \varepsilon\}$$

is an open subset of $X \times Y$ containing the compact subset $\{x_0\} \times K$, hence also a set of the form $U \times K$, where U is a neighborhood of x_0 (Exercise 2.3.2), and this means that for $x \in U$, we $\|F_x - F_{x_0}\|_\infty \leq \varepsilon$.

Finally, the continuity of the function F follows from the continuity of the linear functional $C(K) \to \mathbb{C}, h \mapsto \int_K h \, d\nu$ with respect to $\|\cdot\|_\infty$ on $C(K)$ (cf. Exercise 2.3.1). $\square$

Proposition 2.3.8. *Let G be a topological group and $\sigma \colon G \times X \to X$ be a continuous action of G on the locally compact space X. Further, let μ be a Radon measure on X whose measure class is G-invariant and for which the Radon–Nikodym derivative $\delta(g) := \frac{d(\sigma_g)_*\mu}{d\mu}$ can be realized by a continuous function*

$$\widetilde{\delta} \colon G \times X \to \mathbb{R}^\times, \quad \widetilde{\delta}(g,x) := \delta(g)(x).$$

Then the unitary representation $(\pi, L^2(X,\mu))$, defined by

$$(\pi(g)f)(x) := \sqrt{\delta(g)(x)} f(g^{-1}.x)$$

is continuous.

Proof. In Proposition 2.3.6 we have seen that $C_c(X)$ is dense in $L^2(X,\mu)$. In view of Lemma 1.2.6, it therefore suffices to show that for $f, h \in C_c(X)$, the function

$$\begin{aligned} \pi_{f,h} \colon G \to \mathbb{C}, \quad & g \mapsto \langle \pi(g)f, h \rangle \\ &= \int_X \sqrt{\delta(g)(x)} f(g^{-1}.x)\overline{h(x)} \, d\mu(x) = \int_{\mathrm{supp}(h)} \sqrt{\delta(g)(x)} f(g^{-1}.x)\overline{h(x)} \, d\mu(x) \end{aligned}$$

is continuous. This follows from Lemma 2.3.7. $\square$

Corollary 2.3.9. *Let G be a topological group, $\sigma\colon G \times X \to X$ be a continuous action of G on the locally compact space X and μ be a G-invariant Radon measure on X. Then the unitary representation $(\pi, L^2(X, \mu))$, defined by*

$$\pi(g)f := f \circ \sigma_g^{-1}$$

is continuous.

Example 2.3.10. (a) The translation representation of $G = \mathbb{R}^n$ on $L^2(\mathbb{R}^n, dx)$, given by

$$(\pi(x)f)(y) := f(x + y)$$

is continuous.

(b) On the circle group $G = \mathbb{T}$, we consider the Radon measure $\mu_{\mathbb{T}}$, given by

$$\int_{\mathbb{T}} f(z)\, d\mu_{\mathbb{T}}(z) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, dt.$$

Then the unitary representation of $\mathbb{T}$ on $L^2(\mathbb{T}, \mu_{\mathbb{T}})$, given by

$$(\pi(t)f)(z) := f(tz)$$

is continuous.

Exercises for Section 2.3

Exercise 2.3.1. Let μ be a Radon measure on the locally compact space X and $K \subseteq X$ be a compact subset. Show that the integral

$$I\colon C(K) \to \mathbb{C}, \quad f \mapsto \int_K f(x)\, d\mu(x)$$

satisfies

$$|I(f)| \leq \|f\|_\infty \mu(K).$$

In particular, I is continuous.

Exercise 2.3.2. Let X and Y be Hausdorff spaces and $K \subseteq X$, resp., $Q \subseteq Y$ be a compact subset. Then for each open subset $U \subseteq X \times Y$ containing $K \times Q$, there exist open subsets $U_K \subseteq X$ containing K and $U_Q \subseteq Y$ containing Y with

$$K \times Q \subseteq U_K \times U_Q \subseteq U.$$

2.4 Haar Measure on Locally Compact Groups

In this section G always denotes a locally compact group.

Definition 2.4.1. (a) A positive Radon measure μ on G is called *left invariant* if

$$\int_G f(gx)\ d\mu(x) = \int_G f(x)\ d\mu(x) \quad \text{for} \quad f \in C_c(G), g \in G.$$

We likewise define right invariance by

$$\int_G f(xg)\ d\mu(x) = \int_G f(x)\ d\mu(x) \quad \text{for} \quad f \in C_c(G), g \in G.$$

(b) A positive left invariant Radon measure μ on G is called a *(left) Haar integral*, resp., a *(left) Haar measure*, if $0 \neq f \geq 0$ for $f \in C_c(G)$ implies

$$\int_G f(x)\, d\mu(x) > 0.$$

In the following we shall denote Haar measures on G by μ_G.

Remark 2.4.2. One can show that every locally compact group G possesses a Haar measure and that for two Haar measures μ and μ' there exists a $\lambda > 0$ with $\mu' = \lambda\mu$ ([Neu90], [HiNe91]).

If G is compact and μ a Haar measure on G, then $\mu(G)$ is finite positive, so that we obtain a unique Haar probability measure on G. We call this Haar measure *normalized*.

Example 2.4.3. (a) If G is a discrete group, then $C_c(G)$ is the space of finitely supported functions on G, and the counting measure

$$\int_G f\, d\mu := \sum_{g \in G} f(g)$$

is a Haar measure on G. If, in addition, G is finite, then

$$\int_G f\, d\mu := \frac{1}{|G|} \sum_{g \in G} f(g)$$

is a normalized Haar measure.

(b) For $G = \mathbb{Z}$ we obtain in particular a Haar measure by

$$\int_{\mathbb{Z}} f\, d\mu_{\mathbb{Z}} := \sum_{n \in \mathbb{Z}} f(n).$$

(c) On $G = \mathbb{R}^n$, the Riemann, resp., Lebesgue integral defines a Haar measure by

$$\int_{\mathbb{R}^n} f\, d\mu_G := \int_{\mathbb{R}^n} f(x)\ dx.$$

(d) On the circle group $G = \mathbb{T}$,

$$\int_{\mathbb{T}} f\, d\mu_{\mathbb{T}} := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\ dt$$

is a Haar measure.

(e) If $G = (\mathbb{R}^\times, \cdot)$ is the multiplicative group of real numbers, then

$$\int_{\mathbb{R}^\times} f d\mu := \int_{\mathbb{R}^\times} \frac{f(x)}{|x|}\, dx$$

is a Haar measure on G. Note that a continuous function with compact support on $\mathbb{R}^\times$ vanishes in a neighborhood of 0, so that the integral is defined.

Lemma 2.4.4. *If μ_G is a Haar measure on G and $h \in C(G)$ with $\int_G fh\, d\mu_G = 0$ for all $f \in C_c(G)$, then $h = 0$.*

Proof. Let $g \in G$. Then there exists a function $f \in C_c(G)$ with $0 \leq f$ and $f(g) > 0$ (Urysohn's Theorem). Now $f\overline{h} \in C_c(G)$ satisfies $f\overline{h} \cdot h = f|h|^2 \geq 0$, so that $\int_G f|h|^2\, d\mu_G = 0$ implies $f|h|^2 = 0$, and therefore $h(g) = 0$. $\square$

Proposition 2.4.5. *Let μ_G be a Haar measure on G. Then there exists a continuous homomorphism*

$$\Delta_G \colon G \to (\mathbb{R}_+^\times, \cdot) \quad \textit{with} \quad (\rho_g)_*\mu_G = \Delta_G(g)^{-1}\mu_G \quad \textit{for} \quad g \in G.$$

Proof. Since left and right multiplications on G commutes, the Radon measure $(\rho_g)_*\mu_G$ is also left invariant and satisfies

$$\int_G f\, d\big((\rho_g)_*\mu_G\big) = \int_G f \circ \rho_g\, d\mu_G > 0$$

for $0 \neq f \geq 0$ (cf. (2.2)). Therefore $(\rho_g)_*\mu_G$ is a left Haar measure, and hence there exists $\Delta_G(g) \in]0, \infty[$ with $(\rho_g)_*\mu_G = \Delta_G(g)^{-1}\mu_G$ (Remark 2.4.2).

Let $0 \neq f \in C_c(G)$ with $f \geq 0$. To see that

$$\Delta_G(g)^{-1} = \frac{1}{\int_G f d\mu_G} \int_G (f \circ \rho_g)\, d\mu_G$$

depends continuously on g, we note that for a fixed $g \in G$, we actually integrate only over $\mathrm{supp}(f)g^{-1}$. For any compact neighborhood K of g_0, the subset $\mathrm{supp}(f)K^{-1}$ of G is compact (it is the image of the compact product set $\mathrm{supp}(f) \times K$ under the continuous map $(x, y) \mapsto xy^{-1}$), and for any $g \in K$ we have

$$\Delta_G(g)^{-1} = \frac{1}{\int_G f d\mu_G} \int_{\mathrm{supp}(f)K^{-1}} (f \circ \rho_g)\, d\mu_G,$$

so that the continuity in g_0 follows from Lemma 2.3.7. That Δ_G is a homomorphism is an immediate consequence of the definition:

$$\begin{aligned}(\rho_{gh})_*\mu_G &= (\rho_h\rho_g)_*\mu_G = (\rho_h)_*(\rho_g)_*\mu_G \\ &= \Delta_G(g)^{-1}(\rho_h)_*\mu_G = \Delta_G(g)^{-1}\Delta_G(h)^{-1}\mu_G\end{aligned}$$

(cf. Exercise 2.2.7). $\square$

Definition 2.4.6. The function Δ_G is called the *modular factor of* G. Clearly, it does not depend on the choice of the Haar measure μ_G. A locally compact group G is called *unimodular* if $\Delta_G = 1$, i.e., each left invariant Haar measure is also right invariant, hence biinvariant.

Proposition 2.4.7. *A locally compact group G is unimodular i f it satisfies one of the following conditions:*

(a) *G is compact.*

(b) *G is abelian.*

(c) *Its commutator group (G, G) is dense.*

Proof. (a) In this case $\Delta_G(G)$ is a compact subgroup of $\mathbb{R}^\times_+$, hence equal to $\{1\}$.

(b) Follows from the fact that $\rho_g = \lambda_g$ for any $g \in G$.

(c) Since $\mathbb{R}^\times_+$ is abelian, $(G, G) \subseteq \ker \Delta_G$. If (G, G) is dense, the continuity of Δ_G implies that $\Delta_G = 1$. □

Lemma 2.4.8. *Let G be a locally compact group, μ_G a Haar measure and Δ_G be the modular factor. Then we have for $f \in L^1(G, \mu_G)$ the following formulas:*

(a) $\int_G f(xg)\, d\mu_G(x) = \Delta_G(g)^{-1} \int_G f(x)\, d\mu_G(x)$.

(b) $\Delta_G^{-1} \cdot \mu_G$ *is a right invariant measure on* G.

(c) $\int_G f(x^{-1})\, d\mu_G(x) = \int_G f(x)\Delta_G(x)^{-1}\, d\mu_G(x)$.

Proof. (a) is the definition of the modular factor.

(b) Using (a), we obtain $(\rho_g)_*\mu_G = \Delta_G(g)^{-1}\mu_G$. Since we also have $(\rho_g)_*\Delta_G = \Delta_G(g)^{-1}\Delta_G$, (b) follows.

(c) Let $I(f) := \int_G f(x^{-1})\Delta_G(x)^{-1}\, d\mu_G(x)$. Then (b) implies that

$$\begin{aligned} I(f \circ \lambda_g) &= \int_G f(g^{-1}x^{-1})\Delta_G(x)^{-1}\, d\mu_G(x) = \int_G f((xg)^{-1})\Delta_G(x)^{-1}\, d\mu_G(x) \\ &= \int_G f(x^{-1})\Delta_G(x)^{-1}\, d\mu_G(x) = I(f), \end{aligned}$$

so that I is left invariant. For $0 \leq f \neq 0$ in $C_c(G)$ we also have $I(f) > 0$, so that I is a Haar integral. In view of the Uniqueness of Haar measure, there exists a $C > 0$ with

$$I(f) = \int_G f(x^{-1})\Delta_G(x)^{-1}\, d\mu_G(x) = C \int_G f(x)\, d\mu_G(x) \quad \text{for} \quad f \in C_c(G).$$

It remains to show that $C = 1$. We apply the preceding relation to the compactly supported function $\widetilde{f}(x) := f(x^{-1})\Delta_G(x)^{-1}$ to find

$$C \int_G \widetilde{f}(x)\, d\mu_G(x) = \int_G f(x)\Delta_G(x)\Delta_G(x)^{-1}\, d\mu_G(x) = \int_G f(x)\, d\mu_G(x),$$

which leads to $C = 1/C$, and hence to $C = 1$. □

Proposition 2.4.9. *Let G be a locally compact group and μ_G a (left) Haar measure on G. On $L^2(G) := L^2(G, \mu_G)$ we have two continuous unitary representations of G. The* left regular representation

$$\pi_l(g)f := f \circ \lambda_g^{-1}$$

and the right regular representation

$$\pi_r(g)f := \sqrt{\Delta_G(g)} f \circ \rho_g.$$

Proof. The continuity of the left regular representation follows from Corollary 2.3.9. For the right regular representation we apply Proposition 2.3.8 to the left action of G on G defined by $\sigma_g(x) := xg^{-1} = \rho_{g^{-1}}x$. Then

$$\widetilde{\delta}(g, x) := \frac{d\big((\sigma_g)_*\mu_G\big)}{d\mu_G}(x) = \frac{d\big((\rho_g^{-1})_*\mu_G\big)}{d\mu_G}(x) = \Delta_G(g)$$

is a continuous function on $G \times G$, which implies the continuity of π_r. □

Corollary 2.4.10. *For a locally compact group G, the left regular representation is injective. In particular, G has a faithful continuous unitary representation.*

Proof. For $g \neq \mathbf{1}$, pick disjoint open neighborhoods U of $\mathbf{1}$ and V of g with $gU \subseteq V$. Let $0 \leq f \in C_c(G)$ be non-zero with $\mathrm{supp}(f) \subseteq U$. Then

$$\langle \pi_l(g)f, f\rangle = \int_G f(g^{-1}x)\overline{f(x)}\, d\mu_G(x) = 0$$

because $\mathrm{supp}(\pi_l(g)f) = g\,\mathrm{supp}(f) \subseteq gU \subseteq V$ intersects U trivially. On the other hand the definition of Haar measure implies $\|f\|_2 > 0$, so that $\pi_l(g) \neq \mathbf{1}$. □

Exercises for Section 2.4

Exercise 2.4.1. Let $\lambda = dX$ denote Lebesgue measure on the space $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of real $(n \times n)$-matrices. Show that a Haar measure on $\mathrm{GL}_n(\mathbb{R})$ is given by

$$d\mu_{\mathrm{GL}_n(\mathbb{R})}(g) = \frac{1}{|\det(g)|^n} d\lambda(g).$$

Hint: Calculate the determinant of the linear maps $\lambda_g \colon M_n(\mathbb{R}) \to M_n(\mathbb{R}), x \mapsto gx$.

Exercise 2.4.2. Let $G = \mathrm{Aff}_1(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$ denote the affine group of $\mathbb{R}$, where (b, a) corresponds to the affine map $\varphi_{b,a}(x) := ax + b$. This group is sometimes called the *$ax+b$-group.* Show that a Haar measure on this group is obtained by

$$\int_G f(b, a)\, d\mu_G(b, a) := \int_{\mathbb{R}} \int_{\mathbb{R}^\times} f(b, a)\, db \frac{da}{|a|^2}.$$

Show further that $\Delta_G(b, a) = |a|^{-1}$, which implies that G is not unimodular.

Exercise 2.4.3. We consider the group $G := \mathrm{GL}_2(\mathbb{R})$ and the *real projective line*

$$\mathbb{P}_1(\mathbb{R}) = \{[v] := \mathbb{R}v \colon 0 \neq v \in \mathbb{R}^2\}$$

of 1-dimensional linear subspaces of $\mathbb{R}^2$. We write $[x : y]$ for the line $\mathbb{R}\begin{pmatrix} x \\ y \end{pmatrix}$. Show that:

(a) We endow $\mathbb{P}_1(\mathbb{R})$ with the quotient topology with respect to the map $q \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{P}_1(\mathbb{R}), v \mapsto [v]$. Show that $\mathbb{P}_1(\mathbb{R})$ is homeomorphic to $\mathbb{S}^1$. Hint: Consider the squaring map on $\mathbb{T} \subseteq \mathbb{C}$.

(b) The map $\mathbb{R} \to \mathbb{P}_1(\mathbb{R}), x \mapsto [x : 1]$ is injective and its complement consists of the single point $\infty := [1 : 0]$ (the horizontal line). We thus identify $\mathbb{P}_1(\mathbb{R})$ with the one-point compactification of $\mathbb{R}$. These are the so-called *homogeneous coordinates* on $\mathbb{P}_1(\mathbb{R})$.

(c) The natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}_1(\mathbb{R})$ by $g.[v] := [gv]$ is given in the coordinates of (b) by

$$g.x = \sigma_g(x) := \frac{ax+b}{cx+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(d) There exists a unique Radon measure μ with total mass π on $\mathbb{P}_1(\mathbb{R})$ which is invariant under the group $\mathrm{O}_2(\mathbb{R})$. Hint: Identify $\mathbb{P}_1(\mathbb{R})$ with the compact group $\mathrm{SO}_2(\mathbb{R})/\{\pm\mathbf{1}\} \cong \mathbb{T}$.

(e) Show that, in homogeneous coordinates, we have $d\mu(x) = \frac{dx}{1+x^2}$. Hint: $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.0 = -\tan x$, and the image of Lebesgue measure on $]-\pi/2, \pi/2[$ under tan is $\frac{dx}{1+x^2}$.

(f) Show that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}_1(\mathbb{R})$ preserves the measure class of μ. Hint: Show that $\sigma_g(x) := \frac{ax+b}{cx+d}$ satisfies $\sigma_g'(x) = \frac{1}{(cx+d)^2}$ and derive the formula

$$\delta(\sigma_g)(x) = \frac{d((\sigma_g)_*\mu)}{d\mu} = \frac{1+x^2}{(a-cx)^2+(b-dx)^2}, \quad \delta(\sigma_g)(\infty) = \frac{1}{c^2+d^2}.$$

(g) The density function also has the following metric interpretation with respect to the euclidean norm on $\mathbb{R}^2$:

$$\delta(\sigma_g)([v]) = \frac{\|g^{-1}v\|^2}{\|v\|^2}.$$

The corresponding unitary representations of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{P}_1(\mathbb{R}), \mu)$ defined by

$$\pi_s(g)f := \delta(\sigma_g)^{\frac{1}{2}+is}(\sigma_g)_*f$$

(cf. Example 2.2.7) form the so-called *spherical principal series*.

Exercise 2.4.4. Let X be a locally compact space, μ a positive Radon measure on X, $\mathcal{H}$ a Hilbert space and $f \in C_c(X, \mathcal{H})$ be a compactly supported continuous function.

(a) Prove the existence of the $\mathcal{H}$-valued integral

$$I := \int_X f(x)\, d\mu(x),$$

i.e., the existence of an element $I \in \mathcal{H}$ with

$$\langle v, I \rangle = \int_X \langle v, f(x) \rangle \, d\mu(x) \quad \text{for} \quad v \in \mathcal{H}.$$

Hint: Verify that the right hand side of the above expression is defined and show that it defines a continuous linear functional on $\mathcal{H}$.

(b) Show that, if μ is a probability measure, then

$$I \in \overline{\mathrm{conv}(f(X))}.$$

Hint: Use the Hahn–Banach Separation Theorem.

Exercise 2.4.5. Let $\sigma \colon K \times \mathcal{H} \to \mathcal{H}, (k, v) \mapsto k.v$ be a continuous action of the compact group K by affine maps on the Hilbert space $\mathcal{H}$. Show that σ has a fixed point. Hint: For any orbit K-orbit $K.v$, define the *center of mass* by

$$c(v) := \int_K k.v \, d\mu_K(k),$$

where μ_K is a normalized Haar measure on K (cf. Exercise 2.4.4).

Conclude that for any continuous unitary representation $(\pi, \mathcal{H})$ of K each continuous 1-cocycle $f \colon K \to \mathcal{H}$ is a coboundary, i.e., of the form $f(k) = \pi(k)v - v$ for some $v \in \mathcal{H}$. Hint: Use Exercise 2.2.2.

Chapter 3

Reproducing Kernel Spaces

In Chapter 2 we have seen how Hilbert spaces and continuous unitary representations can be constructed on L^2-spaces. An L^2-space $L^2(X,\mu)$ of a measure space $(X,\mathfrak{S})$ has the serious disadvantage that its elements are not functions on X, they are only equivalence classes of functions modulo those vanishing on μ-zero sets. However, many important classes of unitary representations can be realized in spaces of continuous functions. In particular for infinite dimensional Lie groups, this is the preferred point of view because measure theory on infinite dimensional spaces has serious defects that one can avoid by using other methods.

The main concept introduced in this chapter is that of a reproducing kernel Hilbert space. These are Hilbert spaces $\mathcal{H}$ of *functions* on a set X for which all point evaluations $\mathcal{H} \to \mathbb{C}, f \mapsto f(x)$, are continuous linear functions. Representing these functions according to the Fréchet–Riesz Theorem by an element $K_x \in \mathcal{H}$, we obtain a function

$$K\colon X \times X \to \mathbb{C}, \quad K(x,y) := K_y(x)$$

called the *reproducing kernel of* $\mathcal{H}$. Typical questions arising in this context are: Which functions on $X \times X$ are reproducing kernels (such kernels are called positive definite) and, if we have a group action on X, how can we construct unitary representations on reproducing kernel spaces.

Throughout this chapter $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

3.1 Hilbert Spaces with Continuous Point Evaluations

Definition 3.1.1. Let X be a set.

(a) Consider a Hilbert space $\mathcal{H}$ which is contained in the space $\mathbb{K}^X$ of $\mathbb{K}$-valued functions on X. We say that $\mathcal{H}$ has *continuous point evaluations* if for each $x \in X$ the linear functional

$$\mathrm{ev}_x \colon \mathcal{H} \to \mathbb{K}, \quad f \mapsto f(x)$$

is continuous. In view of the Fréchet–Riesz Theorem, this implies the existence of some $K_x \in \mathcal{H}$ with

$$f(x) = \langle f, K_x \rangle \quad \text{for} \quad f \in \mathcal{H}, x \in X.$$

The corresponding function

$$K \colon X \times X \to \mathbb{C}, \quad K(x,y) := K_y(x)$$

is called the *reproducing kernel of $\mathcal{H}$*. As we shall see below, $\mathcal{H}$ is uniquely determined by K, so that we shall denote it by $\mathcal{H}_K$ to emphasize this fact.

(b) A function $K \colon X \times X \to \mathbb{K}$ is called a *positive definite kernel* if for each finite subset $\{x_1, \ldots, x_n\} \subseteq X$ the matrix $\big(K(x_i, x_j)\big)_{i,j=1,\ldots,n}$ is positive semidefinite. For a function $K \colon X \times X \to \mathbb{C}$ we write $K^*(x,y) := \overline{K(y,x)}$ and say that K is *hermitian* (or *symmetric* for $\mathbb{K} = \mathbb{R}$) if $K^* = K$.

We write $\mathcal{P}(X, \mathbb{K})$ for the set of positive definite kernels on the set X.

Remark 3.1.2. (a) Over $\mathbb{K} = \mathbb{C}$, the positive definiteness of a kernel K already follows from the requirement that for all choices $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{C}$ we have

$$\sum_{j,k=1}^{n} c_j \overline{c_k} K(x_j, x_k) \geq 0$$

because this implies that K is hermitian (Exercise 3.1.1).

For $\mathbb{K} = \mathbb{R}$, the requirement of the kernel to be hermitian is not redundant. Indeed, the matrix

$$(K_{ij})_{i,j=1,2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

considered as a kernel on the two element set $X = \{1, 2\}$, satisfies

$$\sum_{j,k=1}^{2} c_j c_k K(x_j, x_k) = 0$$

for $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{R}$, but K is not hermitian.

(b) For any positive definite kernel $K \in \mathcal{P}(X)$ and $x, y \in \mathcal{P}(X)$, the positive definiteness of the hermitian matrix

$$\begin{pmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{pmatrix}$$

implies in particular that

$$|K(x,y)|^2 \leq K(x,x) K(y,y) \tag{3.1}$$

In the following we call a subset S of a Hilbert space $\mathcal{H}$ *total* if it spans a dense subspace.

Theorem 3.1.3. (Characterization Theorem) *The following assertions hold for a function $K\colon X \times X \to \mathbb{K}$:*

(a) *If K is the reproducing kernel of $\mathcal{H} \subseteq \mathbb{K}^X$, then the following assertions hold:*

  (1) *K is positive definite.*

  (2) *$\{K_x\colon x \in X\}$ is total in $\mathcal{H}$.*

  (3) *For any orthonormal basis $(e_j)_{j\in J}$, we have $K(x,y) = \sum_{j\in J} e_j(x)\overline{e_j(y)}$.*

(b) *If K is positive definite, then $\mathcal{H}_K^0 := \mathrm{span}\{K_x\colon x \in X\} \subseteq \mathbb{K}^X$ carries a unique positive definite hermitian form satisfying*

$$\langle K_y, K_x\rangle = K(x,y) \quad \text{for} \quad x, y \in X. \tag{3.2}$$

The completion $\mathcal{H}_K$ of $\mathcal{H}_K^0$ permits an injection

$$\iota\colon \mathcal{H}_K \to \mathbb{K}^X, \quad \iota(v)(x) := \langle v, K_x\rangle$$

whose image is a Hilbert space with reproducing kernel K that we identify with $\mathcal{H}_K$.

(c) *K is positive definite if and only if there exists a Hilbert space $\mathcal{H} \subseteq \mathbb{K}^X$ with reproducing kernel K.*

Proof. (a)(1) That K is hermitian follows from

$$K(y,x) = K_x(y) = \langle K_x, K_y\rangle = \overline{\langle K_y, K_x\rangle} = \overline{K(x,y)}.$$

For $c \in \mathbb{K}^n$ we further have

$$\sum_{j,k} \overline{c_j} c_k K(x_j, x_k) = \sum_{j,k} \overline{c_j} c_k \langle K_{x_k}, K_{x_j}\rangle = \|\sum_k c_k K_{x_k}\|^2 \geq 0.$$

This proves (1).

(2) If $f \in \mathcal{H}$ is orthogonal to each K_x, then $f(x) = 0$ for each $x \in X$ implies $f = 0$. Therefore $\{K_x\colon x \in X\}$ spans a dense subspace.

(3) If $(e_j)_{j\in J}$ is an ONB of $\mathcal{H}$, then we have for each $y \in X$ the relation

$$K_y = \sum_{j\in J} \langle K_y, e_j\rangle e_j = \sum_{j\in J} \overline{e_j(y)} e_j,$$

and therefore

$$K(x,y) = K_y(x) = \sum_{j\in J} \overline{e_j(y)} e_j(x).$$

(b) We want to put

$$\Big\langle \sum_j c_j K_{x_j}, \sum_k d_k K_{x_k} \Big\rangle := \sum_{j,k} c_j \overline{d_k} K(x_k, x_j), \tag{3.3}$$

so that we have to show that this is well-defined.

So let $f = \sum_j c_j K_{x_j}$ and $h = \sum_k d_k K_{x_k} \in \mathcal{H}_K^0$. Then we obtain for the right hand side

$$\sum_{j,k} c_j \overline{d_k} K(x_k, x_j) = \sum_{j,k} c_j \overline{d_k} K_{x_j}(x_k) = \sum_k \overline{d_k} f(x_k). \tag{3.4}$$

This expression does not depend on the representation of f as a linear combination of the K_{x_j}. Similarly, we see that the right hand side does not depend on the representation of h as a linear combination of the K_{x_k}. Therefore

$$\langle f, h \rangle := \sum_{j,k} c_j \overline{d_k} K(x_k, x_j)$$

is well-defined. Since K is positive definite, we thus obtain a positive semidefinite hermitian form on $\mathcal{H}_K^0$. From (3.4) we obtain for $h = K_x$ the relation

$$\langle f, K_x \rangle = f(x) \quad \text{for} \quad x \in X, f \in \mathcal{H}_K^0.$$

If $\langle f, f \rangle = 0$, then the Cauchy–Schwarz inequality yields

$$|f(x)|^2 = |\langle f, K_x \rangle|^2 \leq K(x,x)\langle f, f \rangle = 0,$$

so that $f = 0$. Therefore $\mathcal{H}_K^0$ is a pre-Hilbert space.

Now let $\mathcal{H}_K$ be the completion of $\mathcal{H}_K^0$. Then

$$\iota\colon \mathcal{H}_K \to \mathbb{K}^X, \quad \iota(v)(x) := \langle v, K_x \rangle$$

is an injective linear map because the set $\{K_x \colon x \in X\}$ is total in $\mathcal{H}_K^0$, hence also in $\mathcal{H}_K$. Now $\mathcal{H}_K \cong \iota(\mathcal{H}_K) \subseteq \mathbb{K}^X$ is a Hilbert space with continuous point evaluations and reproducing kernel K.

(c) follows (a) and (b). □

Lemma 3.1.4. (Uniqueness Lemma for Reproducing Kernel Spaces) *If $\mathcal{H} \subseteq \mathbb{K}^X$ is a Hilbert space with continuous point evaluations and reproducing kernel K, then $\mathcal{H} = \mathcal{H}_K$.*

Proof. Since K is the reproducing kernel of $\mathcal{H}$, it contains the subspace $\mathcal{H}_K^0 := \operatorname{span}\{K_x \colon x \in X\}$ of $\mathcal{H}_K$, and the inclusion $\eta\colon \mathcal{H}_K^0 \to \mathcal{H}$ is isometric because the scalar products on the pairs (K_x, K_y) coincide. Now η extends to an isometric embedding $\widehat{\eta}\colon \mathcal{H}_K \to \mathcal{H}$, and since $\mathcal{H}_K^0$ is also dense in $\mathcal{H}$, we see that $\widehat{\eta}$ is surjective. For $f \in \mathcal{H}_K$ we now have

$$\widehat{\eta}(f)(x) = \langle \widehat{\eta}(f), K_x \rangle_{\mathcal{H}} = \langle f, K_x \rangle_{\mathcal{H}_K} = f(x),$$

so that $\widehat{\eta}(f) = f$, and we conclude that $\mathcal{H}_K = \mathcal{H}$. □

Definition 3.1.5. The preceding lemma justifies the notation $\mathcal{H}_K$ for the unique Hilbert subspace of $\mathbb{K}^X$ with continuous point evaluations and reproducing kernel K. We call it the reproducing kernel Hilbert space defined by K.

Lemma 3.1.6. *If $\mathcal{H}_K \subseteq \mathbb{K}^X$ is a reproducing kernel space and $S \subseteq X$ a subset with*

$$K(x,x) \leq C \quad \textit{for} \quad x \in S,$$

then

$$|f(x)| \leq \sqrt{C}\|f\| \quad \textit{for} \quad x \in S, f \in \mathcal{H}_K.$$

In particular, convergence in $\mathcal{H}_K$ implies uniform convergence on S.

Proof. For $f \in \mathcal{H}_K$ and $x \in S$, we have

$$|f(x)| = |\langle f, K_x\rangle| \leq \|f\| \cdot \|K_x\| = \|f\|\sqrt{\langle K_x, K_x\rangle} = \|f\|\sqrt{K(x,x)} \leq \sqrt{C}\|f\|.$$

□

Exercises for Section 3.1

Exercise 3.1.1. Show that, if $A \in M_2(\mathbb{C})$ satisfies

$$c^*Ac \geq 0 \quad \text{for} \quad c \in \mathbb{C}^2,$$

then $A^* = A$.

Exercise 3.1.2. Let X be a non-empty set and $T \subseteq X \times X$ be a subset containing the diagonal. Then the characteristic function χ_T of T is a positive definite kernel if and only if T is an equivalence relation.

Exercise 3.1.3. Show that if K is a positive definite kernel and $c > 0$, then $\mathcal{H}_{cK} = \mathcal{H}_K$ as subspaces of $\mathbb{K}^X$. Explain how their scalar products are related.

3.2 Basic Properties of Positive Definite Kernels

The key advantage of Hilbert spaces with continuous point evaluations is that they can be completely encoded in the function K, which is a much less complex object than an infinite dimensional Hilbert space. Before we discuss some important examples of positive definite kernels, we take a closer look at the closure properties of the set $\mathcal{P}(X)$ of all positive definite kernels under several operations.

Proposition 3.2.1. (Permanence properties of positive definite kernels) *The set $\mathcal{P}(X)$ of positive definite kernels on $X \times X$ has the following properties:*

(a) *$\mathcal{P}(X)$ is a convex cone in $\mathbb{K}^{X\times X}$, i.e., $K, Q \in \mathcal{P}(X)$ and $\lambda \in \mathbb{R}_+$ imply*

$$K + Q \in \mathcal{P}(X) \quad \textit{and} \quad \lambda K \in \mathcal{P}(X).$$

(b) *The cone $\mathcal{P}(X)$ is closed under pointwise limits. In particular, if $(K_j)_{j\in J}$ is a family of positive definite kernels on X and all sums $K(x,y) := \sum_{j\in J} K_j(x,y)$ exist, then K is also positive definite.*

(c) *If μ is a positive measure on $(J,\mathfrak{S})$ and $(K_j)_{j\in J}$ is a family of positive definite kernels such that for $x,y \in X$ the functions $j \to K_j(x,y)$ are measurable and the functions $j \mapsto K_j(x,x)$ are integrable, then*

$$K(x,y) := \int_J K_j(x,y)\, d\mu(j)$$

is also positive definite.

(d) (Schur) *$\mathcal{P}(X)$ is closed under pointwise multiplication: If $K, Q \in \mathcal{P}(X)$, then the kernel*

$$(KQ)(x,y) := K(x,y)Q(x,y)$$

is also positive definite.

(e) *If $K \in \mathcal{P}(X)$, then $\overline{K}$ and $\operatorname{Re} K \in \mathcal{P}(X)$.*

Proof. A hermitian kernel K is positive definite if

$$S(K) := \sum_{j,k=1}^{n} K(x_i,x_j)c_i\overline{c_j} \geq 0$$

holds for $x_1,\ldots,x_n \in X$ and $c_1,\ldots,c_n \in \mathbb{K}$.

(a) follows from $S(K+Q) = S(K) + S(Q)$ and $S(\lambda K) = \lambda S(K)$.

(b) follows from $S(K_j) \to S(K)$ if $K_j \to K$ holds pointwise on $X \times X$ and the fact that the set of positive semidefinite $(n\times n)$-matrices is closed. For $K = \sum_j K_j$ we likewise have $S(K) = \sum_{j\in J} S(K_j) \geq 0$.

(c) To see that the functions $j \mapsto K_j(x,y)$ are integrable, we first observe that the positive definiteness of the kernels K_j implies that

$$|K_j(x,y)| \leq \sqrt{K_j(x,x)}\sqrt{K_j(y,y)}$$

(Remark 3.1.2) and since the functions $j \mapsto \sqrt{K_j(x,x)}$ are square integrable by assumption, the product $\sqrt{K_j(x,x)}\sqrt{K_j(y,y)}$ is integrable. Now the assertion follows from $S(K) = \int_J S(K_j)\, d\mu(j) \geq 0$, because μ is a positive measure.

(d) We have to show that the pointwise product $C = (a_{ij}b_{ij})$ of two positive semidefinite matrices A and B is positive semidefinite.

On the Hilbert space $\mathcal{H} := \mathbb{K}^n$, the operator defined by B is orthogonally diagonalizable with non-negative eigenvalues. Let $f_1,\ldots,f_n$ be an ONB of eigenvalues for B and $\lambda_1,\ldots,\lambda_n$ be the corresponding eigenvalues. Then

$$Bv = \sum_{j=1}^{n} \langle v, f_j\rangle Bf_j = \sum_{j=1}^{n} \lambda_j \langle v, f_j\rangle f_j = \sum_{j=1}^{n} \lambda_j f_j^* v \cdot f_j = \sum_{j=1}^{n} \lambda_j f_j f_j^* v$$

(where we use matrix products) implies $B = \sum_j \lambda_j f_j f_j^*$, and since the λ_j are non-negative, it suffices to prove the assertion for the special case $B = vv^*$ for some $v \in \mathbb{K}^n$, i.e., $b_{ij} = v_i \overline{v_j}$. Then we obtain for $d \in \mathbb{K}^n$

$$\sum_{i,j} d_i \overline{d_j} c_{ij} = \sum_{i,j} d_i \overline{d_j} v_i \overline{v}_j a_{ij} = \sum_{i,j} (d_i v_i) \overline{d_j v_j} a_{ij} \geq 0,$$

and thus C is positive semidefinite.

(e) Since K is hermitian, we have $\overline{K}(x,y) = K(y,x)$, and this kernel is positive definite. In view of (a), this implies that $\operatorname{Re} K = \frac{1}{2}(K + \overline{K})$ is also positive definite. □

Corollary 3.2.2. *If $f(z) := \sum_{n=0}^\infty a_n z^n$ is a power series with $a_n \geq 0$ converging for $|z| < r$ and $K \in \mathcal{P}(X)$ is a positive definite kernel with $|K(x,y)| < r$ for $x, y \in X$, then the kernel*

$$(f \circ K)(x,y) := f(K(x,y)) = \sum_{n=0}^\infty a_n K(x,y)^n$$

is positive definite.

Proof. This follows from Proposition 3.2.1(b) because Proposition 3.2.1(d) implies that the kernels $K(x,y)^n$ are positive definite. □

3.3 Realizing Hilbert Spaces as Reproducing Kernel Spaces

At this point we know how to get new positive definite kernels from given ones, but we should also have a more effective means to recognize positive definite kernels quickly.

Remark 3.3.1. (a) For any map $\gamma \colon X \to \mathcal{H}$ of a set X into a Hilbert space $\mathcal{H}$, the kernel $K_\gamma(x,y) := \langle \gamma(y), \gamma(x) \rangle$ is positive definite because it clearly is hermitian, and for $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{K}$, we have

$$\sum_{i,j=1}^n c_j \overline{c_k} K_\gamma(x_j, x_k) = \sum_{i,j=1}^n c_j \overline{c_k} \langle \gamma(x_k), \gamma(x_j) \rangle = \| \sum_{i=1}^n \overline{c_k} \gamma(x_k) \|^2 \geq 0.$$

(b) For $K \in \mathcal{P}(X)$ and $f \colon X \to \mathbb{C}$, the kernel

$$Q(x,y) := f(x) K(x,y) \overline{f(y)}$$

is also positive definite. In fact, it is the product of K with the kernel $f(x)\overline{f(y)}$ whose positive definiteness follows from (a), applied to the function

$$\gamma = f \colon X \to \mathbb{C} = \mathcal{H}$$

and Proposition 3.2.1(d).

(c) If $K \in \mathcal{P}(X)$ and $\varphi\colon Y \to X$ is a function, then the kernel

$$\varphi^* K\colon Y \times Y \to \mathbb{C}, \quad (x,y) \mapsto K(\varphi(x), \varphi(y))$$

is also positive definite. This is a direct consequence of the definitions.

Definition 3.3.2. Let $\mathcal{H}$ be a Hilbert space. A triple $(X, \gamma, \mathcal{H})$ consisting of a set X and a map $\gamma\colon X \to \mathcal{H}$ is called a *realization triple* if $\gamma(X)$ spans a dense subspace of $\mathcal{H}$. Then $K(x,y) := \langle \gamma(y), \gamma(x)\rangle$ is called the corresponding positive definite kernel.

Theorem 3.3.3. (Realization Theorem for Positive Definite Kernels) *For each positive definite kernel K on X, there exists a realization triple $(X, \gamma, \mathcal{H})$ with reproducing kernel K. For any other such triple $(X, \gamma', \mathcal{H}')$, there exists a unique isometry $\varphi\colon \mathcal{H} \to \mathcal{H}'$ with $\varphi \circ \gamma = \gamma'$.*

Proof. **Existence:** Let $\mathcal{H}_K \subseteq \mathbb{K}^X$ be the reproducing kernel Hilbert space with kernel K (Theorem 3.1.3). Then $\gamma\colon X \to \mathcal{H}_K, \gamma(x) := K_x$ satisfies

$$\langle \gamma(y), \gamma(x)\rangle = \langle K_y, K_x\rangle = K_y(x) = K(x,y).$$

Uniqueness: Let $c_1, \ldots, c_n$ and $x_1, \ldots, x_n \in X$. Then

$$\begin{aligned}\Big\| \sum_{i=1}^n c_j \gamma'(x_j)\Big\|^2 &= \sum_{j,k=1}^n c_j \overline{c_k} \langle \gamma'(x_j), \gamma'(x_k)\rangle \\ &= \sum_{j,k=1}^n c_j \overline{c_k} K(x_k, x_j) = \Big\| \sum_{i=1}^n c_j \gamma(x_j)\Big\|^2.\end{aligned}$$

We may therefore define a linear map

$$\varphi : \operatorname{span} \gamma(X) \to \mathcal{H}', \quad \varphi\Big(\sum_{i=1}^n c_i \gamma(x_i)\Big) := \sum_{i=1}^n c_i \gamma'(x_i).$$

As the preceding calculation shows, φ is isometric, hence extends to an isometry $\varphi\colon \mathcal{H} \to \mathcal{H}'$, and by definition it satisfies $\varphi \circ \gamma = \gamma'$. That φ is surjective follows from the density of $\operatorname{span}(\gamma'(X))$ in $\mathcal{H}'$ and the uniqueness of φ follows from the density of $\operatorname{span}(\gamma(X))$ in $\mathcal{H}$. $\square$

Definition 3.3.4. (a) For a positive definite kernel $K\colon X \times X \to \mathbb{C}$, the realization triple $(X, \gamma, \mathcal{H}_K)$ with $\gamma(x) = K_x$, used in the previous proof, is called the *canonical realization triple*.

(b) For a realization triple $(X, \gamma, \mathcal{H})$, the corresponding unitary map

$$\varphi_\gamma\colon \mathcal{H} \to \mathcal{H}_K \quad \text{with} \quad \varphi_\gamma(\gamma(x)) = K_x$$

satisfies

$$\varphi_\gamma(v)(x) = \langle \varphi_\gamma(v), K_x\rangle = \langle v, \gamma(x)\rangle.$$

The isomorphism φ_γ is called a *realization of $\mathcal{H}$ as a reproducing kernel space.*

Proposition 3.3.5. *Let K be a continuous positive definite kernel on the topological space X. Then the map $\gamma\colon X \to \mathcal{H}_K, x \mapsto K_x$, is continuous and $\mathcal{H}_K$ consists of continuous functions.*

Proof. The continuity of γ follows from the continuity of

$$\|K_x - K_y\|^2 = K(x,x) + K(y,y) - K(x,y) - K(y,x).$$

Since the scalar product is a continuous function $\mathcal{H} \times \mathcal{H} \to \mathbb{K}$, the continuity of each $f \in \mathcal{H}_K$ now follows from $f(x) = \langle f, K_x \rangle = \langle f, \gamma(x) \rangle$ and the continuity of γ. $\square$

Examples 3.3.6. (a) If $\mathcal{H}$ is a Hilbert space, then the kernel

$$K(x,y) := \langle y, x \rangle$$

on $\mathcal{H}$ is positive definite (Remark 3.3.1(a)). A corresponding realization is given by the map $\gamma = \mathrm{id}_{\mathcal{H}}$. In particular, $\mathcal{H} \cong \mathcal{H}_K \subseteq \mathbb{K}^{\mathcal{H}}$.

(b) The kernel $K(x,y) := \langle x, y \rangle = \overline{\langle y, x \rangle}$ is also positive definite (Proposition 3.2.1(e)). To identify the corresponding Hilbert space, we consider the dual space $\mathcal{H}'$ of continuous linear functionals on $\mathcal{H}$. According to the Fréchet–Riesz Theorem, every element of $\mathcal{H}'$ has the form $\gamma_v(x) := \langle x, v \rangle$ for a uniquely determined $v \in \mathcal{H}$, and the map

$$\gamma\colon \mathcal{H} \to \mathcal{H}', \quad v \mapsto \gamma_v$$

is an antilinear isometry. In particular, $\mathcal{H}'$ also is a Hilbert space, and the scalar product on $\mathcal{H}'$ (which is determined uniquely by the norm via polarization) is given by

$$\langle \gamma_y, \gamma_x \rangle := \langle x, y \rangle = K(x,y).$$

Therefore $\gamma\colon \mathcal{H} \to \mathcal{H}'$ yields a realization of the kernel K, which leads to $\mathcal{H}_K \cong \mathcal{H}'$.

(c) If $(e_j)_{j\in J}$ is an orthonormal basis in $\mathcal{H}$, then the map

$$\gamma\colon J \to \mathcal{H}, \quad j \mapsto e_j$$

has total range, and $K(i,j) := \delta_{ij} = \langle e_i, e_j \rangle$ is the corresponding positive definite kernel on J. The element $v \in \mathcal{H}$ then corresponds to the function

$$\varphi_\gamma(v)\colon J \to \mathbb{K}, \quad j \mapsto \langle v, e_j \rangle$$

of its coefficients in the expansion $v = \sum_{j\in J} \langle v, e_j \rangle e_j$, and the map

$$\varphi_\gamma\colon \mathcal{H} \to \ell^2(J, \mathbb{K}), \quad v \mapsto (\langle v, e_j \rangle)_{j\in J}$$

is an isomorphism of Hilbert spaces. We conclude that $\mathcal{H}_K \cong \ell^2(J, \mathbb{K}) \subseteq \mathbb{K}^J$ is the corresponding reproducing kernel space.

(d) Let $(X, \mathfrak{S}, \mu)$ be a finite measure space and $\mathcal{H} = L^2(X, \mu)$. Then the map

$$\gamma\colon \mathfrak{S} \to L^2(X,\mu), \quad E \mapsto \chi_E$$

has total range because the step functions form a dense subspace of $L^2(X, \mu)$. We thus obtain a realization

$$\varphi_\gamma\colon L^2(X,\mu) \to \mathcal{H}_K \subseteq \mathbb{C}^{\mathfrak{S}}, \quad \varphi_\gamma(f)(E) = \langle f, \chi_E\rangle = \int_E f\, d\mu,$$

of $L^2(X, \mu)$ as a reproducing kernel space on $\mathfrak{S}$ whose kernel is given by

$$K(E, F) = \langle \chi_F, \chi_E\rangle = \mu(E \cap F).$$

(e) If $\mathcal{H}$ is a complex Hilbert space, then the kernel $K(z, w) := e^{\langle z,w\rangle}$ is also positive definite (Corollary 3.2.2. The corresponding Hilbert space $\mathcal{H}_K \subseteq \mathbb{C}^{\mathcal{H}}$ is called the *(symmetric) Fock space* $\mathcal{F}(\mathcal{H})$ *of* $\mathcal{H}$. As we shall see below, it plays an important role in representations theory, and in particular in mathematical physics.

We also note that the same argument shows that for each $\lambda \geq 0$, the kernel $e^{\lambda\langle z,w\rangle}$ is positive definite.

(f) Let $\mathcal{H}$ be a Hilbert space $\mathcal{D} := \{z \in \mathcal{H}\colon \|z\| < 1\}$ be the open unit ball. For each $s \geq 0$, we find with Corollary 3.2.2 that the kernel

$$\begin{aligned} K(z, w) := (1 - \langle z, w\rangle)^{-s} &= \sum_{n=0}^\infty \binom{-s}{n}(-1)^n\langle z, w\rangle^n \\ &= \sum_{n=0}^\infty \frac{(-s)(-s-1)\cdots(-s-n+1)}{n!}(-1)^n\langle z, w\rangle^n \\ &= \sum_{n=0}^\infty \frac{s(s+1)\cdots(s+n-1)}{n!}\langle z, w\rangle^n \end{aligned}$$

is positive definite.

We shall see below how these kernels can be used to obtain interesting unitary representations of certain Lie groups.

Remark 3.3.7. (Extending $\gamma\colon X \to \mathcal{H}$ to measures) Let $(X, \mathfrak{S})$ be a measurable space and $K \in \mathcal{P}(X)$ be a positive definite measurable kernel. Let μ be a measure on $(X, \mathfrak{S})$ satisfying

$$c_K := \int_X \sqrt{K(x,x)}\, d\mu(x) < \infty. \tag{3.5}$$

Each $f \in \mathcal{H}_K$ satisfies

$$|f(x)| \leq \|f\| \cdot \|K_x\| = \|f\|\sqrt{K(x,x)}.$$

We also observe that the measurability of all functions K_x implies the measurability of all functions in $\mathcal{H}_K$ because they are pointwise limits of sequence of linear combinations of the K_x (cf. Theorem 3.1.3). We conclude that $f \in L^1(X, \mu)$ for each $f \in \mathcal{H}_K$, and that we have an estimate

$$\Big| \int_X f(x)\, d\mu(x) \Big| \leq c_K \|f\|.$$

Therefore integration defines a continuous linear functional on $\mathcal{H}_K$, and we thus obtain a uniquely determined element $K_\mu \in \mathcal{H}_K$ with

$$\int_X f(x)\, d\mu(x) = \langle f, K_\mu \rangle \quad \text{for} \quad f \in \mathcal{H}_K.$$

This relation can be rewritten as $\langle f, K_\mu \rangle = \int_X \langle f, K_y \rangle\, d\mu(y)$, so that

$$K_\mu = \int_X K_y\, d\mu(y)$$

in the sense of weak integrals. Evaluating this relation in $x \in X$, leads to

$$K_\mu(x) = \int_X K_y(x)\, d\mu(y) = \int_X K(x, y)\, d\mu(y).$$

We further obtain

$$0 \leq \langle K_\mu, K_\mu \rangle = \int_X K_\mu(y)\, d\mu(y) = \int_X \int_X K(y, x)\, d\mu(x)\, d\mu(y).$$

Note that for the point measure δ_x in X we have $K_{\delta_x} = K_x$, so that the assignment $\mu \mapsto K_\mu$ can be considered as an extension of the map $\gamma_K \colon X \to \mathcal{H}_K, x \mapsto K_x$, to a certain set of measures μ, restricted by the relation (3.5).

If μ is a Radon measure on a locally compact space, K is continuous and $f \in C_c(X)$, then

$$\int_X |f(x)| \sqrt{K(x, x)}\, d\mu(x) < \infty,$$

so that we obtain in particular the relation

$$\int_X \int_X f(x) \overline{f(y)} K(x, y)\, d\mu(x)\, d\mu(y) \geq 0, \tag{3.6}$$

which shows that

$$\langle f, h \rangle := \int_X \int_X f(x) \overline{h(y)} K(x, y)\, d\mu(x)\, d\mu(y) \geq 0$$

defines a positive definite hermitian form on $C_c(X)$.

Exercises for Section 3.3

Exercise 3.3.1. Let $\mathcal{H}_K \subseteq \mathbb{K}^X$ be a reproducing kernel Hilbert space and $\mathcal{H}_K = \widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ be a direct Hilbert space sum. Show that there exist positive definite kernels $K^j \in \mathcal{P}(X)$ with $K = \sum_{j\in J} K^j$ and $\mathcal{H}_j = \mathcal{H}_{K^j}$ for $j \in J$. Hint: Consider $\mathcal{H}_j$ as a Hilbert space with continuous point evaluations and let K^j be its reproducing kernel.

Exercise 3.3.2. Let $X = [a,b]$ be a compact interval in $\mathbb{R}$ and $K\colon [a,b]^2 \to \mathbb{C}$ be a continuous function. Then K is positive definite if and only if

$$\int_a^b \int_a^b c(x)\overline{c(y)}K(x,y)\,dx\,dy \geq 0 \quad \text{for each} \quad c \in C([a,b],\mathbb{C}).$$

Exercise 3.3.3. Show that for $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$ and $z \in \mathbb{C}$, the following integral exists and verify the formula:

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{xz - \frac{ax^2}{2}}\,dx = \frac{1}{\sqrt{a}} e^{\frac{z^2}{2a}},$$

where $\sqrt{a}$ refers to the canonical branch of the square root on the right half plane with $\sqrt{1} = 1$. Hint: Assume first that $a, z \in \mathbb{R}$. Then use a dominated convergence argument to verify that the integral depends holomorphically on z and a.

Exercise 3.3.4. Fix $a > 0$ and define $\gamma\colon \mathbb{C} \to L^2(\mathbb{R}, dx)$ by $\gamma(z) := e^{xz - \frac{ax^2}{2}}$. Show that

$$\langle \gamma(z), \gamma(w)\rangle = \sqrt{\frac{\pi}{a}} e^{\frac{(z+\overline{w})^2}{4a}}$$

and that $\gamma(\mathbb{C})$ is total in $L^2(\mathbb{R})$. Use this to derive an isomorphism φ_γ of $L^2(\mathbb{R}, dx)$ with a reproducing kernel space of holomorphic functions on $\mathbb{C}$.

Exercise 3.3.5. Let $(X, \mathfrak{S}, \mu)$ be a probability space. Show that on $X = \mathfrak{S}$, the kernel

$$K(E,F) := \mu(E \cap F) - \mu(E)\mu(F)$$

is positive definite. Hint: Consider the hyperplane $\{f \in L^2(X,\mu)\colon \int_X f\,d\mu = 0\}$.

Exercise 3.3.6. Show that on $X := [0,1]$, the kernel $K(x,y) := \min(x,y) - xy$ is positive definite.

Exercise 3.3.7. On the interval $[0,1] \subseteq \mathbb{R}$, we consider $\mathcal{H} = L^2([0,1], dx)$ and the map

$$\gamma\colon [0,1] \to \mathcal{H}, \quad \gamma(x) := \chi_{[0,x]}.$$

Show that:

(a) $K(x,y) := \langle \gamma(y), \gamma(x)\rangle = \min(x,y)$.

(b) $\mathrm{im}(\gamma)$ is total in $\mathcal{H}$. Hint: The subspace spanned by $\mathrm{im}(\gamma)$ contains all Riemannian step functions (those corresponding to finite partitions of $[0,1]$ into subintervals). From this one derives that its closure contains all continuous functions and then use Proposition 2.3.6.

(c) The reproducing kernel space $\mathcal{H}_K$ consists of continuous functions and

$$\varphi_\gamma \colon L^2([0,1]) \to \mathcal{H}_K, \quad \varphi_\gamma(f)(x) := \int_0^x f(t)\,dt.$$

The space $\mathcal{H}_K$ is also denoted $H^1_*([0,1])$. It is the Sobolev space of all continuous functions on $[0,1]$, vanishing in 0 whose derivatives are L^2-functions.

Exercise 3.3.8. Show that on $X :=]0,\infty[$ the kernel $K(x,y) := \frac{1}{x+y}$ is positive definite. Hint: Consider the elements $e_\lambda(x) := e^{-\lambda x}$ in $L^2(\mathbb{R}_+, dx)$.

Exercise 3.3.9. Let X be a topological space and $Y \subseteq X$ be a dense subspace. Show that, if $K \colon X \times X \to \mathbb{C}$ is a continuous positive definite kernel, then the restriction map

$$r \colon \mathcal{H}_K \to \mathbb{C}^Y, \quad f \mapsto f|_Y$$

induced a unitary isomorphism onto the reproducing kernel space $\mathcal{H}_Q$ with $Q := K|_{Y\times Y}$.

Exercise 3.3.10. Let X be a set and $K \in \mathcal{P}(X,\mathbb{C})$ be a positive definite kernel. Show that

(a) $\mathcal{H}_{\overline{K}} = \overline{\mathcal{H}_K}$ and that the map $\sigma \colon \mathcal{H}_K \to \mathcal{H}_{\overline{K}}, f \mapsto \overline{f}$ is anti-unitary.

(b) The map $\sigma(f) = \overline{f}$ preserves $\overline{\mathcal{H}_K} = \mathcal{H}_K$ and acts isometrically on this space of and only if K is real-valued.

3.4 Inclusions of Reproducing Kernel Spaces

In this subsection, we discuss the order structure on the cone $\mathcal{P}(X) = \mathcal{P}(X,\mathbb{C})$ of complex-valued positive definite kernels on the set X and relate it to the structure of the corresponding Hilbert spaces.

Definition 3.4.1. Let $K \in \mathcal{P}(X)$ and $A \in B(\mathcal{H}_K)$. We define the *symbol of A* as the kernel

$$K^A \colon X \times X \to \mathbb{C}, \quad K^A(x,y) := \langle AK_y, K_x\rangle.$$

Note that $K^{\mathbf{1}} = K$ for the identity operator $\mathbf{1}$ on $\mathcal{H}_K$. Since the K_x, $x \in X$, form a total subset of X, A is uniquely determined by its symbol.

Lemma 3.4.2. *The assignment $B(\mathcal{H}) \mapsto B(V)^{X\times X}$, $A \mapsto K^A$ has the following properties:*

(i) $(K^A)^* = K^{A^*}$ *and* K^A *is hermitian if and only if* A *is hermitian.*

(ii) K^A *is positive definite if and only if* A *is positive.*

Proof. For $x, y \in X$, the relation

$$K^A(y,x) = \langle AK_x, K_y\rangle = \langle K_x, A^*K_y\rangle = \overline{K^{A^*}(x,y)}$$

implies that $(K^A)^* = K^{A^*}$. Since an operator is uniquely determined by its symbol, (i) follows.

(ii) In view of (i), we may assume that A is hermitian. For $c_1, \ldots, c_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in X$ and $w := \sum_j c_j K_{x_j}$, we have

$$\langle Aw, w\rangle = \sum_{j,k=1}^{n} c_j\overline{c_k}\langle AK_{x_j}, K_{x_k}\rangle = \sum_{j,k=1}^{n} c_j\overline{c_k}K^A(x_k, x_j).$$

Therefore A is a positive operator if and only if the kernel K^A is positive definite. $\square$

Example 3.4.3. (a) Let $(e_j)_{j\in J}$ be an orthonormal basis in the Hilbert space $\mathcal{H}$ and $K\colon J \times J \to \mathbb{C}, (i,j) \mapsto \delta_{ij}$ be the reproducing kernel corresponding to the natural realization as functions on J (cf. Example 3.3.6(c)). Then the symbol of $A \in B(\mathcal{H}_K)$ is the function

$$K^A(i,j) = \langle AK_j, K_i\rangle = \langle Ae_j, e_i\rangle =: a_{ij},$$

where $(a_{ij})_{i,j\in J}$ is the matrix of the operator A with respect to the orthonormal basis $(e_j)_{j\in J}$.

(b) We realize the Hilbert space $\mathcal{H}$ as a reproducing kernel space $\mathcal{H}_K$ on $X = \mathcal{H}$ with kernel $K(z,w) = \langle w, z\rangle$ (cf. Example 3.3.6(a)). Then the symbol of an operator $A \in B(\mathcal{H})$ is the kernel $K^A(z,w) = \langle Aw, z\rangle$.

(c) We assume that $(X, \mathfrak{S}, \mu)$ is a measure space and that we have a reproducing kernel space $\mathcal{H}_K \subseteq \mathbb{C}^X$ with an isometric embedding $\mathcal{H}_K \hookrightarrow L^2(X,\mu)$.[1] Since the K_x form a total subset of $\mathcal{H}_K$, this condition is equivalent to $\|K_x\|_2 < \infty$ for each $x \in X$, and the relation

$$K(x,y) = \langle K_y, K_x\rangle = \int_X K_y(z)\overline{K_x(z)}\,d\mu(z) = \int_X K(x,z)K(z,y)\,d\mu(z).$$

For a bounded operator $A \in B(\mathcal{H}_K)$ and $f \in \mathcal{H}_K$, we then have

$$\begin{aligned}(Af)(z) = \langle f, A^*K_z\rangle &= \int_X f(w)\overline{A^*K_z(w)}\ d\mu(w)\\ &= \int_X f(w)\overline{K^{A^*}(w,z)}\ d\mu(w) = \int_X K^A(z,w)f(w)\ d\mu(w).\end{aligned}$$

This means that every bounded operator A on $\mathcal{H}_K$ can be written as a kernel operator with kernel K^A.

[1] The example under (a) is of this type if μ is the counting measure on J and $K(i,j) = \delta_{ij}$.

Definition 3.4.4. (a) If C is a convex cone in the real vector space V, then we define an order on V by $a \ll b$ if $b - a \in C$. We say that b *dominates* a and write $a \prec b$ if $a \ll \lambda b$ for some $\lambda > 0$.

A *face* F *of* C is a convex cone $F \subseteq C$ with the property that

$$x + y \in F, x, y \in C \quad \Rightarrow \quad x, y \in F.$$

This can also be expressed as

$$(F - C) \cap C \subseteq F.$$

Since intersections of families of faces are faces (Exercise 3.4.1), it makes sense to speak of the face generated by a subset of C. The face generated by an element $x \in C$ is the set of all elements of C, dominated by x (Exercise). An *extremal ray* of C is a face of the form $\mathbb{R}^+ x \subseteq C$. That x generates an extremal ray means that $x = y + z$ for $y, z \in C$ implies that $y, z \in \mathbb{R}_+ x$.

(b) In the following, we shall apply all concepts defined under (a) to the convex cone $\mathcal{P}(X)$ of positive definite kernels.

We now relate these notions to the corresponding Hilbert spaces.

Remark 3.4.5. Let $K, Q \in \mathcal{P}(X)$. We describe the space $\mathcal{H}_{K+Q}$. In order to do that, we define a map

$$\gamma \colon X \to \mathcal{H}_K \oplus \mathcal{H}_Q, \quad x \mapsto (K_x, Q_x).$$

Then $\langle \gamma(y), \gamma(x) \rangle = K(x, y) + Q(x, y)$, so that the Realization Theorem 3.3.3 shows that we obtain a realization of the closed subspace of $\mathcal{H}_K \oplus \mathcal{H}_Q$ generated by $\gamma(X)$ as $\mathcal{H}_{K+Q}$. Since

$$\langle (f_1, f_2), \gamma(x) \rangle = f_1(x) + f_2(x),$$

the mapping $\mathcal{H}_K \oplus \mathcal{H}_Q \to \mathcal{H}_{K+Q}, (f_1, f_2) \mapsto f_1 + f_2$ is surjective and an isometry on the closed subspace generated by $\operatorname{im}(\gamma)$. Its kernel is the set of pairs $(f, -f)$, $f \in \mathcal{H}_K \cap \mathcal{H}_Q$. We conclude in particular that

$$\mathcal{H}_{K+Q} = \mathcal{H}_K + \mathcal{H}_Q \cong (\mathcal{H}_K \oplus \mathcal{H}_Q)/(\mathcal{H}_K \cap \mathcal{H}_Q).$$

Lemma 3.4.6. *Let $\mathcal{H}$ be a Hilbert space, $K \in \mathcal{P}(X)$ and $\mathcal{H}_K$ be the corresponding reproducing kernel Hilbert space. Then a linear mapping $A \colon \mathcal{H} \to \mathcal{H}_K$ is continuous if and only if, for each $x \in X$, the map $\mathcal{H} \to \mathbb{C}, v \mapsto (Av)(x)$ is continuous.*

Proof. Since the point evaluations on $\mathcal{H}_K$ are continuous, the continuity of the functions $v \mapsto (Av)(x)$ is clearly necessary for the continuity of A.

Suppose, conversely, that all these functions are continuous. We claim that the graph

$$\Gamma(A) := \{(v, Av) \in \mathcal{H} \times \mathcal{H}_K \colon v \in \mathcal{H}\}$$

is closed. In fact, if $(v_n, Av_n) \to (v, w)$, then we have for each $x \in X$ the relation $(Av_n)(x) \to w(x)$, and also $(Av_n)(x) \to (Av)(x)$, which leads to $Av = w$. Now the continuity of A follows from the Closed Graph Theorem. $\square$

Theorem 3.4.7. *For $L, K \in \mathcal{P}(X)$, the following are equivalent:*

(1) $L \prec K$, *i.e.,* $\lambda K - L \in \mathcal{P}(X)$ *for some* $\lambda > 0$.

(2) $\mathcal{H}_L \subseteq \mathcal{H}_K$.

(3) *There exists a positive operator* $B \in B(\mathcal{H}_K)$ *with* $K^B = L$.

Proof. (1) $\Rightarrow$ (2): If $L \prec K$, then we find a $\lambda > 0$ with $K' := \lambda K - L \in \mathcal{P}(X)$. Then $\lambda K = L + K'$, and therefore $\mathcal{H}_K = \mathcal{H}_{\lambda K} = \mathcal{H}_L + \mathcal{H}_{K'}$ (Remark 3.4.5) implies $\mathcal{H}_L \subseteq \mathcal{H}_K$.

(2) $\Rightarrow$ (3): We claim that the embedding $A\colon \mathcal{H}_L \to \mathcal{H}_K$ is continuous with $L = K^{AA^*}$. For each $x \in X$, the mapping $\mathcal{H}_L \to \mathbb{C}, f \mapsto (Af)(x) = f(x)$ is continuous so that the continuity of A follows from Lemma 3.4.6. The definition of A implies that

$$\langle f, L_x\rangle = f(x) = (Af)(x) = \langle Af, K_x\rangle = \langle f, A^*K_x\rangle$$

for each $f \in \mathcal{H}_L$, and therefore $L_x = A^*K_x$. This in turn leads to

$$L(x,y) = \langle L_y, L_x\rangle = \langle A^*K_y, A^*K_x\rangle = \langle AA^*K_y, K_x\rangle = K^{AA^*}(x,y).$$

(3) $\Rightarrow$ (1): The operator $C := \|B\|\mathbf{1} - B$ is positive, so that

$$K^C = \|B\|K^{\mathbf{1}} - K^B = \|B\|K - L \in \mathcal{P}(X)$$

(Lemma 3.4.2). Hence $L \prec K$. $\square$

Remark 3.4.8. Note that Theorem 3.4.7 implies in particular that the mapping $B \mapsto K^B$ yields a linear isomorphism of the convex cone $B(\mathcal{H}_K)^+$ of positive operators on $\mathcal{H}_K$ onto the face of $\mathcal{P}(X)$ generated by K.

Proposition 3.4.9. *For $0 \neq K \in \mathcal{P}(X)$, the following are equivalent:*

(1) $\dim \mathcal{H}_K = 1$.

(2) $\mathbb{R}^+K$ *is an extremal ray in* $\mathcal{P}(X)$.

(3) *There exists a non-zero function* $f \in \mathbb{C}^X$ *with* $K(x,y) = f(x)\overline{f(y)}$ *for all* $x, y \in X$.

Proof. (1) $\Leftrightarrow$ (2) follows from Remark 3.4.8 and the observation that the cone of positive operators in $\mathcal{H}_K$ is one-dimensional if and only if $\mathcal{H}_K$ itself is one dimensional.

(1) $\Rightarrow$ (3): If $\mathcal{H}_K$ is one-dimensional, then let $f \in \mathcal{H}_K$ denote a unit vector. We claim that $K(x,y) = f(x)\overline{f(y)}$ holds for all $x, y \in X$. In fact, we have $K_y = c(y)f$ for some $c(y) \in \mathbb{C}$, so that $\overline{f(x)} = \langle f, K_x\rangle = \langle f, c(x)f\rangle = \overline{c(x)}$ leads to $K(x,y) = K_y(x) = c(y)f(x) = f(x)\overline{f(y)}$.

(3) $\Rightarrow$ (1): If $K(x,y) = f(x)\overline{f(y)}$, then all functions K_y are multiples of f, so that $\mathcal{H}_K = \mathbb{C}f$ is one-dimensional if K is non-zero. $\square$

Remark 3.4.10. We claim that the extremal rays of the cone $B(\mathcal{H})^+$ of positive operators on $\mathcal{H}$ are of the form $\mathbb{R}_+ P_v$, where

$$P_v x = \langle x, v\rangle v \quad \text{for some} \quad 0 \neq v \in \mathcal{H}.$$

To verify this claim, we identify $\mathcal{H}$ with the reproducing kernel space $\mathcal{H}_K \subseteq \mathbb{C}^{\mathcal{H}}$ with $K(z,w) = \langle w, z\rangle$ (Example 3.3.6(a)). In view of Remark 3.4.8, an operator $A \in B(\mathcal{H})^+$ generates an extremal ray if and only if its symbol K^A corresponds to a minimal non-zero subspace $\mathcal{H}_{K^A}$ of $\mathcal{H} = \mathcal{H}_K$. This is clearly equivalent to $\dim \mathcal{H}_{K^A} = 1$, which in turn is equivalent to

$$K^A(z,w) = \langle Aw, z\rangle = f(z)\overline{f(w)}$$

for some $f \in \mathcal{H}_K = \mathcal{H}$ (Proposition 3.4.9). Since each $f \in \mathcal{H}$ can be represented as $f(z) = \langle v, z\rangle$ for some $v \in \mathcal{H}$ (a continuous antilinear functional), we have

$$f(z)f(w) = \langle v, z\rangle\langle w, v\rangle = \langle P_v(w), z\rangle = K^{P_v}(z,w).$$

This proves that $\mathbb{R}_+ A$ is extremal if and only if $A = P_v$ for some non-zero vector $v \in \mathcal{H}$.

Exercises for Section 3.4

Exercise 3.4.1. Let C be a convex cone in a real vector space. Show that for any family $(F_i)_{i\in I}$ of faces of C, the intersection $\bigcap_{i\in I} F_i$ also is a face.

Exercise 3.4.2. Let C be a convex cone in a real vector space and $f\colon V \to \mathbb{R}$ a linear functional with $f(C) \subseteq \mathbb{R}_+$. Show that $\ker f \cap C$ is a face of C.

Exercise 3.4.3. Let C be a convex cone in a topological vector space V. Show that every proper face F of C is contained in the boundary ∂C. Hint: Show that the face generated by any $x \in C^0$ is all of C by showing that $C \subseteq \bigcup_{\lambda>0}(\lambda x - C)$.

Exercise 3.4.4. We have seen in Example 3.3.6 that the L^2-space for a finite measure μ on $(X, \mathfrak{S})$ can be realized as a reproducing kernel space $\mathcal{H}_\mu \subseteq \mathbb{C}^{\mathfrak{S}}$ with kernel $K(E,F) = \mu(E \cap F)$. Show that for two finite positive measures λ and μ on $(X, \mathfrak{S})$, we have

$$\mathcal{H}_\lambda \subseteq \mathcal{H}_\mu \quad \Longleftrightarrow \quad (\exists C > 0)\mu \leq C\lambda.$$

Hint: Use Theorem 3.4.7 to see that for the corresponding kernels K_μ and K_λ we have $K_\lambda(E,E) \leq CK_\mu(E,E)$ for all $E \in \mathfrak{S}$.

Chapter 4

Commutants and Tensor Products

In this chapter we turn to finer information on unitary representations, resp., representations of involutive semigroups. We have already seen in Lemma 1.3.1 that for a representation $(\pi, \mathcal{H})$ of $(S, *)$, a closed subspace $\mathcal{K}$ of $\mathcal{H}$ is invariant if and only if the corresponding orthogonal projection $P_{\mathcal{K}}$ onto $\mathcal{K}$ belongs to the subalgebra

$$B_S(\mathcal{H}) = \{A \in B(\mathcal{H}) \colon (\forall s \in S)\, A\pi(s) = \pi(s)A\}.$$

This algebra is called the *commutant of* $\pi(S)$ and since its hermitian projections are in one-to-one correspondence with the closed invariant subspaces of $\mathcal{H}$, it contains all information on how the representation $(\pi, \mathcal{H})$ decomposes.

A key result in this context is Schur's Lemma, asserting that $\pi(S)' = \mathbb{C}\mathbf{1}$ if and only if $(\pi, \mathcal{H})$ is irreducible. The proof we give is based on Gelfand's Representation Theorem for commutative C^*-algebras, which is explained in Section 4.1. In Section 4.2, Schur's Lemma is used to get complete information on the commutant of any representation generated by irreducible ones. In Section 4.3 we introduce tensor products of unitary representations. They provide in particular a natural way to deal with infinite multiples of an irreducible representation and to determine their commutants in terms of the so-called multiplicity space.

4.1 Commutative C^*-algebras

Let $\mathcal{A}$ be a commutative Banach-$*$-algebra. We write

$$\widehat{\mathcal{A}} := \mathrm{Hom}(\mathcal{A}, \mathbb{C}) \setminus \{0\},$$

where $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ denotes the set of all morphisms of Banach-$*$-algebras, i.e., continuous linear functionals $\chi \colon \mathcal{A} \to \mathbb{C}$ with the additional property that

$$\chi(ab) = \chi(a)\chi(b) \quad \text{and} \quad \chi(a^*) = \overline{\chi(a)} \quad \text{for} \quad a, b \in \mathcal{A}. \tag{4.1}$$

Thinking of $\mathbb{C}$ as a one-dimensional Hilbert space, we have $\mathbb{C} \cong B(\mathbb{C})$, so that $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ can also be considered as the set of one-dimensional (involutive) representations of the Banach-$*$-algebra $\mathcal{A}$.

Since the set $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ is defined by the equations (4.1), it is a weak-$*$-closed subset of the topological dual space $\mathcal{A}'$. One can also show that $\|\chi\| \leq 1$ for any $\chi \in \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ (Exercise 4.1.1), so that $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ is also bounded, hence weak-$*$-compact by the Banach–Alaoglu Theorem. Therefore $\widehat{\mathcal{A}}$ is a locally compact space.

Since $\widehat{\mathcal{A}} \subseteq \mathbb{C}^{\mathcal{A}}$ carries the weak-$*$-topology, i.e., the topology of pointwise convergence, each element $a \in \mathcal{A}$ defines a continuous function

$$\widehat{a} \colon \widehat{\mathcal{A}} \to \mathbb{C}, \quad \widehat{a}(\chi) := \chi(a).$$

Since $\widehat{a}$ extends to a continuous function on the compact space $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ vanishing in the 0-functional, we have $\widehat{a} \in C_0(\widehat{\mathcal{A}})$ (Exercise 4.1.2), with

$$|\widehat{a}(\chi)| = |\chi(a)| \leq \|\chi\| \|a\| \leq \|a\|$$

(cf. Exercise 4.1.1). We thus obtain a map

$$\mathcal{G} \colon \mathcal{A} \to C_0(\widehat{\mathcal{A}}), \quad a \mapsto \widehat{a},$$

called the *Gelfand transform.* For $a, b \in \mathcal{A}$ and $\chi \in \widehat{\mathcal{A}}$ we have

$$\Gamma(ab)(\chi) = \chi(ab) = \chi(a)\chi(b) = \Gamma(a)(\chi)\Gamma(b)(\chi)$$

and

$$\Gamma(a^*)(\chi) = \chi(a^*) = \overline{\chi(a)} = \Gamma(a)^*(\chi),$$

so that Γ is a morphism of Banach-$*$-algebras, i.e., a continuous homomorphism compatible with the involution.

Theorem 4.1.1. (Gelfand Representation Theorem) *If $\mathcal{A}$ is a commutative C^*-algebra, then the Gelfand transform*

$$\mathcal{G} \colon \mathcal{A} \to C_0(\widehat{\mathcal{A}})$$

is an isometric isomorphism.

For a proof we refer to [Ru73, Thm. 11.18].

Remark 4.1.2. (a) If $\mathcal{A}$ is already of the form $\mathcal{A} = C_0(X)$ for a locally compact space, then one can show that the natural map

$$\eta \colon X \to \widehat{\mathcal{A}}, \quad \eta(x)(f) := f(x)$$

is a homeomorphism, so that we can recover the space X as $\widehat{\mathcal{A}}$.

(b) The image $\mathcal{G}(\mathcal{A})$ of the Gelfand transform is a $*$-subalgebra of $C_0(\widehat{\mathcal{A}})$ separating the points of $\mathcal{A}$ and for each $\chi \in \mathcal{A}$, there exists an element $a \in \mathcal{A}$ with $\widehat{a}(\chi) \neq 0$. Therefore the Stone–Weierstraß Theorem (for locally compact spaces) implies that $\mathcal{G}(\mathcal{A})$ is dense in $C_0(\widehat{\mathcal{A}})$.

Corollary 4.1.3. *If $\mathcal{A}$ is a unital C^*-algebra and* $\dim \mathcal{A} > 1$*, then there exist non-zero commuting elements $a, b \in \mathcal{A}$ with $ab = 0$.*

Proof. Since $\mathcal{A} \neq \mathbb{C}\mathbf{1}$, there exists an element $x \in \mathcal{A} \setminus \mathbb{C}\mathbf{1}$. Writing $x = y + iz$ with $y^* = y$ and $z^* = z$, it follows immediately that $\mathcal{A}$ contains a hermitian element $a \in \mathcal{A} \setminus \mathbb{C}\mathbf{1}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be the closed unital subalgebra of $\mathcal{A}$ generated by a. Then $\mathcal{B}$ is commutative and larger than $\mathbb{C}\mathbf{1}$, hence isomorphic to $C_0(X)$ for some locally compact space X (Theorem 4.1.1). Then X contains at least two points $x \neq y$, and Urysohn's Lemma implies the existence of non-zero elements $a, b \in C_c(X) \subseteq C_0(X) \cong \mathcal{B}$ with $ab = 0$. □

Exercises for Section 4.1

Exercise 4.1.1. Let $\mathcal{A}$ be a Banach algebra and $\chi \colon \mathcal{A} \to \mathbb{C}$ be an algebra homomorphism. Show that:

(a) χ extends to the unital Banach algebra $\mathcal{A}_+ := \mathcal{A} \times \mathbb{C}$ with the multiplication

$$(a, t)(a', t') := (aa' + ta' + t'a, tt')$$

(cf. Exercise 1.1.11).

(b) If $\mathcal{A}$ is unital and $\chi \neq 0$, then

$$\chi(\mathbf{1}) = 1 \quad \text{and} \quad \chi(\mathcal{A}^\times) \subseteq \mathbb{C}^\times.$$

Conclude further that $\chi(B_1(\mathbf{1})) \subseteq \mathbb{C}^\times$ and derive that χ is continuous with $\|\chi\| \leq 1$.

Exercise 4.1.2. Suppose that Y is a compact space $y_0 \in Y$ and $X := Y \setminus \{y_0\}$. Show that the restriction map yields an isometric isomorphism of C^*-algebras:

$$r \colon C_*(Y, \mathbb{C}) := \{f \in C(Y, \mathbb{C}) \colon f(y_0) = 0\} \to C_0(X, \mathbb{C}).$$

Exercise 4.1.3. Let $\mathcal{A}$ be a C^*-algebra. Show that:

(i) If $a = a^* \in \mathcal{A}$ is a hermitian element, then $\|a^n\| = \|a\|^n$ holds for each $n \in \mathbb{N}$. Hint: Consider the commutative C^*-subalgebra generated by a.

(ii) If $\mathcal{B}$ is a Banach-$*$-algebra and $\alpha \colon \mathcal{B} \to \mathcal{A}$ a continuous morphism of Banach-$*$-algebras, then $\|\alpha\| \leq 1$. Hint: Let $C := \|\alpha\|$ and derive with (i) for $b \in \mathcal{B}$ the relation

$$\|\alpha(b)\|^{2n} = \|\alpha(bb^*)\|^n = \|\alpha((bb^*)^n)\| \leq C\|(bb^*)^n\| \leq C\|b\|^{2n}.$$

Finally, use that $C^{1/n} \to 1$.

Exercise 4.1.4. Let $\mathcal{A}$ be a C^*-algebra. We call a hermitian element $a = a^* \in \mathcal{A}$ *positive* if $a = b^2$ for some hermitian element $b = b^* \in \mathcal{A}$. Show that:

(a) Every positive Element $a \in \mathcal{A}$ has a positive square root. Hint: Consider the commutative C^*-subalgebra $\mathcal{B}$ generated by b and recall that $\mathcal{B} \cong C_0(X)$ for some locally compact space.

(b) If $C_0(X)$, X a locally compact space, is generated as a C^*-algebra by some $f \geq 0$, then it is also generated by f^2. Hint: Use the Stone–Weierstraß Theorem.

(c) If b is a positive square root of a, then there exists a commutative C^*-subalgebra of $\mathcal{A}$ containing a and b in which b is positive. Hint: Write $b = c^2$ and consider the C^*-algebra generated by c.

(d) Every positive Element $a \in \mathcal{A}$ has a unique positive square root. Hint: Use (b) and (c) to see that any positive square root of a is contained in the C^*-algebra generated by a; then consider the special case $\mathcal{A} = C_0(X)$.

Exercise 4.1.5. Let $\mathcal{A}$ be a unital C^*-algebra and $a = a^* \in \mathcal{A}$ with $\|a\| < 1$. Show that

$$b := \sqrt{1 - a^2} := \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n a^{2n}$$

is hermitian and satisfies $b^2 = 1 - a^2$. Show further that

$$u := a + i\sqrt{\mathbf{1} - a^2} \in \mathrm{U}(\mathcal{A})$$

and conclude that $\mathcal{A} = \operatorname{span} \mathrm{U}(\mathcal{A})$. Hint: To verify $b^2 = \mathbf{1} - a^2$, it suffices to consider the commutative C^*-algebra generated by a.

4.2 The Commutant of a Representation

4.2.1 Basic Properties of Commutants

Definition 4.2.1. For a subset $S \subseteq B(\mathcal{H})$, we define the *commutant* by

$$S' := \{A \in B(\mathcal{H}) \colon (\forall s \in S)\, sA = As\}.$$

If $(\pi, \mathcal{H})$ is a representation of an involutive semigroup S, then $\pi(S)' = B_G(\mathcal{H}, \mathcal{H})$ is called the *commutant of* $(\pi, \mathcal{H})$. It coincides with the space of self-intertwining operators of the representation $(\pi, \mathcal{H})$ with itself (cf. Definition 1.3.3).

Lemma 4.2.2. *For subsets $E, F \subseteq B(\mathcal{H})$, we have:*

(i) $E \subseteq F' \Leftrightarrow F \subseteq E'$.

(ii) $E \subseteq E''$.

(iii) $E \subseteq F \Rightarrow F' \subseteq E'$.

(iv) $E' = E'''$.

(v) *$E = E''$ if and only if $E = F'$ holds for some $F \subseteq B(\mathcal{H})$.*

Proof. (i) is trivial.

(ii) In view of (i), this is equivalent to $E' \subseteq E'$, hence trivial.

(iii) is also trivial.

(iv) From (ii) we get $E' \subseteq (E')'' = E'''$. Moreover, (ii) and (iii) imply $E''' \subseteq E'$.

(v) If $E = F'$, then $E'' = F''' = F' = E$ is a consequence of (iv). The converse is trivial, □

Lemma 4.2.3. *The commutant E' of a subset $E \subseteq B(\mathcal{H})$ has the following properties:*

(i) *If E is commutative, then so is E''.*

(ii) *E' is a subalgebra of $B(\mathcal{H})$ which is closed in the weak operator topology, hence in particular norm-closed.*

(iii) *If $E^* = E$, then E' is also $*$-invariant, hence in particular a C^*-subalgebra of $B(\mathcal{H})$.*

Proof. (i) That E is commutative is equivalent to $E \subseteq E'$, but this implies $E'' \subseteq E' = E'''$ (Lemma 4.2.2(iv)), which means that E'' is commutative.

(ii) Clearly E' is a linear subspace closed under products, hence a subalgebra of $B(\mathcal{H})$. To see that E' is closed in the weak operator topology, let $v, w \in \mathcal{H}$ and $B \in E$. For $A \in B(\mathcal{H})$ we then have

$$f_{v,w}(AB - BA) = \langle ABv, w\rangle - \langle BAv, w\rangle = (f_{Bv,w} - f_{v,B^*w})(A),$$

which leads to

$$E' = \bigcap_{v,w\in\mathcal{H},B\in E} \ker(f_{Bv,w} - f_{v,B^*w}),$$

which is subspace of $B(\mathcal{H})$ that is closed in the weak operator topology.

(iii) If $A \in E'$ and $B \in E$, then

$$A^*B - BA^* = (B^*A - AB^*)^* = 0$$

follows from $B^* \in E$. Therefore E' is $*$-invariant. Since it is in particular norm closed by (ii), E' is a C^*-subalgebra of $B(\mathcal{H})$. □

Definition 4.2.4. A unital $*$-subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ is called a *von Neumann algebra* if $\mathcal{A} = \mathcal{A}''$.

A von Neumann algebra $\mathcal{A}$ is called a *factor* if its center

$$Z(\mathcal{A}) := \{z \in \mathcal{A} \colon (\forall a \in \mathcal{A})\, az = za\}$$

is trivial, i.e., $Z(\mathcal{A}) = \mathbb{C}\mathbf{1}$.

Remark 4.2.5. (a) In view of Lemma 4.2.3, any von Neumann algebra $\mathcal{A}$ is closed in the weak operator topology.

(b) For every $*$-invariant subset $E \subseteq B(\mathcal{H})$, the commutant E' is a von Neumann algebra because it is also $*$-invariant and $E''' = E'$ (Lemma 4.2.2). In particular, for any von Neumann algebra $\mathcal{A}$, the commutant $\mathcal{A}'$ is also a von Neumann algebra.

(c) Clearly, the center $Z(\mathcal{A})$ of a von Neumann algebra can also be written as

$$Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}' = \mathcal{A}'' \cap \mathcal{A}' = Z(\mathcal{A}').$$

In particular, we see that $\mathcal{A}$ is a factor if and only if its commutant $\mathcal{A}'$ is a factor.

Example 4.2.6. The full algebra $\mathcal{A} = B(\mathcal{H})$ is a von Neumann algebra. In this case $\mathcal{A}' = \mathbb{C}\mathbf{1}$ (Exercise), which implies in particular that $B(\mathcal{H})$ is a factor. Such factors are called *type I factors*.

4.2.2 Schur's Lemma and some Consequences

The fact that commutants are always C^*-algebras is extremely useful in representation theory. We now use the results on commutative C^*-algebras explained in Section 4.1.

Theorem 4.2.7. (Schur's Lemma) *A representation $(\pi, \mathcal{H})$ of an involutive semigroup is irreducible if and only if its commutant is trivial, i.e., $\pi(S)' = \mathbb{C}\mathbf{1}$.*

Proof. If $(\pi, \mathcal{H})$ is not irreducible and $\mathcal{K} \subsetneq \mathcal{H}$ is a proper closed invariant subspace, then the orthogonal projection P onto $\mathcal{K}$ commutes with $\pi(S)$ (Lemma 1.3.1) and $P \notin \mathbb{C}\mathbf{1}$. Therefore $(\pi, \mathcal{H})$ is irreducible if $\pi(S)' = \mathbb{C}\mathbf{1}$.

Suppose, conversely, that $\pi(S)' \neq \mathbb{C}\mathbf{1}$. Then Corollary 4.1.3 applies to the C^*-algebra $\pi(S)'$ (Lemma 4.2.3), so that there exist non-zero commuting $A, B \in \pi(S)'$ with $AB = 0$. Then $\mathcal{K} := \overline{A(\mathcal{H})}$ is a non-zero closed subspace invariant under $\pi(S)$ and satisfying $B\mathcal{K} = \{0\}$. Therefore $(\pi, \mathcal{H})$ is not irreducible. $\square$

Corollary 4.2.8. *Every irreducible representation $(\pi, \mathcal{H})$ of a commutative involutive semigroup $(S, *)$ is one-dimensional.*

Proof. If S is commutative, then $\pi(S) \subseteq \pi(S)'$. If $(\pi, \mathcal{H})$ is irreducible, then $\pi(S)'$ by Schur's Lemma, and therefore $\pi(S) \subseteq \mathbb{C}\mathbf{1}$, so that the irreducibility further implies $\dim \mathcal{H} = 1$. $\square$

Corollary 4.2.9. *Suppose that $(\pi, \mathcal{H})$ is an irreducible representation of an involutive semigroup and $(\rho, \mathcal{K})$ any representation of S. If $B_S(\mathcal{H}, \mathcal{K}) \neq \{0\}$, then $(\pi, \mathcal{H})$ is equivalent to a subrepresentation of $(\rho, \mathcal{K})$. In particular, $B_S(\mathcal{H}, \mathcal{K}) = 0$ if both representations are irreducible and non-equivalent.*

Proof. Let $A \in B_S(\mathcal{H}, \mathcal{K})$ be a non-zero intertwining operator. Then $A^*A \in B_S(\mathcal{H}) = \pi(S)' = \mathbb{C}\mathbf{1}$ by Schur's Lemma. If this operator is non-zero, then $\langle A^*Av, v\rangle = \|Av\|^2 \geq 0$ for $v \in \mathcal{H}$ implies that $A^*A = \lambda\mathbf{1}$ for some $\lambda > 0$. Then

$B := \sqrt{\lambda}^{-1} A$ is another intertwining operator with $B^*B = \mathbf{1}$. Hence $B \colon \mathcal{H} \to \mathcal{K}$ is an isometric embedding. In particular, its image $\mathcal{K}_0$ is a closed non-zero invariant subspace on which the representation induced by ρ is equivalent to $(\pi, \mathcal{H})$. $\square$

Corollary 4.2.10. *If $(\pi, \mathcal{H})$ is an irreducible representation of an involutive semigroup and $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}$ are non-equivalent irreducible subrepresentations, then $\mathcal{H}_1 \bot \mathcal{H}_2$.*

Proof. Let $P \colon \mathcal{H} \to \mathcal{H}_1$ denote the orthogonal projection onto $\mathcal{H}_1$. Since $\mathcal{H}_1$ is invariant under $\pi(S)$, Lemma 1.3.1 implies that $P \in B_S(\mathcal{H}, \mathcal{H}_1)$. Hence $P|_{\mathcal{H}_1} \in B_S(\mathcal{H}_2, \mathcal{H}_1) = \{0\}$ by Corollary 4.2.9. This means that $\mathcal{H}_1 \bot \mathcal{H}_2$. $\square$

At this point it is natural to observe that any representation $(\pi, \mathcal{H})$ of an involutive semigroup S decomposes naturally into a *discrete part* $(\pi_d, \mathcal{H}_d)$ which is a direct sum of irreducible ones and a *continuous part* $(\pi_c, \mathcal{H}_c)$ which does not contain any irreducible subrepresentations.

Proposition 4.2.11. *Let $(\pi, \mathcal{H})$ be a representation of the involutive semigroup S on $\mathcal{H}$ and $\mathcal{H}_d \subseteq \mathcal{H}$ be the closed subspace generated by all irreducible subrepresentations. Then the following assertions hold:*

(i) *$\mathcal{H}_d$ is S-invariant and the representation $(\pi_d, \mathcal{H}_d)$ of S is a direct sum of irreducible ones.*

(ii) *The orthogonal space $\mathcal{H}_c := \mathcal{H}_d^\perp$ carries a representation $(\pi_c, \mathcal{H}_c)$ of S which does not contain any irreducible subrepresentation.*

Proof. It is clear that the subspace $\mathcal{H}_d$ is S-invariant because it is generated by a family of S-invariant subspaces. To see that it is a direct sum of irreducible representations, we apply Zorn's Lemma. Let $\mathcal{H}_j$, $j \in J$, be a maximal set of S-invariant subspaces of $\mathcal{H}$ such that the corresponding representations $(\pi_j, \mathcal{H}_j)$ are irreducible and their sum $\sum_{j \in J} \mathcal{H}_j$ is orthogonal. Set $\mathcal{H}_0 := \overline{\sum_{j \in J} \mathcal{H}_j} \subseteq \mathcal{H}_d$. Then $\mathcal{H}_1 := \mathcal{H}_0^\perp \cap \mathcal{H}_d$ is S-invariant. We write $p \colon \mathcal{H}_d \to \mathcal{H}_1$ for the orthogonal projection. Then p is surjective and if $\mathcal{H}_1 \neq \{0\}$, there exists an irreducible subspace $\mathcal{K} \subseteq \mathcal{H}_d$ with $p(\mathcal{K}) \neq \{0\}$. This means that $B_S(\mathcal{K}, \mathcal{H}_1) \neq \{0\}$ and hence, by Corollary 4.2.9, that the representation on $\mathcal{K}$ is equivalent to an irreducible subrepresentation of $\mathcal{H}_1$. This contradicts the maximality of the family $(\mathcal{H}_j)_{j \in J}$. We conclude that $\mathcal{H}_1 = \{0\}$, and (i) follows from Exercise 1.3.5.

Assertion (ii) follows from the construction of $\mathcal{H}_d$. $\square$

Definition 4.2.12. If $(\pi, \mathcal{H})$ is an irreducible representation, then we write $[\pi]$ for its (unitary) equivalence class.

For a topological group G, we write $\widehat{G}$ for the set of equivalence classes of irreducible unitary representations (cf. Exercise 1.3.8). It is called the *unitary dual of G*.

Let $(\rho, \mathcal{H})$ be a continuous unitary representations of G. For $[\pi] \in \widehat{G}$, we write $\mathcal{H}_{[\pi]} \subseteq \mathcal{H}$ for the closed subspace generated by all irreducible subrepresentations of type $[\pi]$. From Corollaries 4.2.9 and 4.2.10 it follows in particular that

$$\mathcal{H}_{[\pi]} \perp \mathcal{H}_{[\pi']} \quad \text{for} \quad [\pi] \neq [\pi'],$$

so that the discrete part of $(\rho, \mathcal{H})$ is a direct sum

$$\mathcal{H}_d = \widehat{\bigoplus}_{[\pi] \in \widehat{G}} \mathcal{H}_{[\pi]}$$

(Exercise 1.3.5). The subspaces $\mathcal{H}_{[\pi]}$ are called the *isotypic components* of $\mathcal{H}$.

Remark 4.2.13. (Reduction of commutants) Applying Corollary 4.2.9 to the decomposition

$$\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_d = \mathcal{H}_c \oplus \widehat{\bigoplus}_{[\pi] \in \widehat{G}} \mathcal{H}_{[\pi]},$$

we see that

$$B_G(\mathcal{H}_{[\pi]}, \mathcal{H}_c) = \{0\} \quad \text{and} \quad B_G(\mathcal{H}_{[\pi]}, \mathcal{H}_{[\pi']}) = \{0\}$$

for $[\pi] \neq [\pi']$. Therefore $B_G(\mathcal{H})$ preserves each $\mathcal{H}_{[\pi]}$, hence it also preserves $\mathcal{H}_c = \mathcal{H}_d^\perp$ because it is $*$-invariant.

From Exercise 4.2.3 we thus derive that

$$B_G(\mathcal{H}) = \Big\{(A_{[\pi]}) \in \prod_{[\pi] \in \widehat{G}} B_G(\mathcal{H}_{[\pi]}) \colon \sup_{[\pi] \in \widehat{G}} \|A_\pi\| < \infty\Big\} \oplus B_G(\mathcal{H}_c).$$

Using the concept of an ℓ^∞ direct sum of Banach spaces

$$\oplus_{j \in J}^\infty X_j := \Big\{(x_j)_{j \in J} \in \prod_{j \in J} X_j \colon \|x\| := \sup_{j \in J} \|x_j\| < \infty\Big\},$$

it follows that

$$B_G(\mathcal{H}) \cong \Big(\oplus_{[\pi] \in \widehat{G}}^\infty B_G(\mathcal{H}_{[\pi]})\Big) \oplus B_G(\mathcal{H}_c).$$

In the following section we shall determine the commutant of the isotypic components of a representation.

Example 4.2.14. (a) For an abelian topological group A, we have seen in Corollary 4.2.8 that all irreducible unitary representations are one-dimensional, hence given by continuous homomorphisms

$$\chi \colon A \to \mathbb{T}.$$

Since $\mathrm{U}(\mathbb{C}) = \mathrm{U}_1(\mathbb{C}) = \mathbb{T}$ is abelian, two characters define equivalent unitary representations if and only if they coincide. Therefore the unitary dual of A can be identified with the group

$$\widehat{A} := \mathrm{Hom}(A, \mathbb{T})$$

of continuous characters of A.

For any continuous unitary representation $(\pi, \mathcal{H})$ of A and $\chi \in \widehat{A}$, the isotypic subspace

$$\mathcal{H}_\chi := \mathcal{H}_{[\chi]} = \{v \in \mathcal{H} \colon (\forall a \in A)\, \pi(a)v = \chi(a)v\}$$

is the simultaneous eigenspace of A on $\mathcal{H}$ corresponding to the character χ.

Taking all the simultaneous eigenspaces together, we obtain the subspace

$$\mathcal{H}_d = \widehat{\bigoplus}_{\chi \in \widehat{A}} \mathcal{H}_\chi$$

from Proposition 4.2.11.

(b) In general its orthogonal complement $\mathcal{H}_c$ is non-trivial, as the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ shows. To see this, we first observe that, in view of Exercise 4.2.4, every continuous character of $\mathbb{R}$ is of the form $\chi_\lambda(x) = e^{i\lambda x}$. Therefore any eigenfunction $f \in L^2(\mathbb{R})$ satisfies for each $x \in \mathbb{R}$ for almost every $y \in \mathbb{R}$ the relation

$$f(y + x) = e^{i\lambda x} f(y).$$

This implies in particular that the function $|f|$ is, as an element of $L^2(\mathbb{R}, dx)$, translation invariant. We thus obtain

$$\infty > \int_{\mathbb{R}} |f(x)|^2\, dx = \sum_{n \in \mathbb{Z}} \int_0^1 |f(x)|^2\, dx,$$

and therefore f vanishes almost everywhere. This proves that the discrete part is trivial, i.e., $L^2(\mathbb{R}, dx)_d = \{0\}$.

Example 4.2.15. If A is a compact group and μ_A is a normalized Haar measure on A, we have for each non-trivial character $\chi \in \widehat{A}$

$$\int_A \chi \mu_A = 0.$$

In fact, for $a \in A$ we have

$$I := \int_A \chi(x)\, d\mu_A(x) = \int_A \chi(ax)\, d\mu_A(x) = \chi(a) \int_A \chi(x)\, d\mu_A(x) = \chi(a) I,$$

and for $\chi(a) \neq 1$, we derive $I = 0$. We conclude that $\widehat{A}$ is an orthonormal subset of $L^2(A, \mu_A)$ because

$$\langle \chi, \eta \rangle = \int_A \chi \overline{\eta}\, d\mu_A = \delta_{\chi,\eta}.$$

This observation has an important application to the analysis of unitary representations of A because it permits to derive an explicit form for the projection onto the isotypic components.

Let $(\pi, \mathcal{H})$ be a continuous unitary representation of A and $\chi \in \widehat{A}$. For $v \in \mathcal{H}$ we consider the $\mathcal{H}$-valued integral

$$P_\chi(v) := \int_A \overline{\chi(g)} \pi(g) v\, d\mu_A(g).$$

We refer to Exercise 2.4.4 for its existence. Clearly, P_χ defines a linear map $\mathcal{H} \to \mathcal{H}$ with

$$\|P_\chi(v)\| \leq \int_A |\chi(g)| \|\pi(g)v\| \, d\mu_A(g) = \int_A \|v\| \, d\mu_A(g) = \|v\|,$$

so that P_χ is a contraction. We also note that

$$\begin{aligned}\langle P_\chi(v), w\rangle &= \int_A \overline{\chi(g)} \langle \pi(g)v, w\rangle \, d\mu_A(g) = \int_A \overline{\chi(g)} \langle v, \pi(g^{-1})w\rangle \, d\mu_A(g) \\ &= \langle v, \int_A \chi(g)\pi(g^{-1})w \, d\mu_A(g)\rangle = \langle v, \int_A \chi(g^{-1})\pi(g)w \, d\mu_A(g)\rangle \\ &= \langle v, \int_A \overline{\chi(g)}\pi(g)w \, d\mu_A(g)\rangle = \langle v, P_\chi(w)\rangle,\end{aligned}$$

so that $P_\chi^* = P_\chi$. If $v \in \mathcal{H}_\chi$, then

$$P_\chi(v) = \int_A \overline{\chi(g)}\chi(g)v \, d\mu_A(g) = \int_A \overline{\chi(g)}\chi(g) \, d\mu_A(g) \cdot v = v,$$

and for $a \in A$ we have

$$\begin{aligned}\pi(a)P_\chi(v) &= \int_A \overline{\chi(g)}\pi(a)\pi(g)v \, d\mu_A(g) = \int_A \overline{\chi(g)}\pi(ag)v \, d\mu_A(g) \\ &= \int_A \overline{\chi(a^{-1}g)}\pi(g)v \, d\mu_A(g) = \chi(a) \int_A \overline{\chi(g)}\pi(g)v \, d\mu_A(g) = \chi(a)P_\chi(v).\end{aligned}$$

This shows that $P_\chi^2 = P_\chi = P_\chi^*$ is the orthogonal projection onto the eigenspace $\mathcal{H}_\chi$.

As we shall see in the following subsection, $\mathcal{H} = \mathcal{H}_d$ holds for any continuous unitary representation of a compact group, and for compact abelian groups A, this leads to

$$\mathcal{H} = \widehat{\bigoplus}_{\chi \in \widehat{A}} \mathcal{H}_\chi.$$

This decomposition is of particular interest for the right regular representation $(\pi_r, L^2(A))$. In this case we can be more specific. For $f \in L^2(A)$ we have

$$\begin{aligned}P_\chi(f)(h) &= \int_A \overline{\chi(g)} f(hg) \, d\mu_A(g) = \int_A \overline{\chi(h^{-1}g)} f(g) \, d\mu_A(g) \\ &= \chi(h) \int_A \overline{\chi(g)} f(g) \, d\mu_A(g).\end{aligned}$$

This proves in particular that $L^2(A, \mu_A)_\chi = \mathbb{C}\chi$, so that we obtain

$$L^2(A, \mu_A) = \widehat{\bigoplus}_{\chi \in \widehat{A}} \mathbb{C}\chi,$$

and since the characters form an orthonormal system, the map

$$\mathcal{F}\colon L^2(A,\mu_A) \to \ell^2(\widehat{A},\mathbb{C}), \quad \mathcal{F}(f)(\chi) := \widehat{f}(\chi) := \int_A \overline{\chi(g)} f(g)\, d\mu_A(g)$$

is unitary. This map is called the *Fourier transform* and its unitarity is called the *Plancherel Theorem*.

Example 4.2.16. The preceding example specializes in particular to the right regular representation of the circle group $G = \mathbb{T}$ on $\mathcal{H} = L^2(\mathbb{T},\mu_{\mathbb{T}})$.

To see how this representation decomposes, we first recall from Example 4.2.14(b) that each character of $\mathbb{R}$ is of the form $\chi_\lambda(x) = e^{i\lambda x}$ for some $\lambda \in \mathbb{R}$. Since $q\colon \mathbb{R} \to \mathbb{T}, t \mapsto e^{ix}$ factors through an isomorphism $\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{T}$ of topological groups, it follows that any character $\chi \in \widehat{\mathbb{T}}$ satisfies

$$\chi(e^{ix}) = e^{i\lambda x}$$

for some $\lambda \in \mathbb{R}$ and any $x \in \mathbb{R}$. Then $\chi(1) = 1$ implies $\lambda \in \mathbb{Z}$, so that $\chi_n(z) = z^n$ for some $n \in \mathbb{Z}$ and all $z \in \mathbb{T}$. This proves that

$$\widehat{\mathbb{T}} = \{\chi_n \colon n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

In this case the Fourier transform is given by

$$\mathcal{F}\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z},\mathbb{C}), \quad \widehat{f}(n) = \int_{\mathbb{T}} z^{-n} f(z)\, d\mu_{\mathbb{T}}(z)$$

and the corresponding expansion in $L^2(\mathbb{T})$ has the form

$$f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\chi_n, \quad f(z) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) z^n \quad \text{for} \quad z \in \mathbb{T}.$$

Identifying $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$ and $L^2(\mathbb{T})$ with 2π-periodic functions, this takes the familiar form for Fourier series:

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{inx}, \qquad \widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)\, dx,$$

where the series on the left convergence in $L^2([0,2\pi])$.

Remark 4.2.17. (a) There exist abelian topological groups A with faithful continuous unitary representations for which all continuous characters are trivial, i.e., $\widehat{A} = \{\mathbf{1}\}$. For these groups, the two fundamental problems discussed in the introduction make no sense.

To see how such groups arise in concrete situations, let $(X,\mathfrak{S},\mu)$ be a finite measure space. An $E \in \mathfrak{S}$ is called an *atom* if $\mu(F) \in \{0,\mu(E)\}$ for any measurable subset $F \subseteq E$. Accordingly, μ is said to be *atomic* if all elements of $\mathfrak{S}$ with positive measure are atoms and *non-atomic* if there is no atom.

In [GN01] it is shown that if $(X, \mathfrak{S}, \mu)$ is a finite non-atomic measure space, then the group

$$\mathcal{M}(X, \mathbb{T}) = \{f \in L^\infty(X, \mu) \colon |f| = 1\},$$

endowed with the weak operator topology obtained from the embedding

$$L^\infty(X, \mu) \hookrightarrow B(L^2(X, \mu))$$

(cf. Lemma 2.1.5), is an abelian topological group for which all continuous characters are trivial.

(b) Another pathology that can occur for an (abelian) topological group G is that all its continuous unitary representations are trivial. Such topological groups are called *exotic*. In Chapter 2 of [Ba91] one finds various constructions of such group of the type $G = E/\Gamma$, where E is a Banach space and $\Gamma \subseteq E$ is a discrete subgroup. Note that for any such exotic group G, in particular all characters are trivial.

From Exercise 4.2.4, we immediately derive that all characters of $G = E/\Gamma$ are of the form

$$\chi(v + \Gamma) = e^{2\pi i \alpha(v)},$$

where $\alpha \in E'$ is a continuous linear functional satisfying $\alpha(\Gamma) \subseteq \mathbb{Z}$. If G is exotic, resp., $\widehat{G} = \{\mathbf{1}\}$, then the subgroup Γ has the strange property that for any continuous linear functional $\alpha \in E'$ with $\alpha(\Gamma) \subseteq \mathbb{Z}$ we have $\alpha = 0$. It is not hard to see that this never happens if $\dim E < \infty$.

4.2.3 Discrete Decomposability for Compact Groups

Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the compact group G and μ_G be a normalized Haar measure on G. We assume that $\mathcal{H} \neq \{0\}$ and want to show that $\mathcal{H} = \mathcal{H}_d$, i.e., that $\mathcal{H}$ decomposes as a direct sum of irreducible representations. This will follow, as soon as we can show that $\mathcal{H}$ contains a non-zero finite dimensional G-invariant subspace because every finite dimensional representation is a direct sum of irreducible ones (Proposition 1.3.11).

If $0 \neq A = A^* \in B_G(\mathcal{H})$ is a non-zero compact intertwining operator, then the Spectral Theorem for compact hermitian operators implies that $\mathcal{H} = \widehat{\bigoplus}_{\lambda \in \mathbb{R}} \mathcal{H}_\lambda(A)$ is the orthogonal direct sum of the eigenspaces

$$\mathcal{H}_\lambda(A) := \ker(A - \lambda \mathbf{1})$$

and if $\lambda \neq 0$, then $\dim \mathcal{H}_\lambda(A) < \infty$. Since A is non-zero, it has a non-zero eigenvalue λ, and therefore $\mathcal{H}_\lambda(A)$ is a finite dimensional subspace of $\mathcal{H}$ which is G-invariant (Exercise 1.3.9). It therefore remains to construct a non-zero hermitian compact element of $B_G(\mathcal{H})$.

Proposition 4.2.18. *For each $A \in B(\mathcal{H})$, there exists a unique operator $A_G \in B_G(\mathcal{H})$ with the property that*

$$\langle A_G v, w\rangle = \int_G \langle \pi(g) A \pi(g)^{-1} v, w\rangle \, d\mu_G \quad \textit{for} \quad v, w \in \mathcal{H}.$$

Moreover, $(A_G)^* = (A^*)_G$ and if A is compact, then A_G is also compact.

We also write this operator as an operator-valued integral

$$\int_G \pi(g)A\pi(g)^{-1}\, d\mu_G := A_G.$$

Proof. On $\mathcal{H}$ we consider the sesquilinear form defined by

$$F(v,w) := \int_G \langle \pi(g)A\pi(g)^{-1}v, w\rangle\, d\mu_G(g).$$

Then

$$\begin{aligned}|F(v,w)| &\leq \int_G \|\pi(g)A\pi(g)^{-1}\|\|v\|\|w\|\, d\mu_G(g)\\ &= \int_G \|A\|\|v\|\|w\|\, d\mu_G(g) = \|A\|\|v\|\|w\|,\end{aligned}$$

and we conclude the existence of a unique bounded operator $A_G \in B(\mathcal{H})$ with

$$F(v,w) = \langle A_G v, w\rangle \quad \text{for} \quad v,w \in \mathcal{H}$$

(Exercise in Functional Analysis). To see that A_G commutes with each $\pi(g)$, we calculate

$$\begin{aligned}\langle \pi(g)A_G\pi(g)^{-1}v, w\rangle &= \int_G \langle \pi(g)\pi(h)A\pi(h)^{-1}\pi(g)^{-1}v, w\rangle\, d\mu_G(h)\\ &= \int_G \langle \pi(gh)A\pi(gh)^{-1}v, w\rangle\, d\mu_G(h)\\ &= \int_G \langle \pi(h)A\pi(h)^{-1}v, w\rangle\, d\mu_G(h) = \langle A_G v, w\rangle.\end{aligned}$$

Next we note that

$$\begin{aligned}\langle (A_G)^*v, w\rangle = \langle v, A_G w\rangle &= \int_G \langle v, \pi(g)A\pi(g)^{-1}w\rangle\, d\mu_G(g)\\ &= \int_G \langle \pi(g)A^*\pi(g)^{-1}v, w\rangle\, d\mu_G(g) = \langle (A^*)_G v, w\rangle.\end{aligned}$$

Finally, we assume that A is compact, i.e., the image of the closed unit ball $B \subseteq \mathcal{H}$ is relatively compact. We have to show that the same holds for A_G. Since G is compact, the set $B' := \overline{\pi(G)AB} \subseteq \pi(G)\overline{AB}$ is also compact because the action $G \times \mathcal{H} \to \mathcal{H}, (g,v) \mapsto \pi(g)v$ is continuous (Exercise 1.2.2). The closed convex hull $K := \overline{\mathrm{conv}(B')}$ is also compact (Exercise 4.2.6).

We claim that $A_G(B) \subseteq K$, and this will imply that A_G is compact. So let $v \in \mathcal{H}$ and $c \in \mathbb{R}$ with $\mathrm{Re}\langle v, x\rangle \leq c$ for all $x \in K$. Then we obtain for $w \in B$:

$$\begin{aligned}\mathrm{Re}\langle v, A_G w\rangle &= \mathrm{Re} \int_G \langle v, \pi(g)A\pi(g)^{-1}w\rangle\, d\mu_G(g)\\ &= \int_G \mathrm{Re}\langle v, \pi(g)A\pi(g)^{-1}w\rangle\, d\mu_G(g) \leq \int_G c\, d\mu_G(g) = c.\end{aligned}$$

Since

$$K = \{x \in \mathcal{H} \colon (\forall v \in \mathcal{H}) \operatorname{Re}\langle v, x\rangle \leq \sup \operatorname{Re}\langle v, K\rangle\}$$

by the Hahn–Banach Separation Theorem, it follows that $A_G w \in K$, and thus $A_G B \subseteq K$. $\square$

Combining the preceding proposition with the discussion above, we obtain:

Proposition 4.2.19. *If $(\pi, \mathcal{H})$ is a non-zero continuous unitary representation of the compact group G, then $\mathcal{H}$ contains a non-zero finite dimensional G-invariant subspace.*

Proof. We have to show the existence of a non-zero compact hermitian intertwining operator. So let $v_0 \in \mathcal{H}$ be a unit vector and consider the orthogonal projection $P(v) := \langle v, v_0\rangle v_0$ onto $\mathbb{C}v_0$. Then $\dim(\operatorname{im}(P)) = 1$ implies that P is compact, and since it is an orthogonal projection, we also have $P^* = P$. Therefore

$$P_G(v) := \int_G (\pi(g) P \pi(g)^{-1}) v \, d\mu_G(g) = \int_G \langle \pi(g)^{-1} v, v_0\rangle \pi(g) v_0 \, d\mu_G(g)$$

from Proposition 4.2.18 is a compact hermitian operator. To see that it is non-zero, we simply observe that

$$\langle P_G v_0, v_0\rangle = \int_G \langle \pi(g)^{-1} v_0, v_0\rangle \langle \pi(g) v_0, v_0\rangle \, d\mu_G(g) = \int_G |\langle \pi(g) v_0, v_0\rangle|^2 \, d\mu_G(g) > 0$$

follows from $\langle \pi(\mathbf{1}) v_0, v_0\rangle > 0$ and the defining property of the Haar measure μ_G. $\square$

Theorem 4.2.20. (Fundamental Theorem on Unitary Representations of Compact Groups—Abstract Peter–Weyl Theorem) *If $(\pi, \mathcal{H})$ is a continuous unitary representation of the compact group G, then $(\pi, \mathcal{H})$ is a direct sum of irreducible representations and all irreducible representations of G are finite dimensional.*

Proof. Writing $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c$ for the decomposition into discrete and continuous part (Proposition 4.2.11), we use Proposition 4.2.19 to see that if $\mathcal{H}_c \neq \{0\}$, then it contains a finite dimensional invariant subspace, contradicting the definition of $\mathcal{H}_c$ (Proposition 1.3.11). Therefore $\mathcal{H}_c = \{0\}$ and thus $\mathcal{H} = \mathcal{H}_d$, so that the first part follows from Proposition 4.2.11. Applying Proposition 4.2.11 to an irreducible representation $(\pi, \mathcal{H})$ of G, we thus get $\dim \mathcal{H} < \infty$. $\square$

Exercises for Section 4.2

Exercise 4.2.1. (Concrete commutants) Let $(X, \mathfrak{S}, \mu)$ be a finite measure space, $\mathcal{H} := L^2(X, \mu)$ the corresponding Hilbert space and

$$\pi \colon L^\infty(X, \mu) \to B(L^2(X, \mu)), \quad \pi(f)g := fg$$

be the homomorphism from Lemma 2.1.5. Show that:

(1) $1 \in L^2(X, \mu)$ is a cyclic vector for π, i.e., not contained in a proper closed subspace invariant under $L^\infty(X, \mu)$.

(2) If $B \in \pi(L^\infty(X, \mu))'$, then

  (a) $B(f) = B(1)f$ for $f \in L^2(X, \mu)$. Hint: Verify this relation first for bounded functions f.

  (b) $B(1)$ is bounded. Hint: Apply B to the characteristic function of the set $E_n := \{x \in X \colon n \leq |B(1)|(x) \leq n + 1\}$.

  (c) $B = \pi(B(1))$.

(5) $\pi(L^\infty(X, \mu)) = \pi(L^\infty(X, \mu))'$ is its own commutant, hence in particular a von Neumann algebra.

Exercise 4.2.2. Let $(\pi, \mathcal{H})$ be an irreducible representation of the involutive semigroup $(S, *)$ and $\pi_n := \oplus_{j=1}^n \pi$ be the n-fold direct sum of π with itself on $\mathcal{H}^n = \oplus_{j=1}^n \mathcal{H}$. Show that

$$\pi_n(S)' \cong M_n(\mathbb{C}).$$

Hint: Write operators on $\mathcal{H}^n$ as matrices with entries in $B(\mathcal{H})$ (cf. Exercise 1.3.6) and evaluate the commuting condition.

Exercise 4.2.3. Let $(\mathcal{H}_j)_{j \in J}$ be a family of Hilbert spaces and $A_j \in B(\mathcal{H}_j)$. Suppose that $\sup_{j \in J} \|A_j\| < \infty$. Then $A(x_j) := (A_j x_j)$ defines a bounded linear operator on $\widehat{\oplus}_{j \in J} \mathcal{H}_j$ with

$$\|A\| = \sup_{j \in J} \|A_j\|.$$

If, conversely, $\mathcal{H} = \widehat{\oplus}_{j \in J} \mathcal{H}_j$ is a Hilbert space direct sum and $A \in B(\mathcal{H})$ preserves each subspace $\mathcal{H}_j$, then the restrictions $A_j := A|_{\mathcal{H}_j}$ are bounded operators in $B(\mathcal{H}_j)$ satisfying $\|A\| = \sup_{j \in J} \|A_j\|$.

Exercise 4.2.4. Let V be a real topological vector space. Show that each continuous character $\chi \colon V \to \mathbb{T}$ is of the form $\chi(v) = e^{i\alpha(v)}$ for some continuous linear functional $\alpha \in V'$. Hint: Let $U \subseteq V$ be a circular 0-neighborhood (circular means that $\lambda U \subseteq U$ for $|\lambda| \leq 1$; such neighborhoods form a local basis in 0) with $\operatorname{Re} \chi(v) > 0$ for $v \in U + U$. Define a continuous (!) function

$$L \colon U \to]-\pi, \pi[\subseteq \mathbb{R} \quad \text{by} \quad e^{iL(u)} = \chi(u).$$

Observe that $L(x + y) = L(x) + L(y)$ for $x, y \in U$ and use this to see that

$$\alpha(x) := \lim_{n \to \infty} nL\Big(\frac{x}{n}\Big)$$

is an additive extension of L to V. Now it remains to observe that continuous additive maps $V \to \mathbb{R}$ are linear functionals (prove $\mathbb{Q}$-linearity first).

Exercise 4.2.5. Let G be a countable group acting in a measure preserving fashion on the σ-finite measure space $(X, \mathfrak{S}, \mu)$. The measure μ is said to be *ergodic* (with respect to this action) if any G-invariant subset $E \in \mathfrak{S}$ either

$$\mu(E) = 0 \quad \text{or} \quad \mu(E^c) = 0.$$

Show that, if μ is ergodic, then the unitary representation of $\mathcal{M}(X, \mathbb{T}) \rtimes G$ on $L^2(X, \mu)$ by

$$(\pi(\theta, g)f)(x) := \theta(x)f(g^{-1}.x)$$

(cf. Remark 2.2.5) is irreducible. We suggest the following steps:

(i) Any element of the commutant of $\pi(\mathcal{M}(X, \mathbb{T}))$ coincides with $\rho(h)f := hf$ for some $h \in L^\infty(X, \mu)$. Hint: Exercise 4.1.5 implies $\mathcal{M}(X, \mathbb{T})$ has the same commutant as $L^\infty(X, \mu)$; then use Exercise 4.2.1.

(ii) If $\rho(h)$ commutes with $\pi(G)$, then h coincides μ-almost everywhere with a constant function. Hint: Here we need the countability of G to select for an invariant class in $L^\infty(X, \mu)$ a G-invariant measurable function in this class.

(iii) Use Schur's Lemma to conclude that π is irreducible because

$$\pi(\mathcal{M}(X, \mathbb{T}) \rtimes G)' = \mathbb{C}\mathbf{1}.$$

Exercise 4.2.6. Show that if B is a compact subset of a Banach space E, then its closed convex hull $K := \overline{\mathrm{conv}(B)}$ is also compact. Hint: Since we are dealing with metric spaces, it suffices to show precompactness, i.e., that for each $\varepsilon > 0$, there exists a finite subset $F \subseteq K$ with $K \subseteq B_\varepsilon(F) := F + B_\varepsilon(0)$. Since B is compact, there exists a finite subset $F_B \subseteq B$ with $B \subseteq B_\varepsilon(F_B)$. Then $\mathrm{conv}(B) \subseteq \mathrm{conv}(F_B) + B_\varepsilon(0)$, and since $\mathrm{conv}(F_B)$ is compact (why?), $\mathrm{conv}(F_B) \subseteq B_\varepsilon(F)$ for a finite subset $F \subseteq \mathrm{conv}(F_B)$. This leads to $\mathrm{conv}(B) \subseteq F + B_{2\varepsilon}(0)$, which implies implies $K \subseteq F + B_{\leq 2\varepsilon}(0)$.

Exercise 4.2.7. Show that for each $n \in \mathbb{N}$ the unitary group

$$\mathrm{U}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \colon \mathbf{1} = g^*g = gg^*\}$$

is compact.

Exercise 4.2.8. Let $\mathcal{H}$ be a complex Hilbert space and $G \subseteq \mathrm{U}(\mathcal{H})_s$ be a closed subgroup. Show that G is compact if and only if $\mathcal{H}$ can be written as an orthogonal direct sum $\mathcal{H} = \widehat{\bigoplus}_{j \in J} \mathcal{H}_j$ of finite dimensional G-invariant subspaces. Hint: Use Tychonov's Theorem and Exercise 4.2.7 to see that for any family of finite dimensional Hilbert spaces $(\mathcal{H}_j)_{j \in J}$, the topological group $\prod_{j \in J} \mathrm{U}(\mathcal{H}_j)_s$ is compact.

4.3 Tensor Products of Unitary Representations

In this section we define tensor products of Hilbert spaces and of unitary representations. Our motivation for discussing tensor products are twofold. First, they provide a natural construction for new unitary representations of product groups $G_1 \times G_2$ from representations of the factor groups G_1 and G_2. Second, they can also be used to give natural descriptions of isotypical representations of a group G in terms of a multiplicity space on which G acts trivially and the corresponding irreducible representation.

4.3.1 Tensor Products of Hilbert Spaces

Definition 4.3.1. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then

$$K\big((x,y),(x',y')\big) := \langle x', x\rangle\langle y', y\rangle$$

defines on the product set $\mathcal{H} \times \mathcal{K}$ a positive definite kernel because it is a product of two positive definite kernels (Proposition 3.2.1(d)). The corresponding Hilbert space in $\mathbb{C}^{\mathcal{H}\times\mathcal{K}}$ is called the *tensor product* of $\mathcal{H}$ and $\mathcal{K}$ and is denoted by $\mathcal{H}\widehat{\otimes}\mathcal{K}$.

Then the realization map

$$\gamma \colon \mathcal{H} \times \mathcal{K} \to \mathcal{H}\widehat{\otimes}\mathcal{K}, \quad \gamma(x,y) := K_{(x,y)}$$

is bilinear and we write

$$x \otimes y := K_{(x,y)}$$

for the image of (x,y) under this map. These elements span a dense subspace and their scalar products are given by

$$\langle x \otimes y, x' \otimes y'\rangle = \langle K_{(x,y)}, K_{(x',y')}\rangle = K((x',y'),(x,y)) = \langle x, x'\rangle\langle y, y'\rangle.$$

Remark 4.3.2. (a) One can obtain a more concrete picture of the tensor product by choosing orthonormal bases $(e_j)_{j\in J}$ in $\mathcal{H}$ and $(f_k)_{k\in K}$ in $\mathcal{K}$. Then the family $(e_j \otimes f_k)_{(j,k)\in J\times K}$ is orthonormal in the tensor product and spans a dense subspace, so that it is an orthonormal basis. That it spans a dense subspace follows directly from the continuity of the bilinear map γ (Proposition 3.3.5), which implies that

$$x \otimes y = \gamma(x,y) = \sum_{j\in J, k\in K} \langle x, e_j\rangle\langle y, f_k\rangle \cdot e_j \otimes f_k.$$

(b) Similarly, we find that the subspaces $e_j \otimes \mathcal{K}$ of $\mathcal{H}\widehat{\otimes}\mathcal{K}$ are pairwise orthogonal and span a dense subspace, so that

$$\mathcal{H}\widehat{\otimes}\mathcal{K} \cong \widehat{\bigoplus}_{j\in J}(e_j \otimes \mathcal{K})$$

(cf. Exercise 1.3.5). In addition, we have

$$\langle e_j \otimes v, e_j \otimes w\rangle = \langle v, w\rangle,$$

so that the inclusion maps

$$\mathcal{K} \to \mathcal{H}\widehat{\otimes}\mathcal{K}, \quad v \mapsto e_j \otimes v$$

are isometric embeddings. This implies that

$$\mathcal{H}\widehat{\otimes}\mathcal{K} \cong \ell^2(J, \mathcal{K})$$

(cf. Example 1.3.8).

(c) With a slight generalization, we can form tensor products of finitely many Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ by using the kernel

$$K(x, x') := \prod_{j=1}^n \langle x'_j, x_j \rangle \quad \text{for} \quad x = (x_1, \ldots, x_n) \in \prod_{j=1}^n \mathcal{H}_j,$$

which leads to a reproducing kernel space

$$\widehat{\otimes}_{j=1}^n \mathcal{H}_j := \mathcal{H}_K \subseteq \mathbb{C}^{\prod_{j=1}^n \mathcal{H}_j}.$$

One easily verifies that

$$\widehat{\otimes}_{j=1}^n \mathcal{H}_j \cong \big(\widehat{\otimes}_{j=1}^{n-1} \mathcal{H}_j\big)\widehat{\otimes}\mathcal{H}_n,$$

so that an alternative construction is to apply the construction of twofold tensor products several times.

To form tensor products of two representations, we first verify that pairs of linear operators define operators on the tensor product space:

Lemma 4.3.3. *Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. Then there exists a unique bounded linear operator $A \otimes B$ on $\mathcal{H}\widehat{\otimes}\mathcal{K}$ with*

$$(A \otimes B)(v \otimes w) := (Av) \otimes (Bw) \quad \textit{for} \quad v \in \mathcal{H}, w \in \mathcal{K}. \tag{4.2}$$

It satisfies

$$\|A \otimes B\| \leq \|A\|\|B\| \quad \textit{and} \quad (A \otimes B)^* = A^* \otimes B^*.$$

Proof. Since the elements $v \otimes w$ span a dense subspace, the operator $A \otimes B$ is uniquely determined by (4.2). It therefore remains to show its existence. To this end, we fist consider the case $A = \mathbf{1}$.

Identifying $\mathcal{H}\widehat{\otimes}\mathcal{K}$ with $\ell^2(J, \mathcal{K})$ (Remark 4.3.2(b)), we see that B defines an operator $\widetilde{B}$ on $\ell^2(J, \mathcal{K}) \cong \widehat{\oplus}_{j \in J}\mathcal{K}$ by $\widetilde{B}(x_j) := (Bx_j)$, and $\|\widetilde{B}\| = \|B\|$ (Exercise 4.2.3). This proves the existence of $\mathbf{1} \otimes B$. We likewise obtain an operator $A \otimes \mathbf{1}$ with $\|A \otimes \mathbf{1}\| = \|A\|$, and we now put

$$A \otimes B := (A \otimes \mathbf{1})(\mathbf{1} \otimes B).$$

It satisfies

$$(A \otimes B)(v \otimes w) = (A \otimes \mathbf{1})(\mathbf{1} \otimes B)(v \otimes w) = (A \otimes \mathbf{1})(v \otimes Bw) = Av \otimes Bw$$

and

$$\|A \otimes B\| = \|(A \otimes \mathbf{1})(\mathbf{1} \otimes B)\| \leq \|A \otimes \mathbf{1}\|\|\mathbf{1} \otimes B\| \leq \|A\| \cdot \|B\|.$$

From

$$\begin{aligned}\langle (A \otimes B)(v \otimes w), v' \otimes w' \rangle &= \langle Av, v' \rangle \langle Bw, w' \rangle = \langle v, A^* v' \rangle \langle w, B^* w' \rangle \\ &= \langle v \otimes w, (A^* \otimes B^*)(v' \otimes w') \rangle\end{aligned}$$

we derive that $(A \otimes B)^* = A^* \otimes B^*$. □

Lemma 4.3.4. *Let $(\pi_j, \mathcal{H}_j)$ be a continuous unitary representation of G_j for $j = 1, 2$. Then*

$$(\pi_1 \otimes \pi_2)(g_1, g_2) := \pi_1(g_1) \otimes \pi_2(g_2)$$

defines a continuous unitary representation of the product group $G_1 \times G_2$ in $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$.

Proof. From the uniqueness part of Lemma 4.3.3 we derive that

$$(\pi_1(g_1) \otimes \pi_2(g_2))(\pi_1(h_1) \otimes \pi_2(h_2)) = \pi_1(g_1 h_1) \otimes \pi_2(g_2 h_2)$$

and

$$(\pi_1(g_1) \otimes \pi_2(g_2))^* = \pi_1(g_1^{-1}) \otimes \pi_2(g_2^{-1}).$$

Therefore $\pi_1 \otimes \pi_2$ is a unitary representation of $G_1 \times G_2$.

For $v_j, w_j \in \mathcal{H}_j$, we further have

$$\langle (\pi_1 \otimes \pi_2)(g_1, g_2)(v_1 \otimes w_1), v_2 \otimes w_2 \rangle = \langle \pi_1(g_1) v_1, w_1 \rangle \langle \pi_2(g_2) v_2, w_2 \rangle,$$

which is a continuous function on $G_1 \times G_2$. Now the continuity of $\pi_1 \otimes \pi_2$ follows from the fact that the elements $v_1 \otimes v_2, v_1 \in \mathcal{H}_1, v_2 \in \mathcal{H}_2$, form a total subset (Lemma 1.2.6). □

Definition 4.3.5. If $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are unitary representations of the same group G, then we define their *tensor product* as the representation on $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$, given by

$$(\pi_1 \otimes \pi_2)(g) := \pi_1(g) \otimes \pi_2(g).$$

This corresponds to the restriction of the tensor product representation of $G \times G$ to the diagonal $\Delta_G = \{(g, g) \colon g \in G\} \cong G$.

Definition 4.3.6. Let $(\rho, \mathcal{H})$ be a continuous unitary representation of G and $(\pi, \mathcal{H}_\pi)$ be an irreducible one. Then the Banach space $\mathcal{M}_\pi := B_G(\mathcal{H}_\pi, \mathcal{H})$ is called the *multiplicity space for π in ρ.*

From Schur's Lemma we know that $B_G(\mathcal{H}_\pi) = \mathbb{C}\mathbf{1}$, so that we obtain a sesquilinear map

$$\langle \cdot, \cdot \rangle \colon \mathcal{M}_\pi \times \mathcal{M}_\pi \to \mathbb{C}, \quad B^* A = \langle A, B \rangle \mathbf{1}.$$

If $A \neq 0$ and $v \in \mathcal{H}_\pi$ is a unit vector, then $Av \neq 0$ because otherwise $A\pi(G)v = \pi(G)Av = 0$ leads to $A = 0$. We therefore obtain

$$\langle A, A \rangle = \langle A^* A v, v \rangle = \|Av\|^2 > 0,$$

showing that $\langle \cdot, \cdot \rangle$ is positive definite on $\mathcal{M}_\pi$, turning $\mathcal{M}_\pi$ into a pre-Hilbert space. This argument also shows that the evaluation map

$$\mathrm{ev}_v \colon \mathcal{M}_\pi \to \mathcal{H}, \quad A \mapsto Av$$

is an isometric embedding.

Proposition 4.3.7. *The multiplicity space $\mathcal{M}_\pi$ is a Hilbert space and the evaluation map induces a unitary map*

$$\mathrm{ev} \colon \mathcal{M}_\pi \widehat{\otimes} \mathcal{H}_\pi \to \mathcal{H}_{[\pi]}, \quad A \otimes v \mapsto Av$$

which is an equivalence of unitary representations if $\mathcal{M}_\pi \widehat{\otimes} \mathcal{H}_\pi$ is endowed with the representation $\mathbf{1} \otimes \pi$ of G.

Proof. Replacing $\mathcal{H}$ by the isotypic component $\mathcal{H}_{[\pi]}$ does not change the space $\mathcal{M}_\pi$ (Remark 4.2.13), so that we may w.l.o.g. assume that $\mathcal{H} = \mathcal{H}_{[\pi]}$. In view of Proposition 4.2.11, we can write $\mathcal{H}$ as $\widehat{\bigoplus}_{j \in J} \mathcal{H}_\pi \cong \ell^2(J, \mathbb{C}) \widehat{\otimes} \mathcal{H}_\pi$ for some set J. For each $x \in \ell^2(J, \mathbb{C})$ we then obtain an element $A_x \in \mathcal{M}_\pi$ by $A_x(w) := x \otimes w$ for $w \in \mathcal{H}_\pi$. In particular, we obtain for the unit vector v from above the relation $A_x(v) = x \otimes v$ with $\|A_x\|^2 = \|v \otimes x\|^2 = \|x\|^2$. Conversely, we have for each $j \in J$ an intertwining operator $P_j \colon \mathcal{H} \to \mathcal{H}_\pi, x \otimes w \mapsto \langle x, e_j \rangle e_j \otimes w$, corresponding to the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_j \cong e_j \otimes \mathcal{H}_\pi$. For each $A \in \mathcal{M}_\pi$ we then have $P_j A \in B_G(\mathcal{H}_\pi) = \mathbb{C}\mathbf{1}$, so that

$$Av = \sum_{j \in J} P_j Av \in \sum_{j \in J} \mathbb{C} e_j \otimes v \subseteq \ell^2(J, \mathbb{C}) \otimes v$$

implies that $Av = x \otimes v$ for some $x \in \ell^2(J, \mathbb{C})$. This proves that $A = A_x$. Collecting all this information, we see that

$$\mathrm{ev}_v \colon \mathcal{M}_\pi \to \ell^2(J, \mathbb{C}) \otimes v \cong \ell^2(J, \mathbb{C})$$

is unitary, and therefore that $\mathcal{M}_\pi$ is a Hilbert space isomorphic to $\ell^2(J, \mathbb{C})$.

The preceding discussion also shows that the evaluation map

$$\mathcal{M}_\pi \otimes \mathcal{H}_\pi \to \mathcal{H}, \quad A \otimes v \mapsto Av$$

induces an isomorphism of Hilbert spaces

$$\mathcal{M}_\pi \widehat{\otimes} \mathcal{H}_\pi \to \mathcal{H},$$

and even an equivalence of representations if $\mathcal{M}_\pi \widehat{\otimes} \mathcal{H}_\pi$ is endowed with the representation $\mathbf{1} \otimes \pi$. □

The main advantage of the multiplicity space, as compared to the description of $\mathcal{H}_{[\pi]}$ as a tensor product $\ell^2(J, \mathbb{C}) \widehat{\otimes} \mathcal{H}_\pi$, is that $\mathcal{M}_\pi$ and its scalar product are naturally defined in terms of the irreducible representation $(\pi, \mathcal{H}_\pi)$ and the representation $(\rho, \mathcal{H})$. We don't have to refer to Zorn's Lemma to define it.

Lemma 4.3.8. *If $(\pi, \mathcal{H})$ is an irreducible representation of G and $\rho := \mathbf{1} \otimes \pi$ is the tensor product with the trivial representation $\mathbf{1}$ on $\mathcal{M}$, then*

$$\rho(G)' = B(\mathcal{M}) \otimes \mathbf{1} \cong B(\mathcal{M}).$$

Proof. Let $(e_j)_{j\in J}$ be an ONB in $\mathcal{M}$, so that $\mathcal{M} = \ell^2(J, \mathbb{C})$, $\mathcal{M}\widehat{\otimes}\mathcal{H} \cong \bigoplus_{j\in J} \mathcal{H}$ and $\rho \cong \oplus_{j\in J}\pi$. We write

$$P_j \colon \mathcal{M}\widehat{\otimes}\mathcal{H} \to \mathcal{H}, \quad x \otimes v \mapsto \langle x, e_j\rangle w$$

for the projections which are in particular intertwining operators.

Clearly, $B(\mathcal{M}) \otimes \mathbf{1} \subseteq \rho(G)'$. If, conversely, $A \in \rho(G)'$, then, for $i, j \in J$, $A_{ij} := P_i A P_j^* \in B_G(\mathcal{H}) = \mathbb{C}\mathbf{1}$ by Schur's Lemma. Let $a_{ij} \in \mathbb{C}$ with $A_{ij} = a_{ij}\mathbf{1}$ and $v, w \in \mathcal{H}$ be unit vectors. Then

$$\langle A(e_j \otimes v), e_i \otimes w\rangle = \langle a_{ij} v, w\rangle = a_{ij}\langle v, w\rangle$$

implies that the closed subspaces $\mathcal{M} \otimes v \cong \mathcal{M}$ are A-invariant with

$$A(e_j \otimes v) = \Big(\sum_{i\in I} a_{ij} e_i\Big) \otimes v.$$

Therefore the matrix $(a_{ij})_{i,j\in J}$ defines a bounded operator $\widetilde{A}$ on $\mathcal{M}$ with $A = \widetilde{A} \otimes \mathbf{1}$. This proves that $A \in B(\mathcal{M}) \otimes \mathbf{1}$. □

Proposition 4.3.9. *If $(\pi_j, \mathcal{H}_j)$, $j = 1, 2$, are irreducible unitary representations of the groups G_j, then the tensor product representation $\pi_1 \otimes \pi_2$ of $G_1 \times G_2$ on $\mathcal{H}_1\widehat{\otimes}\mathcal{H}_2$ is irreducible.*

Proof. Lemma 4.3.8 implies that

$$\begin{aligned}(\pi_1 \otimes \pi_2)(G_1 \times G_2)' &= (\pi_1(G_1) \otimes \mathbf{1})' \cap (\mathbf{1} \otimes \pi_2(G))' \\ &= (\mathbf{1} \otimes B(\mathcal{H}_2)) \cap (B(\mathcal{H}_1) \otimes \mathbf{1}).\end{aligned}$$

To see that this is not larger than $\mathbb{C}\mathbf{1}$, let $A \in B(\mathcal{H}_1)$ and $B \in B(\mathcal{H}_2)$ with $C := A \otimes \mathbf{1} = \mathbf{1} \otimes B$. Let $(e_j)_{j\in J}$ be an ONB in $\mathcal{H}_1$ and $(f_k)_{k\in K}$ be an ONB in $\mathcal{H}_2$. Then this operator preserves all subspaces $\mathcal{H}_1 \otimes f_k$ and $e_j \otimes \mathcal{H}_2$, so that

$$(\mathcal{H}_1 \otimes f_k) \cap (e_j \otimes \mathcal{H}_2) = \mathbb{C}e_j \otimes f_k$$

(Remark 4.3.2(a)) implies that all elements $e_j \otimes f_k$ are eigenvectors. Write $C(e_j \otimes f_k) = c_{jk}(e_j \otimes f_k)$. Then $C = A \otimes \mathbf{1}$ implies that c_{jk} does not depend on k, and $C = \mathbf{1} \otimes B$ implies that it does not depend on j. Therefore $C = c\mathbf{1}$ for $c = c_{jk}$. □

4.3.2 Types of Representations

Definition 4.3.10. Let $(\pi, \mathcal{H})$ be a representation of an involutive semigroup $(S, *)$. It is said to be

(i) *multiplicity free* if its commutant $\pi(S)'$ is commutative, i.e.,

$$\pi(S)' \subseteq \pi(S)''.$$

(ii) *a factor representation* or *primary* if

$$Z(\pi(S)') := \pi(S)' \cap \pi(S)'' = \mathbb{C}\mathbf{1},$$

i.e., the von Neumann algebra $\pi(S)'$, resp., $\pi(S)''$ is a factor (cf. Remark 4.2.5).

(iii) *a factor representation of type I* if $\mathcal{H} = \mathcal{H}_{[\rho]}$ for an irreducible representation $(\rho, \mathcal{H}_\rho)$, i.e.,

$$\mathcal{H} \cong \mathcal{M}_\rho \widehat{\otimes} \mathcal{H}_\rho \quad \text{with} \quad \pi(s) = \mathbf{1} \otimes \rho(s) \quad \text{for} \quad s \in S.$$

(iv) A topological group G is said to be *tame* or of *type I* if all its unitary factor representations are of type I.

Remark 4.3.11. To understand the terminology introduced above, it is instructive to consider the special case where $\mathcal{H} = \mathcal{H}_d$, i.e., the representation $(\rho, \mathcal{H})$ is a direct sum of irreducible representations. We combine Remark 4.2.13 and Lemma 4.3.8 to see that

$$\rho(G)' = B_G(\mathcal{H}) \cong \oplus^\infty_{[\pi] \in \widehat{G}} B_G(\mathcal{H}_{[\pi]}) \cong \oplus^\infty_{[\pi] \in \widehat{G}} B(\mathcal{M}_\pi)$$

is an ℓ^∞-direct sum. In view of $Z(B(\mathcal{M})) = \mathbb{C}\mathbf{1}$ for any non-zero Hilbert space $\mathcal{K}$ (Example 4.2.6), we have

$$Z(B_G(\mathcal{H})) \cong \oplus^\infty_{[\pi] \in \widehat{G}} \mathbb{C}\mathbf{1} \cong \ell^\infty(J, \mathbb{C}) \quad \text{for} \quad J := \{[\pi] \in \widehat{G} \colon \mathcal{H}_{[\pi]} \neq \{0\}\}.$$

(a) That the representation of $\mathcal{H}$ is multiplicity free means that $B_G(\mathcal{H})$ is commutative, which in turn is equivalent to $\dim M_\pi \leq 1$ for each $[\pi] \in \widehat{G}$. This means that the representation on $\mathcal{H}_{[\rho]}$ is irreducible and that $(\rho, \mathcal{H})$ is a direct sum of pairwise non-equivalent irreducible representations.

(b) We also see that $(\rho, \mathcal{H})$ is a factor representation if and only if $\ell^\infty(J, \mathbb{C}) \cong \mathbb{C}$, i.e., $|J| = 1$. This means that $\mathcal{H} = \mathcal{H}_{[\pi]}$ is an isotypic representation. In particular, the isotypic components of any representation are factor representations. By definition, these are the factor representations of type I.

(c) If $(\pi, \mathcal{H})$ is a finite dimensional factor representation, then it is of type I because it is a direct sum of irreducible ones (Proposition 1.3.11).

Remark 4.3.12. (a) The Fundamental Theorem on Unitary Representations of Compact Groups 4.2.20 implies in particular that compact groups are tame.

(b) If G is abelian and $(\pi, \mathcal{H})$ a unitary representation, then $\pi(G) \subseteq \pi(G)'$ implies that $\pi(G) \subseteq Z(\pi(G)')$. If, in addition, $(\pi, \mathcal{H})$ is a factor representation, then $\pi(G) \subseteq \mathbb{C}\mathbf{1}$. This implies that $(\pi, \mathcal{H})$ is a factor representation if and only if there exists a character $\chi \in \widehat{G}$ with $\pi(g) = \chi(g)\mathbf{1}$ for $g \in G$. It follows in particular that all factor representations are of type I, so that abelian groups are tame.

(c) The general idea is that the tameness condition for topological groups means that the two fundamental problems of representation theory are well-posed for G. Later we shall make this statement more explicit.

Two central results in the representation theory of locally compact groups assert that a discrete group G is tame if and only if it possesses an abelian normal subgroup of finite index (cf. [Fo05, Thm. 7.8]). This means that if discrete groups are "too large" or "too non-commutative", then they are not tame.

On the positive side, one knows that if $G \subseteq \mathrm{GL}_n(\mathbb{R})$ is a subgroup which is *algebraic* in the sense that it is the common zero set of a family $(p_j)_{j \in J}$ of polynomials in the n^2 matrix entries, then G is tame (cf. [Fo05, Thm. 7.8]).

Proposition 4.3.13. *Suppose that $(\pi, \mathcal{H})$ is an irreducible representation of the product group $G = G_1 \times G_2$. Then $\pi|_{G_1}$ and $\pi|_{G_2}$ are factor representations. If one of these is of type I, then there exist irreducible representations $(\pi_j, \mathcal{H}_j)$ of G_j, $j = 1, 2$, with*

$$\pi \cong \pi_1 \otimes \pi_2.$$

Proof. We identify G_1 and G_2 with the corresponding subgroups of G. In view of Schur's Lemma, we then have

$$\mathbb{C}\mathbf{1} = \pi(G)' = (\pi(G_1)\pi(G_2))' = \pi(G_1)' \cap \pi(G_2)'.$$

Therefore $\pi(G_2) \subseteq \pi(G_1)'$ implies that

$$\pi(G_1)' \cap \pi(G_1)'' \subseteq \pi(G_1)' \cap \pi(G_2)' = \mathbb{C}\mathbf{1},$$

which means that $\pi|_{G_1}$ is a factor representation. A similar argument shows that $\pi|_{G_2}$ is a factor representation.

If $\pi|_{G_1}$ is of type I, then we accordingly have $\mathcal{H} \cong \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$, where $\pi(g_1) = \pi_1(g_1) \otimes \mathbf{1}$ for $g_1 \in G_1$ and an irreducible representation $(\pi_1, \mathcal{H}_1)$ of G_1. Then Lemma 4.3.8 implies that

$$\pi(G_1)' = (\pi_1(G_1) \otimes \mathbf{1})' = \mathbf{1} \otimes B(\mathcal{H}_2),$$

and since $\pi(G_2) \subseteq \pi(G_1)'$, we thus obtain a unitary representation $\pi_2 \colon G_2 \to \mathrm{U}(\mathcal{H}_2)$ with

$$\pi(g_2) = \mathbf{1} \otimes \pi_2(g_2) \quad \text{for} \quad g_2 \in G_2.$$

Now $\mathbf{1} \otimes \pi_2(G_2)' \subseteq \pi(G)' = \mathbb{C}\mathbf{1}$ implies that $\pi_2(G_2)' = \mathbb{C}\mathbf{1}$, so that $(\pi_2, \mathcal{H}_2)$ is irreducible, and we clearly have $\pi \cong \pi_1 \otimes \pi_2$. □

Corollary 4.3.14. *If $G = G_1 \times G_2$ is a direct product group and one factor is tame, then the map*

$$\Gamma \colon \widehat{G}_1 \times \widehat{G}_2 \to \widehat{G}, \quad ([\pi_1], [\pi_2]) \mapsto [\pi_1 \otimes \pi_2]$$

is a bijection.

Proof. First we observe that Γ is well defined because equivalent unitary representations have equivalent tensor products (Exercise). Next, the preceding Proposition 4.3.13 asserts that Γ is surjective. To see that it is also injective, assume w.l.o.g. that G_1 is tame. If the representations $\pi_1 \otimes \pi_2$ and $\rho_1 \otimes \rho_2$ of G are equivalent, then the restrictions to G_1 are equivalent isotypic representations, so that Remark 4.2.13 implies that $\pi_1 \sim \rho_1$. The same argument applies to the restriction to G_2, which leads to $\pi_2 \sim \rho_2$. $\square$

Exercises for Section 4.3

Exercise 4.3.1. We have defined the tensor product $\mathcal{H}\widehat{\otimes}\mathcal{K}$ of two Hilbert spaces as a space of functions on the product $\mathcal{H} \times \mathcal{K}$, defined by the kernel

$$K((x', y'), (x, y)) = \langle x, x' \rangle \langle y, y' \rangle.$$

Show that $\mathcal{H}\widehat{\otimes}\mathcal{K}$ consists of continuous maps which are biantilinear, i.e., antilinear in each argument.

Exercise 4.3.2. Show that if $(\pi, \mathcal{H})$ is a factor representation of G and there exists an irreducible subrepresentation $\mathcal{H}_1 \subseteq \mathcal{H}$, then $(\pi, \mathcal{H})$ is of type I. Hint: Consider the decomposition $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c$ into continuous and discrete part and show that $\mathcal{H}_c$ is trivial.

Chapter 5

Representations on Reproducing Kernel Spaces

In this chapter we combine the concepts of the preceding two chapters. First we explain how group actions on a space X lead to unitary representations on reproducing kernel spaces on X (Section 5.1) and discuss a variety of examples in Section 5.2. A key advantage of this general setup is that it specializes to many interesting settings. In particular, we shall see in Section 5.3 how cyclic continuous unitary representations are encoded in positive definite functions.

In Section 5.1 we also describe the commutant of a representation on a reproducing kernel space in terms of invariance conditions on certain kernels. This technique provides a simple direct way for verifications of irreducibility in many important contexts.

5.1 From Cocycles to Unitary Representations

If X is a set, then the group S_X of all bijections of X, the *symmetric group on* X, acts by $(\varphi, \theta) \mapsto \varphi_*\theta = \theta \circ \varphi^{-1}$ on the group $(\mathbb{K}^\times)^X$, so that we can form the semidirect product group $(\mathbb{K}^\times)^X \rtimes S_X$ with the multiplication

$$(\theta, \varphi)(\theta', \varphi') = (\theta \cdot (\varphi_*\theta'), \varphi\varphi') \quad \text{and} \quad (\theta, \varphi)^{-1} = ((\varphi^{-1})_*\theta^{-1}, \varphi^{-1}).$$

This semidirect product group acts in a natural way on the vector space $\mathbb{K}^X$ of complex-valued functions on X, given by

$$\big(\pi(\theta, \varphi)f\big)(x) := \theta(x)f(\varphi^{-1}(x)), \quad \pi(\theta, \varphi)f = \theta \cdot \varphi_* f.$$

If $\mathcal{H}_K \subseteq \mathbb{K}^X$ is a reproducing kernel space, we are now interested in a characterization of those pairs (θ, φ) for which $\pi(\theta, \varphi)$ leaves $\mathcal{H}_K$ invariant and induces a unitary operator on this space.

Lemma 5.1.1. *Let $\mathcal{H}_K \subseteq \mathbb{K}^X$ be a reproducing kernel space, $\theta\colon X \to \mathbb{K}^\times$ a function and $\varphi\colon X \to X$ a bijection. Then $\pi(\theta,\varphi)$ preserves $\mathcal{H}_K$ and restricts to a unitary operator on this space if and only if*

$$K(\varphi(x),\varphi(y)) = \theta(\varphi(x))K(x,y)\overline{\theta(\varphi(y))} \quad \textit{for} \quad x,y \in X, \tag{5.1}$$

which is equivalent to

$$\pi(\theta,\varphi)K_x = \overline{\theta(\varphi(x))}^{-1}K_{\varphi(x)} \quad \textit{for} \quad x \in X. \tag{5.2}$$

Proof. Condition (5.2) can be written as

$$\theta(\varphi(x))K(x,y) = \theta(\varphi(x))K_y(x) = \big(\pi(\theta,\varphi)K_y\big)(\varphi(x)) = \overline{\theta(\varphi(y))}^{-1}K_{\varphi(y)}(\varphi(x))$$

for $x,y \in X$, and this is equivalent to (5.1).

If $\mathcal{H}_K$ is invariant under $\pi(\theta,\varphi)$ and it restricts to a unitary operator on $\mathcal{H}_K$, we obtain for $f \in \mathcal{H}_K$ and $x \in X$:

$$\begin{aligned}&\langle f, \pi(\theta,\varphi)K_x\rangle = \langle \pi(\theta,\varphi)^{-1}f, K_x\rangle = (\pi(\theta,\varphi)^{-1}f)(x)\\&= (\pi(((\varphi^{-1})_*\theta)^{-1},\varphi^{-1})f)(x) = \theta(\varphi(x))^{-1}f(\varphi(x)) = \langle f, \overline{\theta(\varphi(x))}^{-1}K_{\varphi(x)}\rangle\end{aligned}$$

which is (5.2).

Suppose, conversely, that (5.1) and (5.2) hold. For

$$\gamma(x) := K_x \quad \text{and} \quad \gamma'(x) := \overline{\theta(\varphi(x))}^{-1}K_{\varphi(x)},$$

we then find that

$$\begin{aligned}\langle\gamma'(y),\gamma'(x)\rangle &= \theta(\varphi(x))^{-1}\overline{\theta(\varphi(y))}^{-1}\langle K_{\varphi(y)},K_{\varphi(x)}\rangle\\&= \theta(\varphi(x))^{-1}\overline{\theta(\varphi(y))}^{-1}K(\varphi(x),\varphi(y)) = K(x,y) = \langle\gamma_y,\gamma_x\rangle.\end{aligned}$$

Therefore $\pi(\theta,\varphi)$ coincides on $\mathcal{H}_K^0$ with the unique unitary map $\Phi\colon \mathcal{H}_K \to \mathcal{H}_K$ satisfying $\Phi\circ\gamma = \gamma'$ (Theorem 3.3.3). In Definition 3.3.4 we have seen that Φ is given on any $f \in \mathcal{H}_K$ by the formula

$$\begin{aligned}\Phi(f)(\varphi(x)) &= \langle\Phi(f),K_{\varphi(x)}\rangle = \langle\Phi(f),\overline{\theta(\varphi(x))}\Phi(K_x)\rangle\\&= \theta(\varphi(x))\langle f,K_x\rangle = \theta(\varphi(x))f(x) = (\pi(\theta,\varphi)f)(\varphi(x)).\end{aligned}$$

This proves that $\pi(\theta,\varphi)$ leaves $\mathcal{H}_K$ invariant and restricts to a unitary map on this space. □

Definition 5.1.2. Composing in each argument with φ^{-1}, (5.1) can also be written as

$$K(x,y) = \theta(x)K(\varphi^{-1}(x),\varphi^{-1}(y))\overline{\theta(y)} \quad \text{for} \quad x,y \in X,$$

and this means that K is invariant under the action of the group $(\mathbb{K}^\times)^X \rtimes S_X$ on the set $\mathbb{K}^{X\times X}$ of kernels, given by

$$((\theta,\varphi).K)(x,y) := \theta(x)K(\varphi^{-1}(x),\varphi^{-1}(y))\overline{\theta(y)}.$$

In particular, the stabilizer of K with respect to this action is a subgroup

$$\mathrm{Aut}(X,K) := \{(f,\varphi) \in (\mathbb{K}^\times)^X \rtimes S_X : (f,\varphi).K = K\},$$

called the *automorphism group of the pair* (X,K).

From the preceding Lemma 5.1.1, we immediately derive that, if K is positive definite, then

$$(\pi_K(\theta,\varphi)f)(x) := \theta(x)f(\varphi^{-1}(x)) \tag{5.3}$$

defines a unitary representation of $\mathrm{Aut}(X,K)$ on the reproducing kernel Hilbert space $\mathcal{H}_K$ and that it is the maximal subgroup of $(\mathbb{K}^\times)^X \rtimes S_X$ with this property.

Example 5.1.3. If $\mathbb{K} = \mathbb{C}$ and $K(x,y) = \delta_{x,y}$ is the δ-kernel on the set X, then every bijection $\varphi \in S_X$ preserves this kernel, which leads to

$$\mathrm{Aut}(X,K) = \mathbb{T}^X \rtimes S_X.$$

Definition 5.1.4. Let $\sigma\colon G \times X \to X, (g,x) \mapsto \sigma_g(x) = g.x$ be an action of G on X. Then σ defines a homomorphism $\sigma\colon G \to S_X, g \mapsto \sigma_g$, and to obtain a homomorphism

$$\widetilde{\sigma} = (J,\sigma)\colon G \to (\mathbb{K}^\times)^X \rtimes S_X,$$

the map $J\colon G \to (\mathbb{K}^\times)^X$ needs to be a 1-cocycle, i.e.,

$$J(gh) = J(g) \cdot g_* J(h) \quad \text{for} \quad g,h \in G$$

(cf. (2.6)). Here we simply write g_* instead of $(\sigma_g)_*$. In the following we often write $J(g,x) := J(g)(x)$, so that the cocycle property for the function $J\colon G \times X \to \mathbb{K}^\times$ reads

$$J(gh,x) = J(g,x)J(h,g^{-1}.x) \quad \text{for} \quad g,h \in G, x \in X. \tag{5.4}$$

Remark 5.1.5. The cocycle condition implies in particular that $J(\mathbf{1},x) = J(\mathbf{1},x)^2$, so that $J(\mathbf{1},x) = 1$ holds for each $x \in X$. This in turn implies that

$$J(g,x)^{-1} = J(g^{-1},g^{-1}.x) \quad \text{for} \quad g \in G, x \in X. \tag{5.5}$$

Proposition 5.1.6. *Let $K \in \mathcal{P}(X,\mathbb{K})$ be a positive definite kernel, $\sigma\colon G \times X \to X$ be a group action and $J\colon G \times X \to \mathbb{K}^\times$ be a 1-cocycle. Then*

$$(\pi_K(g)f)(x) := J(g,x)f(g^{-1}.x)$$

defines a unitary representation of G on $\mathcal{H}_K$ if and only if K satisfies the invariance condition

$$K(g.x,g.y) = J(g,g.x)K(x,y)\overline{J(g,g.y)} \quad \textit{for} \quad g \in G, x,y \in X, \tag{5.6}$$

which is equivalent to

$$\pi_K(g)K_x = \overline{J(g^{-1},x)}K_{g.x} \quad \textit{for} \quad g \in G, x \in X. \tag{5.7}$$

If these conditions are satisfied, we further have:

(a) *If, in addition, X is a topological space, G a topological group, and σ, J and K are continuous, then the representation $(\pi_K, \mathcal{H}_K)$ of G is continuous.*

(b) *Any G-invariant closed subspace $\mathcal{K} \subseteq \mathcal{H}_K$ is a reproducing kernel space $\mathcal{H}_Q$ whose kernel Q satisfies*

$$Q(g.x, g.y) = J(g, g.x)Q(x, y)\overline{J(g, g.y)} \quad \textit{for} \quad g \in G, x, y \in X. \tag{5.8}$$

Proof. The fist part follows immediately from Lemma 5.1.1, applied to $(\theta, \varphi) = (J(g), \sigma_g)$ for $g \in G$ and the relation $J(g, g.x)^{-1} = J(g^{-1}, x)$ (Remark 5.1.5).

(a) We apply Lemma 1.2.6 to the total subset $E := \{K_x : x \in X\}$. For $x, y \in X$ we have

$$\langle \pi(g)K_y, K_x\rangle = (\pi(g)K_y)(x) = \overline{J(g^{-1}, y)}K_{g.y}(x) = \overline{J(g^{-1}, y)}K(x, g.y),$$

which depends continuously on g. Therefore the representation $(\pi_K, \mathcal{H}_K)$ is continuous.

(b) Since the inclusion $\mathcal{K} \to \mathcal{H}_K$ is continuous, $\mathcal{K}$ has continuous point evaluations, hence is a reproducing kernel space $\mathcal{H}_Q$ (Lemma 3.1.4). By assumption, $\mathcal{H}_Q = \mathcal{K}$ is invariant under the unitary G-action defined by

$$(\pi_K(g)f)(x) = J(g, x)f(g^{-1}.x),$$

so that (5.8) follows from the first part of the proof. □

Definition 5.1.7. If (5.6) is satisfied, the cocycle J is called a *multiplier for the kernel K*.

Remark 5.1.8. The preceding proposition applies in particular if the kernel K is *G-invariant*, i.e.,

$$K(g.x, g.y) = K(x, y) \quad \text{for} \quad g \in G, x, y \in X.$$

Then we may use the cocycle $J = 1$ and obtain a unitary representation of G on $\mathcal{H}_K$ by

$$(\pi(g)f)(x) := f(g^{-1}.x), \quad f \in \mathcal{H}_K, x \in X, g \in G.$$

Commutants and Invariant Kernels

Definition 5.1.9. Let $\sigma : G\times X \to X$ be a group action and $J : G\times X \to \mathbb{K}^\times$ be a corresponding cocycle. We write $\mathcal{P}(X, \sigma, J)$ for the set of all positive definite kernels $K \in \mathcal{P}(X, \mathbb{K})$ satisfying the *J-invariance condition* (5.6):

$$K(g.x, g.y) = J(g, g.x)K(x, y)\overline{J(g, g.y)} \quad \text{for} \quad g \in G, x, y \in X. \tag{5.9}$$

Since this condition is linear in K, $\mathcal{P}(X, \sigma, J)$ is closed under sums and positive scalar multiplication, hence a convex cone.

Remark 5.1.10. If K is J-invariant positive definite and $\mathcal{H}_K = \mathcal{H}_1 \oplus \mathcal{H}_2$ is a G-invariant orthogonal decomposition into two closed subspaces and $K = K_1 + K_2$ the corresponding decomposition of K with $\mathcal{H}_j = \mathcal{H}_{K_j}$ (Exercise 3.3.1), then Proposition 5.1.6(b) implies that $K_j \in \mathcal{P}(X, \sigma, J)$ for $j = 1, 2$.

Proposition 5.1.11. (a) *For $K, L \in \mathcal{P}(X, \sigma, J)$, the relation $\mathcal{H}_L \subseteq \mathcal{H}_K$ is equivalent to the existence of a positive operator $B \in B_G(\mathcal{H}_K)$ with $L = K^B$.*

(b) *For $B \in B(\mathcal{H}_K)$, the J-invariance of K^B is equivalent to $B \in B_G(\mathcal{H}_K)$.*

Proof. (cf. [Dix64, p. 35]) In view of Theorem 3.4.7, it remains to show that an operator $B \in B(\mathcal{H}_K)$ commutes with G if and only if its symbol $K^B(x, y) = \langle BK_y, K_x \rangle = (BK_y)(x)$ is J-invariant. As we have seen in Proposition 5.1.6, the invariance condition is equivalent to

$$\pi_K(g)K_x = \overline{J(g^{-1}, x)}K_{g.x} \quad \text{for} \quad g \in G, x \in X.$$

Since $K^B_x = BK_x$, the invariance of K^B is equivalent to

$$\pi_K(g)BK_x = \overline{J(g^{-1}, x)}BK_{g.x} = B\pi_K(g)K_x \quad \text{for} \quad g \in G, x \in X.$$

Since the K_x span a dense subspace of $\mathcal{H}_K$, this condition is equivalent to $B \in \pi_K(G)'$. □

The following theorem provides an important criterion for irreducibility of representations on reproducing kernel spaces. However, it is still quite abstract and therefore not easy to apply. However, we shall see below how it can be turned into an effective tool for actions on Hilbert spaces of holomorphic functions.

Theorem 5.1.12. (Irreducibility Criterion for Reproducing Kernel Spaces) *If $0 \neq K \in \mathcal{P}(X, \sigma, J)$, then the representation $(\pi_K, \mathcal{H}_K)$ of G is irreducible if and only if $\mathbb{R}^+ K$ is an extremal ray of the convex cone $\mathcal{P}(X, \sigma, J)$.*

Proof. According to Proposition 5.1.11 and Theorem 3.4.7, the face of $\mathcal{P}(X, \sigma, J)$ generated by K is in one-to-one correspondence with the cone of positive operators in the commutant $\pi(G)'$. The kernel K generates an extremal ray if and only if this face is one-dimensional, i.e., if and only if the commutant $\pi(G)'$ is one-dimensional which in turn means that $\pi(G)' = \mathbb{C}\mathbf{1}$. Since by Schur's Lemma (Theorem 4.2.7) the latter condition is equivalent to the irreducibility of the representation $(\pi, \mathcal{H})$, the assertion follows. □

Remark 5.1.13. Let $K \in \mathcal{P}(X, \sigma, J)$, so that we have a unitary representation $(\pi_K, \mathcal{H}_K)$ of G satisfying

$$\pi(g)K_x = \overline{J(g^{-1}, x)}K_{g.x} \in \mathbb{C}^\times K_{g.x}, \quad g \in G, x \in X.$$

This implies in particular that

$$K(x, x) = \|K_x\|^2 = |J(g^{-1}, x)|^2 K(g.x, g.x).$$

If all vectors K_x are non-zero, i.e., $K(x,x) > 0$ for each $x \in X$, then we can normalize these vectors and obtain

$$\gamma(x) := \frac{1}{\sqrt{K(x,x)}} K_x, \quad x \in X.$$

Then $(X, \gamma, \mathcal{H}_K)$ also is a realization triple, and the corresponding kernel is given by

$$Q(x,y) := \langle \gamma(y), \gamma(x) \rangle = \frac{K(x,y)}{\sqrt{K(x,x)}\sqrt{K(y,y)}},$$

so that the map

$$\varphi_\gamma \colon \mathcal{H}_K \to \mathcal{H}_Q, \quad \varphi_\gamma(f)(x) := \langle f, \gamma(x) \rangle = \frac{f(x)}{\sqrt{K(x,x)}}$$

is unitary.

We now transfer the unitary representation $(\pi_K, \mathcal{H}_K)$ to $\mathcal{H}_Q$ by

$$\pi_Q(g) := \varphi_\gamma \circ \pi_K(g) \circ \varphi_\gamma^{-1},$$

so that

$$(\pi_Q(g)f)(x) = \frac{1}{\sqrt{K(x,x)}} J(g,x) \sqrt{K(g^{-1}.x, g^{-1}.x)} f(g^{-1}.x) = J_Q(g,x) f(g^{-1}.x)$$

for

$$J_Q(g,x) = J(g,x) \frac{\sqrt{K(g^{-1}.x, g^{-1}.x)}}{K(x,x)} = \frac{J(g,x)}{|J(g,x)|} \in \mathbb{T}.$$

We thus obtain an equivalent unitary representation $(\pi_Q, \mathcal{H}_Q)$ with a $\mathbb{T}$-valued multiplier. If the original multiplier has positive values, i.e., $J(g,x) > 0$ for $g \in G$ and $x \in X$, then $J_Q = 1$ and the kernel Q on X is G-invariant.

Exercises for Section 5.1

Exercise 5.1.1. Let $K \colon X \times X \to \mathbb{C}$ be a positive definite kernel and $\theta \colon X \to \mathbb{C}^\times$ a function. Determine necessary and sufficient conditions on θ such that

$$\theta(x) K(x,y) \overline{\theta(y)} = K(x,y) \quad \text{for} \quad x, y \in X.$$

Hint: Consider the subset $X_1 := \{x \in X \colon K(x,x) > 0\}$ and its complement X_0 separately.

Exercise 5.1.2. Let $K, Q \in \mathcal{P}(X, \mathbb{C})$ be positive definite kernels on X and $\theta \colon X \to \mathbb{C}^\times$. Show that

$$m_\theta \colon \mathcal{H}_K \to \mathcal{H}_Q, \quad f \mapsto \theta f$$

defines a unitary map if and only if

$$Q(x,y) = \theta(x) K(x,y) \overline{\theta(y)} \quad \text{for} \quad x, y \in X.$$

Exercise 5.1.3. Let $(V, \|\cdot\|)$ be a normed space,

$$\mathbb{P}(V) := \{[v] := \mathbb{R}v \colon 0 \neq v \in V\}$$

be the space of one-dimensional subspace of V (the projective space). Show that

(a) $g.[v] := [gv]$ defines an action of $\mathrm{GL}(V)$ on $\mathbb{P}(V)$.

(b) $J \colon \mathrm{GL}(V) \times \mathbb{P}(V) \to \mathbb{R}^\times, J(g, [v]) := \frac{\|g^{-1}v\|}{\|v\|}$ is a 1-cocycle with respect to this action.

5.2 Some Examples

Let $\mathcal{H}$ be a complex Hilbert space. We consider the kernel $K(z, w) := e^{\langle z, w\rangle}$ corresponding to the Fock space $\mathcal{F}(\mathcal{H}) := \mathcal{H}_K \subseteq \mathbb{C}^{\mathcal{H}}$. Fock spaces play a central role in operator theory and mathematical physics, in particular in Quantum Field Theory (QFT). In this section we discuss several interesting unitary representations of groups on $\mathcal{F}(\mathcal{H})$.

Since our approach is based on reproducing kernels, we start with groups acting on $\mathcal{H}$, and then discuss the cocycles that are needed to make the kernel invariant under the group action.

5.2.1 The Schrödinger Representation of the Heisenberg Group

The simplest group acting on $\mathcal{H}$ is the group of translations. For $v \in \mathcal{H}$, we write $\tau_v(x) := x + v$ for the corresponding translation. We want to associate to τ_v a unitary operator on the Hilbert space $\mathcal{H}_K$. Since the kernel K is not translation invariant, this requires a function $\theta_v \colon \mathcal{H} \to \mathbb{C}^\times$ with $(\theta_v, \tau_v) \in \mathrm{Aut}(\mathcal{H}, K)$.

To find this function, we observe that

$$\begin{aligned} K(z+v, w+v) &= e^{\langle z+v, w+v\rangle} = e^{\langle z,v\rangle} e^{\langle z,w\rangle} e^{\langle v,w\rangle} e^{\langle v,v\rangle} \\ &= e^{\langle z,v\rangle + \frac{1}{2}\langle v,v\rangle} K(z,w) e^{\langle v,w\rangle + \frac{1}{2}\langle v,v\rangle}. \end{aligned}$$

Therefore

$$\theta_v(z) := e^{\langle z-v,v\rangle + \frac{1}{2}\langle v,v\rangle} = e^{\langle z,v\rangle - \frac{1}{2}\langle v,v\rangle}$$

satisfies

$$K(z+v, z+w) = \theta_v(z+v) K(z,w) \overline{\theta_w(z+v)},$$

which means that $(\theta_v, \tau_v) \in \mathrm{Aut}(\mathcal{H}, K)$. Hence

$$(\pi(v)f)(z) := \theta_v(z) f(z-v) = e^{\langle z,v\rangle - \frac{1}{2}\langle v,v\rangle} f(z-v)$$

defines a unitary operator on $\mathcal{H}_K$ (Lemma 5.1.1).

However, this assignment does not define a unitary representation $(\mathcal{H},+) \to \mathrm{U}(\mathcal{H}_K)$ because we have in $\mathrm{Aut}(\mathcal{H},K)$ the relation

$$(\theta_v,\tau_v)(\theta_w,\tau_w) = (\theta_v \cdot (\tau_v)_*\theta_w, \tau_{v+w}) \neq (\theta_{v+w},\tau_{v+w})$$

because

$$\begin{aligned}(\theta_v \cdot (\tau_v)_*\theta_w)(z) &= e^{\langle z,v\rangle - \frac{1}{2}\langle v,v\rangle} e^{\langle z-v,w\rangle - \frac{1}{2}\langle w,w\rangle} \\ &= e^{\langle z,v+w\rangle - \frac{1}{2}\langle v+w,v+w\rangle} e^{-\langle v,w\rangle + \frac{1}{2}(\langle v,w\rangle + \langle w,v\rangle)} \\ &= \theta_{v+w}(z) e^{\frac{1}{2}(\langle w,v\rangle - \langle v,w\rangle)} = \theta_{v+w}(z) e^{-\frac{i}{2}\operatorname{Im}\langle v,w\rangle}.\end{aligned}$$

This leads us to the *Heisenberg group of* $\mathcal{H}$, which is given by

$$\mathrm{Heis}(\mathcal{H}) := \mathbb{R} \times \mathcal{H} \quad \text{and} \quad (t,v)(s,w) := \Big(t+s-\frac{1}{2}\operatorname{Im}\langle v,w\rangle, v+w\Big).$$

It is easy to verify that this defines a group structure on $\mathbb{R} \times \mathcal{H}$ with

$$(0,v)(0,w) = \Big(-\frac{1}{2}\operatorname{Im}\langle v,w\rangle, v+w\Big).$$

Proposition 5.2.1. *The group* $\mathrm{Heis}(\mathcal{H})$ *is a topological group with respect to the product topology on* $\mathbb{R} \times \mathcal{H}$*,* $\sigma(t,v)(z) := z+v$ *defines a continuous action of* $\mathrm{Heis}(\mathcal{H})$ *on* $\mathcal{H}$ *and*

$$J((t,v),z) := e^{it}\theta_v(z) = e^{it+\langle z,v\rangle - \frac{1}{2}\langle v,v\rangle}$$

is a continuous cocycle. Further,

$$(\pi(t,v)f)(z) := e^{it}\theta_v(z)f(z-v) = e^{it+\langle z,v\rangle-\frac{1}{2}\langle v,v\rangle}f(z-v)$$

defines a continuous unitary representation of $\mathrm{Heis}(\mathcal{H})$ *on* $\mathcal{F}(\mathcal{H})$*.*

Proof. The continuity of the group operations on $\mathrm{Heis}(\mathcal{H})$ is clear and the continuity of the action on $\mathcal{H}$ is trivial.

From the preceding calculations we know that the map

$$\mathrm{Heis}(\mathcal{H}) \to \mathrm{Aut}(\mathcal{H},K), \quad (t,v) \mapsto (e^{it}\theta_v, \tau_v)$$

is a homomorphism, and this implies that J is a cocycle. Its continuity is clear, and therefore Proposition 5.1.6 implies that π defines a continuous unitary representation of $\mathrm{Heis}(\mathcal{H})$ on $\mathcal{H}_K = \mathcal{F}(\mathcal{H})$. □

Remark 5.2.2. (a) If $\sigma\colon \mathcal{H} \to \mathcal{H}$ is an antilinear isometric involution, then

$$\mathcal{H}^\sigma := \{v \in \mathcal{H} \colon \sigma(v) = v\}$$

is a *real form of* $\mathcal{H}$, i.e., a closed real subspace for which

$$\mathcal{H} = \mathcal{H}^\sigma \oplus i\mathcal{H}^\sigma$$

is orthogonal with respect to the real scalar product $(x, y) := \mathrm{Re}\langle x, y\rangle$. To verify this claim, we recall the relation

$$\langle z, w\rangle = \langle \sigma(w), \sigma(z)\rangle$$

from Exercise 1.1.1. For $z, w \in \mathcal{H}^\sigma$ it implies that $\langle z, w\rangle \in \mathbb{R}$, and for $z \in \mathcal{H}^\sigma, w \in i\mathcal{H}^\sigma = \mathcal{H}^{-\sigma}$ we obtain $\langle z, w\rangle \in i\mathbb{R}$, so that $(z, w) = 0$.

This observation has the interesting consequence that $\{0\} \times \mathcal{H}^\sigma$ is a subgroup of the Heisenberg group $\mathrm{Heis}(\mathcal{H})$ and that

$$(\pi(v)f)(x) = \theta_v(x)f(x - v) = e^{\langle v,x\rangle - \frac{1}{2}\langle v,v\rangle} f(x - v).$$

defines a unitary representation of $\mathcal{H}^\sigma$ on the Fock space $\mathcal{F}(\mathcal{H})$. We shall see later that $\mathcal{F}(\mathcal{H})$ is isomorphic to the reproducing kernel space on $\mathcal{H}^\sigma$ defined by the real-valued kernel $K(z, w) = e^{(z,w)}$.

(b) For a real Hilbert space $\mathcal{H}$, the situation is simpler. Then we have the relation $\theta_{v+w} = \theta_v \cdot (\tau_v)_*\theta_w$, so that

$$(\pi(v)f)(x) = \theta_v(x)f(x - v)$$

defines a unitary representation of the additive group $(\mathcal{H}, +)$ on the reproducing kernel space $\mathcal{H}_K$ with kernel $K(z, w) = e^{(z,w)}$.

5.2.2 The Fock Representation of the Unitary Group

Proposition 5.2.3. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{F}(\mathcal{H}) := \mathcal{H}_K \subseteq \mathbb{C}^{\mathcal{H}}$ be the Fock space on $\mathcal{H}$ with the reproducing kernel $K(z, w) = e^{\langle z,w\rangle}$. Further, let $\mathcal{F}_m(\mathcal{H}) \subseteq \mathcal{F}(\mathcal{H})$ denote the subspace of those functions in $\mathcal{F}(\mathcal{H})$ which are homogeneous of degree m, i.e., $f(\lambda z) = \lambda^m f(z)$ for $\lambda \in \mathbb{C}$, $z \in \mathcal{H}$. Then the following assertions hold:*

(i) *The action $\big(\pi(g)f\big)(v) := f(g^{-1}v)$ defines a continuous unitary representation of $\mathrm{U}(\mathcal{H})_s$ on $\mathcal{F}(\mathcal{H})$. The closed subspaces $\mathcal{F}_m(\mathcal{H})$ are invariant under this action and their reproducing kernel is given by $K^m(z, w) = \frac{1}{m!}\langle z, w\rangle^m$.*

(ii) *Let $(e_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}$. Then the functions*

$$p_{\mathbf{m}}(z) = z^{\mathbf{m}} := \prod_{j\in J} z_j^{m_j} \quad \text{for} \quad z_j := \langle z, e_j\rangle, \mathbf{m} \in \mathbb{N}_0^{(J)},$$

form a complete orthogonal system in $\mathcal{F}(\mathcal{H})$ and $\|p_{\mathbf{m}}\|^2 = \mathbf{m}! := \prod_{j\in J} m_j!$.

Proof. Since the action of $\mathrm{U}(\mathcal{H})_s$ on $\mathcal{H}$ given by $(g, v) \mapsto gv$ is continuous (Exercise 1.2.2), it follows from the invariance of K under this action that $\big(\pi(g)f\big)(v) = f(g^{-1}v)$ defines a continuous unitary action of $\mathrm{U}(\mathcal{H})_s$ on $\mathcal{F}(\mathcal{H})$ (Proposition 5.1.6). It is clear that the subspaces $\mathcal{F}_m(\mathcal{H})$ are invariant under this action.

Next we consider the action of the subgroup $T := \mathbb{T}\mathbf{1} \subseteq \mathrm{U}(\mathcal{H})$ on $\mathcal{F}(\mathcal{H})$. For $m \in \mathbb{Z}$, let

$$\mathcal{F}(\mathcal{H})_m := \{f \in \mathcal{F}(\mathcal{H})\colon (\forall t \in \mathbb{T})(\forall z \in \mathcal{H})\, f(tz) = t^m f(z)\}$$

be the common eigenspace corresponding to the character $t\mathbf{1} \mapsto t^{-m}$ of T (cf. Example 4.2.16). According to the discussion in Example 4.2.15 and Theorem 4.2.20, we have an orthogonal decomposition

$$\mathcal{F}(\mathcal{H}) = \widehat{\oplus}_{m\in\mathbb{Z}} \mathcal{F}(\mathcal{H})_m.$$

In view of Exercise 3.3.1, we have a corresponding decomposition $K = \sum_{m\in\mathbb{Z}} K^m$ of the reproducing kernel Then $K_x^m \in \mathcal{F}(\mathcal{H})_m$ is the projection of K_x to the subspace $\mathcal{F}(\mathcal{H})_m$, which leads with the discussion in Example 4.2.15 to

$$\begin{aligned} K_x^m(y) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-imt} K_x(e^{it}y)\, dt = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^\infty \frac{1}{n!} e^{-imt}\langle e^{it}y, x\rangle^n\, dt \\ &= \sum_{n=0}^\infty \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{n!} e^{it(n-m)}\langle y, x\rangle^n\, dt = \begin{cases} \frac{1}{m!}\langle y, x\rangle^m, & \text{for } m \in \mathbb{N}_0 \\ 0, & \text{for } m < 0. \end{cases} \end{aligned}$$

We conclude that $K^m(z,w) = \frac{1}{m!}\langle z, w\rangle^m$ for $m \in \mathbb{N}_0$ and that $\mathcal{F}(\mathcal{H})_m = 0$ for $m < 0$. We also see that $\mathcal{F}(\mathcal{H})_m \subseteq \mathcal{F}_m(\mathcal{H})$ for all $m \in \mathbb{N}_0$ and therefore obtain $\mathcal{F}_m(\mathcal{H}) = \mathcal{F}(\mathcal{H})_m$ for each $m \in \mathbb{N}$ because the inclusion $\mathcal{F}_m(\mathcal{H}) \subseteq \mathcal{F}(\mathcal{H})_m$ is trivial.

(ii) We consider the topological product group $T := \mathbb{T}^J$ and note that Tychonov's Theorem implies that this group is compact if it is endowed with the product topology. Next we observe that the natural homomorphism

$$\alpha\colon T \to \mathrm{U}(\mathcal{H}), \quad \alpha\big((t_j)_{j\in J}\big)\sum_{j\in J} z_j e_j := \sum_{j\in J} t_j z_j e_j$$

(the action of T by diagonal matrices) defines a continuous unitary representation. In view of the continuity criterion Lemma 1.2.6, this follows from the fact that the functions $T \to \mathbb{C}, t = (t_j)_{j\in J} \mapsto \langle \alpha(t)e_k, e_\ell\rangle = \delta_{\ell k} t_k$ are continuous for all $k, \ell \in J$. Next we use (i) to conclude that $\pi \circ \alpha$ is a continuous unitary representation of T on $\mathcal{F}(\mathcal{H})$. Since T is compact abelian, the Fundamental Theorem on Unitary Representations of Compact Groups (Theorem 4.2.20) shows that the eigenfunctions of T form a total subset of $\mathcal{F}(\mathcal{H})$. So we have to determine these eigenfunctions.

Each continuous character $\chi\colon T \to \mathbb{T}$ is of the form $\chi_{\mathbf{m}}(z) = \prod_{j\in J} z_j^{m_j}$ for $\mathbf{m} \in \mathbb{Z}^{(J)}$, where $\mathbb{Z}^{(J)} \subseteq \mathbb{Z}^J$ denotes the subset of all functions with finite support, i.e., the free abelian group on J (Exercise 5.2.2). Accordingly, we have

$$\mathcal{F}(\mathcal{H}) = \widehat{\bigoplus}_{\mathbf{m}\in\mathbb{Z}^{(J)}} \mathcal{F}(\mathcal{H})_{\mathbf{m}},$$

where

$$\mathcal{F}(\mathcal{H})_{\mathbf{m}} = \{f \in \mathcal{F}(\mathcal{H})\colon (\forall t \in T)\pi(\alpha(t))f = f \circ \alpha(t)^{-1} = \chi_{-\mathbf{m}}(t)f\}$$

(cf. Example 4.2.15). Then we have a corresponding decomposition $K = \sum_{\mathbf{m}\in\mathbb{Z}^{(J)}} K^{\mathbf{m}}$ of the reproducing kernel (Exercise 3.3.1). To determine the kernels $K^{\mathbf{m}}$, we first observe that, in view of (i), $\mathcal{F}(\mathcal{H})_{\mathbf{m}} \subseteq \mathcal{F}(\mathcal{H})_m$ holds for $\sum_{j\in J} m_j = m$.

We recall from Example 4.2.15 the orthogonal projection

$$P_{\mathbf{m}}\colon \mathcal{F}(\mathcal{H})_m \to \mathcal{F}(\mathcal{H})_{\mathbf{m}}, \quad P_{\mathbf{m}}(f)(z) = \int_T \chi_{\mathbf{m}}(t) f(\alpha(t)^{-1}z)\, d\mu_T(t).$$

In particular, we obtain

$$\begin{aligned} K^{\mathbf{m}}_w(z) = P_{\mathbf{m}}(K^m_w)(z) &= \int_T \chi_{\mathbf{m}}(t) K^m_w(\alpha(t)^{-1}z)\, d\mu_T(t) \\ &= \frac{1}{m!}\int_T \chi_{\mathbf{m}}(t^{-1})\langle \alpha(t)z, w\rangle^m\, d\mu_T(t). \end{aligned}$$

To evaluate this expression, we recall the multinomial formula

$$(x_1 + \ldots + x_n)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} x^\alpha, \quad x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \quad \binom{k}{\alpha} := \frac{k!}{\alpha_1!\cdots\alpha_n!},$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We thus obtain for $z, w \in \mathcal{H}$ with the Cauchy Product Formula

$$\langle \alpha(t)z, w\rangle^m = \Big(\sum_{j\in J} t_j z_j \overline{w_j}\Big)^m = \sum_{\mathbf{m}\in\mathbb{N}_0^{(J)}, |\mathbf{m}|=m} \binom{m}{\mathbf{m}} t^{\mathbf{m}} z^{\mathbf{m}} \overline{w}^{\mathbf{m}}$$

with uniform convergence in $t \in T$. This leads to $K^{\mathbf{m}}_w(z) = 0$ for $\mathbf{m} \notin \mathbb{N}_0^{(J)}$, and for $\mathbf{m} \in \mathbb{N}_0^{(J)}$ we get

$$K^{\mathbf{m}}_w(z) = \frac{1}{m!}\binom{m}{\mathbf{m}} z^{\mathbf{m}}\overline{w}^{\mathbf{m}} = \frac{1}{\mathbf{m}!} z^{\mathbf{m}}\overline{w}^{\mathbf{m}} = \frac{1}{\mathbf{m}!} p_{\mathbf{m}}(z)\overline{p_{\mathbf{m}}(w)}.$$

This shows that $\mathcal{F}(\mathcal{H})_{\mathbf{m}} = \mathbb{C}p_{\mathbf{m}}$ (Proposition 3.4.9), and Theorem 3.1.3(3) implies that $\|p_{\mathbf{m}}\|^2 = \mathbf{m}!$. □

Remark 5.2.4. At this point we have unitary representations of the Heisenberg group $\mathrm{Heis}(\mathcal{H})$ and the unitary group $\mathrm{U}(\mathcal{H})$ on the Fock space $\mathcal{F}(\mathcal{H})$. These two representations are compatible in the following sense.

For each $g \in \mathrm{U}(\mathcal{H})$, we obtain a topological automorphism of $\mathrm{Heis}(\mathcal{H})$ by $\alpha(g)(t, v) := (t, gv)$, and we thus obtain a homomorphism

$$\alpha\colon \mathrm{U}(\mathcal{H}) \to \mathrm{Aut}(\mathrm{Heis}(\mathcal{H})),$$

defining a continuous action of $\mathrm{U}(\mathcal{H})_s$ on $\mathrm{Heis}(\mathcal{H})$ (cf. Exercise 1.2.2). Therefore we obtain a topological semidirect product group

$$\mathrm{Heis}(\mathcal{H}) \rtimes_\alpha \mathrm{U}(\mathcal{H}).$$

In view of the relation

$$\begin{aligned}(\pi(g)\pi(t,v)f)(z) &= e^{it+\langle g^{-1}z,v\rangle-\frac{1}{2}\langle v,v\rangle}f(g^{-1}z-v)\\ &= e^{it+\langle z,gv\rangle-\frac{1}{2}\langle gv,gv\rangle}f(g^{-1}(z-gv)) = (\pi(t,gv)\pi(g)f)(z),\end{aligned}$$

we have

$$\pi(g)\pi(t,v)\pi(g)^{-1} = \pi(t,gv),$$

so that the representations of $\mathrm{Heis}(\mathcal{H})$ and $\mathrm{U}(\mathcal{H})$ on the Fock space combine to a continuous unitary representation $\pi(t,v,g) := \pi(t,v)\pi(g)$ of the semidirect product group.

5.2.3 Hilbert Spaces on the Unit Disc

Before we discuss some typical examples of Hilbert spaces on the unit disc, we show that many L^2-spaces of holomorphic functions have continuous point evaluations.

Proposition 5.2.5. *Let $\Omega \subseteq \mathbb{C}$ be an open subset and $\rho\colon \Omega \to \mathbb{R}_+^\times$ be a measurable function such that every point $p \in \Omega$ has a neighborhood on which ρ is bounded below by some $\varepsilon > 0$. Then*

$$\mathcal{H} := \Big\{ f \in \mathcal{O}(\Omega)\colon \int_\Omega |f(z)|^2\rho(z)dz < \infty \Big\}$$

is a Hilbert space with respect to the scalar product

$$\langle f, g\rangle := \int_\Omega f(z)\overline{g(z)}\rho(z)\,dz,$$

and the inclusion $\mathcal{H} \to \mathcal{O}(\Omega)$ is continuous with respect to the topology of uniform convergence on compact subsets sets on $\mathcal{O}(\Omega)$. In particular, the point evaluations on $\mathcal{H}$ are continuous.

Proof. Let $p \in \Omega$ and $r > 0$ such that the closed disc $D := B_{\leq r}(p)$ of radius r is contained in Ω. Then the compactness of D implies that ρ is bounded below by some $\varepsilon > 0$ on D. For $f \in \mathcal{O}(\Omega)$ we obtain from the mean value property

$$\begin{aligned}\frac{1}{\pi r^2}\int_D f(w)\;dw &= \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi} f(p+re^{it})\,dt\,rdr = \frac{1}{\pi r^2}\int_0^r 2\pi f(p)\,rdr\\ &= \frac{2}{r^2}\int_0^r rdr\cdot f(p) = f(p).\end{aligned}$$

Now the Cauchy–Schwarz inequality in $L^2(D,dz)$ leads to

$$\begin{aligned}|f(p)|^2 &\leq \frac{1}{(\pi r^2)^2}\int_D |f(w)|^2\;dw\cdot\int_D 1dw = \frac{1}{\pi r^2}\int_D |f(w)|^2\;dw\\ &\leq \frac{1}{\varepsilon\pi r^2}\int_D |f(w)|^2\rho(w)\;dw \leq \frac{1}{\varepsilon\pi r^2}\|f\|^2.\end{aligned}$$

Applying this to smaller discs $B_s(q) \subseteq B_r(p)$, we find for $q \in B_{r-s}(p)$ the estimate

$$|f(q)|^2 \leq \frac{1}{\varepsilon\pi s^2}\|f\|^2.$$

This estimate proves that every Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ is a Cauchy sequence on $B_{r-s}(p)$ with respect to the sup-norm, hence uniformly convergent to a continuous holomorphic function $f\colon B_{r-s}(p) \to \mathbb{C}$. Therefore the limit $f(z) := \lim_{n\to\infty} f_n(z)$ exists pointwise on Ω and f is holomorphic because it is holomorphic in a neighborhood of any $p \in \Omega$.

Since each compact subset $B \subseteq \Omega$ can be covered by finitely many discs on which the sequence converges uniformly, it converges uniformly on B, so that we obtain

$$\int_B |f(z)|^2\rho(z)\,dz = \lim_{n\to\infty}\int_B |f_n(z)|^2\rho(z)\,dz \leq \lim_{n\to\infty}\|f_n\|^2.$$

Therefore $\int_\Omega |f(z)|^2\rho(z)\,dz \leq \lim_{n\to\infty}\|f_n\|^2$ follows from the Monotone Convergence Theorem, so that $f \in \mathcal{H}$.

To see that $f_n \to f$ holds in $\mathcal{H}$, let $\delta > 0$ and $\|f_n - f_m\| \leq \delta$ for $m, n \geq n_0$. Then for each compact subset $B \subseteq \Omega$ and $m \geq n_0$, we have

$$\begin{aligned}\int_B |f(z) - f_m(z)|^2\rho(z)\,dz &= \lim_{n\to\infty}\int_B |f_n(z) - f_m(z)|^2\rho(z)\,dz\\ &\leq \lim_{n\to\infty}\|f_n - f_m\|^2 \leq \delta^2\end{aligned}$$

and hence $\|f - f_m\| \leq \delta$. This proves that $f_m \to f$, and therefore that $\mathcal{H}$ is complete. □

Example 5.2.6. (a) Let $\mathcal{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ denote the unit disc, and consider for a real $m > 1$ the Hilbert space

$$\mathcal{H}_m = \{f \in \mathcal{O}(\mathcal{D})\colon \|f\|_m < \infty\},$$

where

$$\|f\|_m^2 := \frac{m-1}{\pi}\int_{\mathcal{D}} |f(z)|^2(1-|z|^2)^{m-2}\,dz.$$

Since the measure $(1-|z|^2)^{m-2}dz$ is invariant under the action of $\mathbb{T}$ by scalar multiplication, we obtain with Proposition 2.3.8 a continuous unitary representation of $\mathbb{T}$ on $\mathcal{H}_m$, given by $(t.f)(w) := f(tw)$.

Further, Proposition 5.2.5 implies that $\mathcal{H}_m$ is a reproducing kernel Hilbert space and Proposition 5.1.6 shows that its kernel K^m is $\mathbb{T}$-invariant, i.e.,

$$K^m(tz, tw) = K^m(z, w) \quad \text{for} \quad z, w \in \mathcal{D}, t \in \mathbb{T}.$$

From the Fundamental Theorem on Unitary Representations of Compact Groups (Theorem 4.2.20) we now derive that $\mathcal{H}_m$ is an orthogonal direct sum of the $\mathbb{T}$-eigenspaces $\mathcal{H}_{m,n}$, corresponding to the characters $\chi_n(t) := t^n$. If $f \in \mathcal{H}_{m,n}$, then

$$f(tz) = t^n f(z) \quad \text{for} \quad z \in \mathcal{D}, t \in \mathbb{T},$$

and this implies that $f(wz) = w^n f(z)$ for $|w| \leq 1$ because both sides are holomorphic in w and coincide on $\mathbb{T}$. Therefore $f(z) = cz^n$ for some $c \in \mathbb{C}$, and thus either $\mathcal{H}_{m,n} = \mathbb{C}p_n$ for $p_n(z) = z^n$, or $\mathcal{H}_{m,n} = \{0\}$, where the latter relation holds in particular for $n < 0$ because the functions z^n, $n < 0$, on $\mathcal{D}\setminus\{0\}$ do not extend holomorphically to $\mathcal{D}$.

To see which monomials are contained in $\mathcal{H}_m$, we calculate $\|p_n\|_m^2$:

$$\|p_n\|_m^2 = \frac{m-1}{\pi}\int_{\mathcal{D}} |z|^{2n}(1-|z|^2)^{m-2}\, dz = 2(m-1)\int_0^1 r^{2n+1}(1-r^2)^{m-2}\, dr$$
$$= (m-1)\int_0^1 u^n(1-u)^{m-2}\, du.$$

Let $I_{n,k} := (k+1)\int_0^1 u^n(1-u)^k\, du$. Then integration by parts gives

$$I_{n,k} = \frac{n}{k+2} I_{n-1,k+1}$$

whenever $n > 0$ and $k > -1$. Proceeding further, we obtain $I_{0,k} = 1$ and thus

$$I_{n,k} = \frac{n!}{(k+2)\cdots(k+n+1)} I_{0,k+n} = \binom{k+n+1}{n}^{-1}.$$

We thus obtain $\|p_n\|_m^2 = I_{n,m-2} = \binom{m-1+n}{n}^{-1}$. We know already from above that the orthogonal family $(p_n)_{n\in\mathbb{N}_0}$ is complete. With Theorem 3.1.3(iii), we thus obtain for the reproducing kernel

$$K^m(z,w) = \sum_{n=0}^{\infty} \binom{m+n-1}{n} z^n \overline{w}^n = (1 - z\overline{w})^{-m}$$

(cf. Example 3.3.6(f) for $\mathcal{H} = \mathbb{C}$). For $m = 2$, the Hilbert space $\mathcal{H}_m$ is called the *Bergman space of* $\mathcal{D}$. It is the space of square integrable holomorphic functions on $\mathcal{D}$.

One also obtains an interesting "limit space" for $m = 1$. This can be done as follows. On $\mathcal{O}(\mathcal{D})$ we consider

$$\|f\|^2 := \lim_{\substack{r\to 1\\ r<1}} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2\, dt.$$

To evaluate this expression, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ denote the Taylor series of f about 0 which converges uniformly on each compact subset of $\mathcal{D}$. Hence we can interchange integration and summation and obtain

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2\, dt = \sum_{n,m=0}^{\infty} a_n \overline{a_m} \frac{1}{2\pi}\int_0^{2\pi} r^{n+m} e^{it(n-m)}\, dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Applying the Monotone Convergence Theorem to the sequences $(|a_n|^2 r^{2n})_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}_0)$, we see that $(|a_n|^2)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}_0)$ if and only if $\|f\| < \infty$, and that in this case $\|f\|^2 = \sum_{n=0}^\infty |a_n|^2$. Therefore

$$\mathcal{H}_1 := \{f \in \mathcal{O}(\mathcal{D}) \colon \|f\| < \infty\} \cong \ell^2(\mathbb{N}_0, \mathbb{C})$$

is a Hilbert space and the polynomials form a dense subspace of $\mathcal{H}_1$. Moreover, the monomials $p_n(z) = z^n$ form an orthonormal basis of $\mathcal{H}_1$. Note that $\|p_n\| = 1$ is exactly the limit obtained for general $m > 1$ if m tends to 1. We put

$$K^1(z,w) = \sum_{n=0}^\infty p_n(z)\overline{p_n(w)} = \sum_{n=0}^\infty z^n \overline{w}^n = \frac{1}{1 - z\overline{w}}$$

(cf. Theorem 3.1.3(a)). Then, for $w \in \mathcal{D}$, the functions $K_w(z) = \frac{1}{1-z\overline{w}} = \sum_{n=0}^\infty \overline{w}^n z^n$ are contained in $\mathcal{H}_1$, and for $f(z) = \sum_{n=0}^\infty a_n z^n$, we get

$$\langle f, K_w\rangle = \sum_{n=0}^\infty a_n w^n = f(w).$$

This proves that $\mathcal{H}_1$ has continuous point evaluations and that its reproducing kernel is given by K^1.

The space $\mathcal{H}_1$ is called the *Hardy space of* $\mathcal{D}$ and K^1 is called the *Cauchy kernel*. This is justified by the following observation. For each holomorphic function f on $\mathcal{D}$ extending continuously to the boundary, we obtain the simpler formula for the norm:

$$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 \, dt.$$

We see, in particular, that such a function is contained in $\mathcal{H}_1$ and thus

$$\begin{aligned} f(z) = \langle f, K_z^1\rangle &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{K_z^1(e^{it})}\, dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{1 - ze^{-it}}\, dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{it})}{e^{it} - z}\, e^{it} i dt = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \end{aligned}$$

where the latter integral denotes a complex line integral. This means that the fact that K^1 is the reproducing kernel for $\mathcal{H}_1$ is equivalent to *Cauchy's integral formula*

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We have already seen how the spaces $\mathcal{H}_m$ decompose under the unitary representation of the group $\mathbb{T}$, but the spaces $\mathcal{H}_m$ carry for $m \in \mathbb{N}$ a unitary representation of the larger group

$$G := \mathrm{SU}_{1,1}(\mathbb{C}) := \Big\{ g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a}\end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \colon a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \Big\}.$$

We claim that

$$\sigma_g(z) := g.z := (az + b)(\overline{b}z + \overline{a})^{-1}$$

defines a continuous action of G on $\mathcal{D}$. Note that this expression is always defined because $|z| < 1$ and $|b| < |a|$ implies that $\overline{b}z + \overline{a} \neq 0$. That $\sigma_g(z) \in \mathcal{D}$ for $z \in \mathcal{D}$ follows from

$$|az+b|^2 = |a|^2|z|^2 + (a\overline{b}z + \overline{a}b\overline{z}) + |b|^2 < |b|^2|z|^2 + (a\overline{b}z + \overline{a}b\overline{z}) + |a|^2 = |\overline{b}z + \overline{a}|^2.$$

The relations $\sigma_{\mathbf{1}}(z) = z$ and $\sigma_{gg'} = \sigma_g \sigma_{g'}$ are easily verified (see Exercise 5.2.5). To see that this action is transitive, we note that for $|z| < 1$,

$$g := \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix} \in \mathrm{SU}_{1,1}(\mathbb{C})$$

satisfies $g.0 = z$.

To obtain a unitary action of G on $\mathcal{H}_m$, we have to see how the corresponding kernel K^m transforms under the action of G. For the kernel $Q(z,w) = 1 - z\overline{w}$ an easy calculation shows that

$$\begin{aligned} Q(g.z, g.w) &= 1 - \frac{(az+b)}{(\overline{b}z + \overline{a})} \frac{(\overline{aw} + \overline{b})}{(b\overline{w} + a)} = \frac{(\overline{b}z + \overline{a})(a + b\overline{w}) - (az+b)(\overline{aw} + \overline{b})}{(\overline{b}z + \overline{a})(a + b\overline{w})} \\ &= \frac{(|a|^2 - |b|^2)(1 - z\overline{w})}{(\overline{b}z + \overline{a})(a + b\overline{w})} = \frac{Q(z,w)}{(\overline{b}z + \overline{a})(a + b\overline{w})}. \end{aligned}$$

Finally, we note that,

$$J(g,z) := a - \overline{b}z$$

defines a 1-cocycle for the action of G on $\mathcal{D}$, which can be verified by direct calculation), and

$$J(g, g.z) = a - \overline{b}\frac{az+b}{\overline{b}z + \overline{a}} = \frac{a\overline{b}z + |a|^2 - \overline{b}az - |b|^2}{\overline{b}z + \overline{a}} = \frac{1}{\overline{b}z + \overline{a}},$$

so that we obtain for

$$J^m(g,z) := J(g,z)^{-m}$$

the relation

$$K^m(g.z, g.w) = J^m(g, g.z) K(z,w) \overline{J^m(g, g.w)},$$

and Proposition 5.1.6 show that, for $m \in \mathbb{N}$,

$$(\pi_m(g)f)(z) = J^m(g,z) f(g^{-1}.z) = (a - \overline{b}z)^{-m} f\left(\frac{\overline{a}z - b}{a - \overline{b}z}\right)$$

defines a continuous unitary representation of $G = \mathrm{SU}_{1,1}(\mathbb{C})$ on $\mathcal{H}^m$.

For $m = 2$, the space $\mathcal{H}_2$ is the Bergman space of the disc and

$$J^2(g,z) = \frac{1}{(a - \overline{b}z)^2} = \sigma'_{g^{-1}}(z)$$

for $g \in \mathrm{SU}_{1,1}(\mathbb{C})$, so that the existence of the representation in this case also follows from Exercise 5.2.4.

Remark 5.2.7. From Example 3.3.6 we recall the positive definite kernels

$$K^s(z,w) := (1 - z\overline{w})^{-s}, \quad s > 0,$$

on the open unit disc $\mathcal{D} \subseteq \mathbb{C}$. We have seen in Example 5.2.6 that for $s \in \mathbb{N}$, we have a unitary representation of $\mathrm{SU}_{1,1}(\mathbb{C})$ on the corresponding Hilbert space. The reason for restricting to integral values of s is that otherwise we don't have a corresponding cocycle. However, for $Q(z,w) = 1 - z\overline{w}$, we have

$$Q(g.z, g.w) = \frac{Q(z,w)}{(\overline{b}z + \overline{a})(a + b\overline{w})}. = \frac{Q(z,w)}{|a|^2(1 + (\overline{b}/\overline{a})z)(1 + (b/a)\overline{w})} Q(z,w),$$

and therefore

$$K^s(g.z, g.w) = \theta_g(z)\overline{\theta_g(w)} K^s(z,w)$$

for

$$\theta_g(z) := |a|^s (1 + (\overline{b}/a)z)^s,$$

where, in view of $|b| < |a|$, the right hand side can be defined by a power series converging in $\mathcal{D}$.

As we shall see below, these considerations lead to a projective unitary representation of $\mathrm{SU}_{1,1}(\mathbb{C})$ on $\mathcal{H}_s$ by

$$(\pi_s(g)f)(z) := \theta_g(g^{-1}.z) f(g^{-1}.z).$$

5.2.4 Some Representations of $\mathrm{SU}_2(\mathbb{C})$

In this section we discuss the action of $\mathrm{SU}_2(\mathbb{C})$ on a Hilbert space of polynomials on the complex plane. We shall see later that all these representations are irreducible, and one can even show that they exhaust the irreducible continuous unitary representations of this compact group.

Example 5.2.8. For $m \in \mathbb{N}_0$, let $\mathcal{H}_m = \{f \in \mathcal{O}(\mathbb{C}) \colon \|f\|_m < \infty\}$, where

$$\|f\|_m^2 = \frac{m+1}{\pi} \int_{\mathbb{C}} \frac{|f(z)|^2}{(1 + |z|^2)^{m+2}}\, dz$$

and dz denotes Lebesgue measure on $\mathbb{C}$.

Then exactly the same arguments as in Example 5.2.6, based on the rotation invariance of the density $(1+|z|^2)^{-(m+2)}$, show that $\mathcal{H}_m$ is a reproducing kernel Hilbert space and that the monomial $p_n(z) := z^n$ contained in $\mathcal{H}_m$ form an orthogonal total subset. To determine these monomials, we calculate

$$\begin{aligned}\|p_n\|_m^2 &= \frac{m+1}{\pi} \int_{\mathbb{C}} \frac{|z|^{2n}}{(1 + |z|^2)^{m+2}}\, dz = 2(m+1) \int_0^\infty r^{2n+1}(1 + r^2)^{-m-2}\, dr \\ &= (m+1) \int_0^\infty u^n (1+u)^{-m-2}\, du.\end{aligned}$$

Let $I_{n,k} := (k-1)\int_0^\infty \frac{u^n}{(1+u)^k}\,du$. This integral exists if and only if $k > n+1$. Under this assumption integration by parts yields $I_{n,k} = \frac{n}{k-2} I_{n-1,k-1}$. For $n = 0$, we get $I_{0,k} = -\int_0^\infty (1-k)(1+u)^{-k}\,du = 1$. Therefore

$$I_{n,k} = \frac{n!}{(k-2)(k-3)\cdots(k-n-1)} I_{0,k-n} = \binom{k-2}{n}^{-1}$$

and $\|p_n\|^2 = \binom{m}{n}^{-1}$ for $n \leq m$, so that the monomials $\sqrt{\binom{m}{n}} p_n$, $n = 0, \ldots, m$ form an orthonormal basis in $\mathcal{H}_m$ and in particular $\mathcal{H}_m$ consists of all polynomials of degree $\leq m$.

Since the the point evaluations on the finite dimensional space $\mathcal{H}_m$ are continuous, we have a reproducing kernel which is given by

$$K^m(z,w) = \sum_{n=0}^m \binom{m}{n} z^n \overline{w}^n = (1 + z\overline{w})^m.$$

(Theorem 3.1.3). The group $G = \mathrm{SU}_2(\mathbb{C})$ acts on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by $g.z = \frac{az+b}{-\overline{b}z+\overline{a}}$ for $g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ (Exercise 5.2.5). We claim that we have a unitary representation on $\mathrm{SU}_2(\mathbb{C})$ on the Hilbert space $\mathcal{H}_m$ by

$$(\pi_m(g)f)(z) = (a + \overline{b}z)^m f\left(\frac{\overline{a}z - b}{a + \overline{b}z}\right) = (a + \overline{b}z)^m f(g^{-1}.z).$$

Note that the factor $(a + \overline{b}z)^m$ ensures that, although $f(g^{-1}.z)$ is a rational function, the product is a polynomial of degree $\leq m$.

It is instructive to see how this arises from the identification of $\widehat{\mathbb{C}}$ with the projective line $\mathbb{P}_1(\mathbb{C}) = \mathbb{P}(\mathbb{C}^2)$. Let $\mathcal{F}_m(\mathbb{C}^2)$ be the subspace of the Fock space $\mathcal{F}(\mathbb{C}^2)$ consisting of homogeneous polynomials of degree m. From the inclusion

$$\mathbb{C} \to \mathbb{C}^2, \quad z \mapsto (z, 1),$$

we obtain an inclusion

$$R\colon \mathcal{F}_m(\mathbb{C}^2) \to \mathcal{O}(\mathbb{C}), \quad (Rf)(z) := f(z,1),$$

and the injectivity of R follows from the homogeneity of the elements of $\mathcal{F}_m(\mathbb{C}^2)$. Defining a Hilbert space structure on the image of R in such a way that R is unitary, its image is a reproducing kernel space with kernel

$$K(z,w) = \frac{1}{m!}\langle (z,1),(w,1)\rangle^m = \frac{(1+z\overline{w})^m}{m!}.$$

Since this is, up to a positive factor, the kernel of $\mathcal{H}_m$, it follows that $\mathrm{im}(R) = \mathcal{H}_m$ and that the unitary representation of $\mathrm{SU}_2(\mathbb{C}) \subseteq \mathrm{U}(\mathbb{C}^2)$ on $\mathcal{F}_m(\mathbb{C}^2)$ corresponds to the unique unitary representation π_m on $\mathcal{H}_m$ for which R is an intertwining operator.

For $g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C})$ we then find with $g^{-1} = \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix}$ the formula

$$(\pi_m(g)R(f))(z) := (f \circ g^{-1})(z,1) = f(\overline{a}z - b, \overline{b}z + a) = (a + \overline{b}z)^m (Rf)\Big(\frac{\overline{a}z - b}{\overline{b}z + a}\Big).$$

Exercises for Section 5.2

Exercise 5.2.1. Let V be a real vector space and $\omega\colon V \times V \to \mathbb{R}$ be a bilinear map.

(a) Show that on $\mathbb{R} \times V$ we obtain a group structure by

$$(t,v)(s,w) := (t + s + \omega(v,w), v + w).$$

This group is called the *Heisenberg group* $\mathrm{Heis}(V,\omega)$.

More generally, we obtain for any two abelian groups V and Z and any biadditive map $\omega\colon V \times V \to Z$ a group structure on $Z \times V$ by

$$(t,v)(s,w) := (t + s + \omega(v,w), v + w).$$

(b) Let $\mathcal{H}$ be a complex Hilbert space. How do we have to choose V and ω to obtain an isomorphism $\mathrm{Heis}(V,\omega) \cong \mathrm{Heis}(\mathcal{H})$?

(c) Verify that $Z(\mathrm{Heis}(V,\omega)) = \mathbb{R} \times \mathrm{rad}(\omega_s)$, where

$$\omega_s(v,w) := \omega(v,w) - \omega(w,v) \quad \text{and} \quad \mathrm{rad}(\omega_s) := \{v \in V \colon \omega_s(v,V) = \{0\}\}.$$

(d) Show that for $V = \mathbb{R}^2$ with $\omega(x,y) = x_1 y_2$, the Heisenberg group $H(V,\omega)$ is isomorphic to the matrix group

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Exercise 5.2.2. Let $G = \prod_{j \in J} G_j$ be a product of abelian topological groups and $p_j\colon G \to G_j$ be the projection maps. Show that the map

$$S\colon \bigoplus_{j \in J} \widehat{G_j} \to \widehat{G}, \quad (\chi_j)_{j \in J} \mapsto \prod_{j \in J} (\chi_j \circ p_j)$$

is an isomorphism of abelian groups.

Exercise 5.2.3. On $\mathbb{R}^n$ we consider the vector space $\mathcal{P}_k$ of all homogeneous polynomials of degree k:

$$p(x) = \sum_{|\alpha| = k} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{R}, x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

We associate to such a polynomial p a differential operator by

$$p(\partial) := \sum_{|\alpha|=k} c_\alpha \partial^\alpha, \quad \partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad \partial_i := \frac{\partial}{\partial x_i}.$$

Show that the *Fischer inner product*

$$\langle p, q \rangle := (p(\partial)q)(0)$$

defines on $\mathcal{P}_k$ the structure of a real Hilbert space with continuous point evaluations. Show further that its kernel is given by

$$K(x, y) = \frac{1}{k!}\langle x, y\rangle^k = \frac{1}{k!}\Big(\sum_{j=1}^n x_j y_j\Big)^k.$$

Hint: Show that the monomials $p_\alpha(x) = x^\alpha$ form an orthogonal subset with $\langle p_\alpha, p_\alpha\rangle = \alpha!$ and conclude with Theorem 3.1.3 that $K(x, y) = \sum_{|\alpha|=m} \frac{x^\alpha y^\alpha}{\alpha!}$.

Exercise 5.2.4. Let $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathbb{C}$ be two open subsets and $\varphi\colon \mathcal{D}_1 \to \mathcal{D}_2$ be a biholomorphic map, i.e., φ is bijective and φ^{-1} is also holomorphic. Let $\mathcal{B}(\mathcal{D}) := L^2(\mathcal{D}, dz) \cap \mathcal{O}(\mathcal{D})$ denote the *Bergman space* of $\mathcal{D}$. Show that the map

$$\Phi\colon \mathcal{B}(\mathcal{D}_2) \to \mathcal{B}(\mathcal{D}_1), \quad f \mapsto (\varphi^* f)\cdot \varphi', \quad \varphi^* f = f \circ \varphi$$

is unitary. Hint: For the real linear map $\lambda_z\colon \mathbb{C} \to \mathbb{C}, w \mapsto zw$, we have $\det_{\mathbb{R}}(\lambda_z) = |z|^2$.

Exercise 5.2.5. We consider the group $G := \mathrm{GL}_2(\mathbb{C})$ and the *complex projective line* (the *Riemann sphere*)

$$\mathbb{P}_1(\mathbb{C}) = \{[v] := \mathbb{C}v\colon 0 \neq v \in \mathbb{C}^2\}$$

of 1-dimensional linear subspaces of $\mathbb{C}^2$. We write $[x : y]$ for the line $\mathbb{C}\begin{pmatrix} x \\ y \end{pmatrix}$. Show that:

(a) The map $\mathbb{C} \to \mathbb{P}_1(\mathbb{C}), z \mapsto [z : 1]$ is injective and its complement consists of the single point $\infty := [1 : 0]$ (the horizontal line). We thus identify $\mathbb{P}_1(\mathbb{C})$ with the one-point compactification $\widehat{\mathbb{C}}$ of $\mathbb{C}$. These are the so-called *homogeneous coordinates* on $\mathbb{P}_1(\mathbb{C})$.

(b) The natural action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}_1(\mathbb{C})$ by $g.[v] := [gv]$ is given in the coordinates of (b) by

$$g.z = \sigma_g(z) := \frac{az + b}{cz + d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(c) On $\mathbb{C}^2$ we consider the indefinite hermitian form

$$\beta(z,w) := z_1\overline{w_1} - z_2\overline{w_2} = w^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z.$$

We define

$$\mathrm{U}_{1,1}(\mathbb{C}) := \{g \in \mathrm{GL}_2(\mathbb{C})\colon (\forall z, w \in \mathbb{C}^2)\, \beta(gz, gw) = \beta(z,w)\}.$$

Show that $g \in \mathrm{U}_{1,1}(\mathbb{C})$ is equivalent to

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show further that the above relation is equivalent to

$$\det g \in \mathbb{T}, \quad d = \overline{a} \det g \quad \text{and} \quad c = \overline{b} \det g.$$

In particular, we obtain $|a|^2 - |b|^2 = 1$.

(d) The hermitian form β is negative definite on the subspace $[z_1 : z_2]$ if and only if $|z_1| < |z_2|$, i.e., $[z_1 : z_2] = [z : 1]$ for $|z| < 1$. Conclude that $g.z := \frac{az+b}{cz+d}$ defines an action of $\mathrm{U}_{1,1}(\mathbb{C})$ on the open unit disc $\mathcal{D}$ in $\mathbb{C}$.

5.3 Positive Definite Functions

It was one of our first observations in Section 1.3 that any unitary representation decomposes as a direct sum of cyclic ones. The main point of this section is to describe the bridge between cyclic representations and positive definite kernels on the group G which are invariant under right translations. Such kernels are determined by the function $\varphi := K_{\mathbf{1}}$ via $K(g,h) = \varphi(gh^{-1})$, and φ is said to be positive definite if K has this property. We shall see that any cyclic representation of G is equivalent to one in a reproducing kernel subspace $\mathcal{H}_\varphi \subseteq C(G)$ corresponding to a continuous positive definite function φ, and characterize the irreducible ones geometrically by the condition that φ generates an extremal ray in the cone of continuous positive definite functions on G.

5.3.1 Cyclic Representations

Definition 5.3.1. A function $\varphi\colon S \to \mathbb{C}$ on an involutive semigroup $(S, *)$ is called *positive definite* if the kernel

$$K_\varphi\colon S \times S \to \mathbb{C}, \quad K_\varphi(s,t) := \varphi(st^*)$$

is positive definite. We then write $\mathcal{H}_\varphi \subseteq \mathbb{C}^S$ for the corresponding reproducing kernel Hilbert space.

(b) If G is a group, then (G, η_G) is an involutive semigroup, so that a function $\varphi\colon G \to \mathbb{C}$ is called *positive definite* if the kernel

$$K_\varphi\colon G \times G \to \mathbb{C}, \quad K_\varphi(s,t) := \varphi(st^{-1})$$

is positive definite. We then write $\mathcal{H}_\varphi := \mathcal{H}_{K_\varphi}$ for the corresponding reproducing kernel space in $\mathbb{C}^G$.

A kernel $K\colon G \times G \to \mathbb{C}$ is called *right invariant* if

$$K(xg, yg) = K(x, y) \quad \text{holds for} \quad g, x, y \in G.$$

For any such kernel the function $\varphi := K_{\mathbf{1}}$ satisfies

$$K(x, y) = K(xy^{-1}, \mathbf{1}) = \varphi(xy^{-1}).$$

Conversely, for every function $\varphi\colon G \to \mathbb{C}$, the kernel $K(x, y) := \varphi(xy^{-1})$ is right invariant. Therefore the right invariant positive definite kernels on G correspond to positive definite functions.

(c) For a topological group G, we write $\mathcal{P}(G)$ for the set of continuous positive definite functions on G. The subset

$$\mathcal{S}(G) := \{\varphi \in \mathcal{P}(G) \colon \varphi(\mathbf{1}) = 1\}$$

is called the *set of states of* G.

Clearly, $\mathcal{P}(G)$ is a convex cone and $\mathcal{S}(G) \subseteq \mathcal{P}(G)$ is a convex subset with $\mathcal{P}(G) = \mathbb{R}_+\mathcal{S}(G)$. The extreme points of $\mathcal{S}(G)$ are called *pure states of* G.

Recall from Proposition 3.3.5 that $\mathcal{H}_\varphi \subseteq C(G)$ for each continuous positive definite function $\varphi \in \mathcal{P}(G)$.

(d) If $\mathcal{A}$ is a Banach-$*$-algebra, then a linear functional $f\colon \mathcal{A} \to \mathbb{C}$ is said to be *positive* if it is a positive definite function on the involutive semigroup $((\mathcal{A}, \cdot), *)$, i.e., if the sesquilinear kernel

$$K(a, b) := f(ab^*)$$

is positive definite.

Remark 5.3.2. If $(\pi, \mathcal{H})$ is a unitary representation of the involutive semigroup $(S, *)$ and $v \in \mathcal{H}$, then the function

$$\pi^v\colon S \to \mathbb{C}, \quad s \mapsto \langle \pi(s)v, v\rangle$$

is positive definite because

$$K(s, t) := \pi^v(st^*) = \langle \pi(s)\pi(t^*)v, v\rangle = \langle \pi(t^*)v, \pi(s^*)v\rangle,$$

and the positive definiteness of this kernel follows from Remark 3.3.1. The corresponding realization map is

$$\gamma\colon S \to \mathcal{H}, \quad \gamma(s) = \pi(s^*)v$$

(Theorem 3.3.3). If v is a cyclic vector, then this map has total range, so that

$$\varphi_\gamma\colon \mathcal{H} \to \mathcal{H}_K, \quad \varphi_\gamma(w)(s) := \langle w, \pi(s^*)v\rangle = \langle \pi(s)w, v\rangle$$

is an isomorphism of Hilbert spaces. In view of

$$\varphi_\gamma(\pi(t)w)(s) = \langle \pi(st)w, v\rangle = \varphi_\gamma(w)(st),$$

this map intertwines the representation π of S on $\mathcal{H}$ with the representation of S on $\mathcal{H}_K$ by $(s.f)(x) := f(xs)$.

In the following we write a cyclic unitary representation $(\pi, \mathcal{H})$ with cyclic vector v as a triple $(\pi, \mathcal{H}, v)$.

Proposition 5.3.3. *Let G be a topological group.*

(a) *For every continuous unitary representation $(\pi, \mathcal{H})$ of G and $v \in \mathcal{H}$,*

$$\pi^v(g) := \langle \pi(g)v, v \rangle$$

is a continuous positive definite function.

(b) *Conversely, for every continuous positive definite function $\varphi \colon G \to \mathbb{C}$, the reproducing kernel space $\mathcal{H}_\varphi \subseteq C(G, \mathbb{C})$ with the kernel $K(g,h) := \varphi(gh^{-1})$ carries a continuous unitary representation of G, given by*

$$(\pi_\varphi(g)f)(x) := f(xg),$$

satisfying $\pi_\varphi^\varphi = \varphi$, i.e.,

$$\varphi(g) = \langle \pi_\varphi(g)\varphi, \varphi \rangle \quad \text{for} \quad g \in G.$$

(c) *A continuous unitary representation $(\pi, \mathcal{H})$ of G is cyclic if and only if it is equivalent to some $(\pi_\varphi, \mathcal{H}_\varphi)$ with $\varphi \in \mathcal{P}(G)$.*

(d) *For two cyclic unitary representations $(\pi, \mathcal{H}, v)$ and $(\pi', \mathcal{H}', v')$ of G, there exists a unitary intertwining operator $\Gamma \colon \mathcal{H} \to \mathcal{H}'$ with $\Gamma(v) = v'$ if and only if $\pi^v = (\pi')^{v'}$.*

Proof. (a) follows immediately from Remark 5.3.2.

(b) We first observe that the kernel K is invariant under right multiplications:

$$K(xg, yg) = \varphi(xg(yg)^{-1}) = K(x,y), \quad x, y, g \in G,$$

so that we obtain a continuous unitary representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of G (Proposition 5.1.6). Finally, we note that $K(x,g) = \varphi(xg^{-1})$ leads to $K_g = \varphi \circ \rho_{g^{-1}}$, so that

$$\langle \pi_\varphi(g)\varphi, \varphi \rangle = \langle \pi_\varphi(g)\varphi, K_{\mathbf{1}} \rangle = (\pi_\varphi(g)\varphi)(\mathbf{1}) = \varphi(g).$$

(c) To see that $(\pi_\varphi, \mathcal{H}_\varphi)$ is cyclic, we show that φ is a cyclic vector. In fact, if $f \in \mathcal{H}_\varphi$ is orthogonal to $\pi_\varphi(G)\varphi$, then we have

$$f(g) = (\pi(g)f)(\mathbf{1}) = \langle \pi(g)f, \varphi \rangle = \langle f, \pi(g)^{-1}\varphi \rangle = 0,$$

for each $g \in G$, and therefore $f = 0$.

If, conversely, $(\pi, \mathcal{H})$ is a cyclic continuous unitary representation of G and $v \in \mathcal{H}$ a cyclic vector, then $\varphi := \pi^v \in \mathcal{P}(G)$ by (a), and Remark 5.3.2 implies that the map

$$\varphi_\gamma \colon \mathcal{H} \to \mathcal{H}_\varphi, \quad \varphi_\gamma(w)(g) = \langle \pi(g)w, v \rangle$$

is a unitary intertwining operator. We conclude that $(\pi, \mathcal{H}) \cong (\pi_\varphi, \mathcal{H}_\varphi)$.

(d) If $\Gamma \colon \mathcal{H} \to \mathcal{H}'$ is a unitary intertwining operator with $\Gamma(v) = v'$, then we have for each $g \in G$ the relation

$$(\pi')^{v'}(g) = \langle \pi'(g)v', v' \rangle = \langle \Gamma(\pi(g)v), \Gamma(v) \rangle = \langle \pi(g)v, v \rangle = \pi^v(g).$$

Suppose, conversely, that $\varphi := \pi^v = (\pi')^{v'}$. Then we obtain with Remark 5.3.2 unitary intertwining operators

$$\Gamma\colon \mathcal{H} \to \mathcal{H}_\varphi \quad \text{with} \quad \Gamma(v) = \varphi$$

and

$$\Gamma'\colon \mathcal{H}' \to \mathcal{H}_\varphi \quad \text{with} \quad \Gamma'(v') = \varphi.$$

Then $\Gamma' \circ \Gamma^{-1}\colon \mathcal{H} \to \mathcal{H}'$ is a unitary intertwining operator mapping v to v'. □

Applying the Irreducibility Criterion for Reproducing Kernel Spaces (Theorem 5.1.12) to the right action of G on itself and $J = 1$, we immediately obtain:

Proposition 5.3.4. *For $\varphi \in \mathcal{S}(G)$, the representation $(\pi_\varphi, \mathcal{H}_\varphi)$ is irreducible if and only if φ is an extreme point of $\mathcal{S}(G)$, i.e., a pure state.*

Proof. We only have to observe that $\mathbb{R}_+\varphi$ is an extreme ray of the cone $\mathcal{P}(G)$ if and only if φ is an extreme point of $\mathcal{S}(G)$ (Exercise 5.3.2). □

Example 5.3.5. (a) Let G be an abelian topological group and $\widehat{G}$ its character group. Further, let $\mathfrak{S}$ be the smallest σ-algebra on $\widehat{G}$ for which all the evaluation functions

$$\widehat{g}\colon \widehat{G} \to \mathbb{T}, \quad \chi \mapsto \chi(g)$$

are Borel measurable. [1] It is generated by the inverse images $\widehat{g}^{-1}(E)$, $E \subseteq \mathbb{T}$ an open subset. For any finite measure μ on $(\widehat{G}, \mathfrak{S})$, we obtain a unitary representation of G on $L^2(\widehat{G}, \mu)$ by

$$(\pi(g)f)(\chi) := \widehat{g}(\chi)f(\chi).$$

Suppose that G is first countable, i.e., has a countable basis of $\mathbf{1}$-neighborhoods. We claim that π is continuous. It suffices to verify sequential continuity of the functions

$$\pi_{v,w}\colon G \to \mathbb{C}, \quad \pi_{v,w}(g) = \langle \pi(g)v, w\rangle = \int_{\widehat{G}} \widehat{g}(\chi)v(\chi)\overline{w(\chi)}\, d\mu(\chi).$$

In view of the estimate

$$|\widehat{g}(\chi)v(\chi)\overline{w(\chi)}| = |v(\chi)\overline{w(\chi)}| = |(v\overline{w})(\chi)|$$

and $v\overline{w} \in L^1(\widehat{G}, \mu)$, this follows from Lebesgue's Dominated Convergence Theorem.

It follows in particular that the function

$$\widehat{\mu}(g) := \int_{\widehat{G}} \widehat{g}(\chi)\, d\mu(\chi) = \langle \pi(g)1, 1\rangle$$

[1] In general, this σ-algebra may be much smaller than the σ-algebra of Borel sets for the topology of pointwise convergence. This is due to the fact that arbitrary unions of open sets are open, but for measurable subsets, only countable unions are measurable.

is continuous and positive definite. It is called the *Fourier transform of* μ. In Corollary 6.2.19 below we shall develop tools to verify that 1 is a cyclic vector for this representation (see also Example 6.2.21), [2] so that

$$\mathcal{H}_{\widehat{\mu}} \cong L^2(\widehat{G}, \mu)$$

and we shall also see that for a locally compact abelian group G every positive definite function is the Fourier transform of a unique Radon measure μ on $\widehat{G}$ (Bochner's Theorem).

(b) The preceding construction applies in many special cases. If $G = (V, +)$ is a Banach space, then $\widehat{G} \cong V'$, the topological dual space (Exercise 4.2.4), so that we obtain for any finite Borel measure μ on V' the continuous positive definite function

$$\widehat{\mu}(v) := \int_{V'} e^{i\alpha(v)}\, d\mu(\alpha).$$

For $V = \mathbb{R}^n$ this specializes to

$$\widehat{\mu}(x) := \int_{\mathbb{R}^n} e^{i\langle x, y\rangle}\, d\mu(y),$$

which in probability theory is also called the *characteristic function of the measure* μ.

5.3.2 Spherical Functions

In many situations one is interested in understanding and classifying unitary representations in reproducing kernel spaces $\mathcal{H}_Q$ of continuous functions (and with continuous point evaluations) on a homogeneous space $X := G/K$ of a topological group G and a closed subgroup K, where G acts by $(\pi(g)f)(x) = f(g^{-1}x)$. For any unitary representation $(\pi_Q, \mathcal{H}_Q)$ in such a Hilbert space, we know from Proposition 5.1.6 that its kernel Q on X is G-invariant, i.e.,

$$Q(g.x, g.y) = Q(x, y) \quad \text{for} \quad g \in G, x, y \in X.$$

Hence the problem is to describe the convex cone $\mathcal{P}(X)^G$ of G-invariant positive definite continuous kernels on X. In particular, we are interested in its extremal rays because they correspond to irreducible representations (Theorem 5.1.12).

Choosing the natural base point $x_0 := \mathbf{1}K \in G/K = X$, we see that the transitivity of the G-action on X implies that any invariant kernel Q satisfies

$$Q(g.x_0, h.x_0) = Q(h^{-1}g.x_0, x_0),$$

hence is determined by the function

$$\varphi\colon G \to \mathbb{C}, \quad \varphi(g) := Q(g.x_0, x_0).$$

[2] The main difficulty is to show that the von Neumann algebra generated by the functions $\widehat{g}$, $g \in G$, is all of $L^\infty(\widehat{G}, \mu)$, when we consider these functions as operators on $L^2(\widehat{G}, \mu)$. This requires spectral measures as a tool (cf. Corollary 6.2.19).

Clearly, φ is K-biinvariant because for $h_1, h_2 \in K$ we have

$$\varphi(h_1gh_2) = Q(h_1gh_2.x_0, x_0) = Q(g.x_0, h_1^{-1}.x_0) = Q(g.x_0, x_0).$$

If, conversely, $\varphi\colon G \to \mathbb{C}$ is an K-biinvariant function, then

$$Q(g_1K, g_2K) := \varphi(g_2^{-1}g_1)$$

is a well-defined G-invariant kernel on G/K and $\overline{\varphi}(gK) := \varphi(g)$ defines a K-invariant function on G/K (cf. Exercise 5.3.8 for continuity issues). This leads to the following concept.

Lemma 5.3.6. *The kernel Q on G/K is positive definite if and only if the function $\varphi\colon G \to \mathbb{C}$, defined by $\varphi(g) := Q(g.x_0, x_0)$ is positive definite.*

Proof. We have

$$Q_\varphi(g, h) := \varphi(gh^{-1}) = Q(gh^{-1}.x_0, x_0) = Q(h^{-1}.x_0, g^{-1}.x_0),$$

and this kernel on G is positive definite if and only if Q is positive definite because the map $G \to G/K, g \mapsto g^{-1}.x_0$ is surjective. $\square$

Definition 5.3.7. Let (G, K) be a pair of a topological group G and a subgroup K. A continuous positive definite function $\varphi \in \mathcal{P}(G)$ is said to be *spherical* if it is K-biinvariant and the representation $(\pi_\varphi, \mathcal{H}_\varphi)$ with $(\pi_\varphi(g)f)(x) := f(g^{-1}x)$ is irreducible.

Remark 5.3.8. (K-biinvariant functions and K-fixed vectors) If $\varphi \in \mathcal{P}(G)$ is a K-biinvariant positive definite function on G, then the element $\varphi \in \mathcal{H}_\varphi$ is a K-invariant vector, i.e., an element of $\mathcal{H}_\varphi^K$.

Suppose, conversely, that $(\pi, \mathcal{H})$ is a continuous unitary representation of G and $v_0 \in \mathcal{H}^K$ is a K-fixed vector. Then the orbit map $G \to \mathcal{H}, g \mapsto \pi(g)v$ factors through a continuous G-equivariant map

$$\gamma\colon G/K \to \pi(g)v_0,$$

so that the closed G-invariant subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ generated by v_0 leads to a realization triple $(G/K, \gamma, \mathcal{H}_0)$ for the kernel

$$Q(gK, hK) = \langle \gamma(hK), \gamma(gK) \rangle = \langle \pi(h)v_0, \pi(g)v_0 \rangle = \pi^{v_0}(g^{-1}h)$$

(cf. Definition 3.3.4), and hence to a realization of the representation $(\pi, \mathcal{H}_0)$ in continuous functions on G/K.

Remark 5.3.9. The terminology is due to the special case where X is the sphere

$$\mathbb{S} := \mathbb{S}(V) := \{x \in V : \|x\| = 1\}$$

in a real euclidean space V, such as $\mathbb{R}^n$. On this space the orthogonal group

$$\mathrm{O}(V) := \{g \in \mathrm{GL}(V) : g^\top = g^{-1}\}$$

of linear surjective isometries acts transitively (Exercise 5.3.9) and the corresponding spherical functions on $\mathrm{O}(V)$ correspond to the irreducible representations of $\mathrm{O}(V)$ in reproducing kernel spaces in $C(\mathbb{S})$.

Definition 5.3.10. Let $(\pi, \mathcal{H})$ be a unitary representation of the group G and $K \subseteq G$ a subgroup. For a homomorphism $\chi\colon K \to \mathbb{T}$, we write

$$\mathcal{H}^{K,\chi} := \{v \in \mathcal{H}\colon (\forall k \in K)\, \pi(k)v = \chi(k)v\}$$

for the corresponding common eigenspace in $\mathcal{H}$. For the trivial character $\chi = 1$, we write $\mathcal{H}^K = \mathcal{H}^{K,1}$ for the space of K-fixed vectors in $\mathcal{H}$.

Proposition 5.3.11. (Irreducibility criterion for cyclic representations) *Suppose that the unitary representation $(\pi, \mathcal{H})$ of G is generated by the subspace $\mathcal{H}^{K,\chi}$ for some character $\chi\colon K \to \mathbb{T}$ of a subgroup K. If $\dim \mathcal{H}^{K,\chi} = 1$, then $(\pi, \mathcal{H})$ is irreducible.*

Proof. Let $\mathcal{K} \subseteq \mathcal{H}$ be a closed non-zero invariant subspace and $P\colon \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. Since P commutes with G, we then have $P(\mathcal{H}^{K,\chi}) = \mathcal{K}^{K,\chi}$. Since $\mathcal{K} = P(\mathcal{H})$ is generated, as a unitary G-representation, by $\mathcal{H}^{K,\chi}$, the representation on $\mathcal{K}$ is generated by $P(\mathcal{H}^{K,\chi}) = \mathcal{K}^{K,\chi}$, and we see in particular that $\mathcal{K}^{K,\chi} \neq \{0\}$. As $\mathcal{H}^{K,\chi}$ is one-dimensional, we this leads to $\mathcal{H}^{K,\chi} = \mathcal{K}^{K,\chi} \subseteq \mathcal{K}$, so that $\mathcal{H} = \mathcal{K}$ follows from the fact that $\mathcal{H}^{K,\chi}$ generates $\mathcal{H}$. This proves that the representation $(\pi, \mathcal{H})$ is irreducible. $\square$

Proposition 5.3.12. *Let $Q \in \mathcal{P}(G/K)^G$ be a G-invariant positive definite kernel. Then the following assertions hold:*

(a) *$\mathcal{H}_Q$ is generated by the closed subspace $\mathcal{H}_Q^K$ of K-fixed vectors.*

(b) *If $\dim \mathcal{H}_Q^K = 1$, then the unitary representation $(\pi_Q, \mathcal{H}_Q)$ of G is irreducible.*

Proof. (a) The evaluation in the base point $x_0 = \mathbf{1}K \in G/K$ is fixed under K, so that $Q_{x_0} \in \mathcal{H}_Q^K$. Now $\pi_Q(g).Q_{x_0} = Q_{g.x_0}$ (Proposition 5.1.6) implies that the G-invariant subspace generated by $\mathcal{H}_Q^K$ contains all elements $Q_x, x \in G/K$, hence is dense.

(b) follows by combining (a) with Proposition 5.3.11. $\square$

Definition 5.3.13. Representations $(\pi, \mathcal{H})$ of G with $\dim \mathcal{H}^K = 1$ which are generated by this subspace are called *class one representations*. As we have seen above, class one representations are always irreducible.

In view of Remark 5.3.8 all class one representations can be realized in reproducing kernel spaces $\mathcal{H}_Q \subseteq C(G/K)$ with kernel $Q(xK, yK) = \langle \pi(x^{-1}y)v_K, v_K\rangle$ for any $v_K \in \mathcal{H}^K$. Here the main point of the class one condition is that the vector $v_K \in \mathcal{H}_K$ is unique up to scalar multiples, so that the space $B_G(\mathcal{H}, C(G/K))$ of intertwining operators is one-dimensional, hence the name.

Remark 5.3.14. We have seen in Proposition 5.3.11 that any class one representation is irreducible. The main advantage of class one representations is that they are completely encoded in the single spherical function $\varphi(g) = \langle v_K, \pi(g)v_K\rangle$, where $v_K \in \mathcal{H}^K$ is a unit vector because every other unit vector leads to the same function and $(\pi, \mathcal{H}) \cong (\pi_\varphi, \mathcal{H}_\varphi)$ by Proposition 5.3.3. This fact makes class one representations particularly accessible via the corresponding spherical functions.

The converse is not true in general, i.e., there may be irreducible representations $(\pi_Q, \mathcal{H}_Q)$ in $C(G/K)$ which are not class one. To see this, we choose $K := \{\mathbf{1}\}$, then each irreducible representation $(\pi, \mathcal{H})$ of G can be realized in $C(G)$ by a right invariant kernel defined by a positive definite function (Proposition 5.3.3), but in this case $\mathcal{H}^K = \mathcal{H}$ is not one-dimensional in general.

To obtain situations where all irreducible representation are type one, so that their classification is more manageable, one has to look for conditions ensuring that K is "large" in G. One such requirement is that for every continuous unitary representation $(\pi, \mathcal{H})$ of G and the orthogonal projection $P_K \colon \mathcal{H} \to \mathcal{H}^K$, the set of all operators

$$S := \{P_K \pi(g) P_K \colon g \in G\} \subseteq B(\mathcal{H}^K)$$

is commutative. Then (G, K) is called a *Gelfand pair.*

If $\mathcal{K} \subseteq \mathcal{H}^K$ is an S-invariant closed subspace, then $P_K(\pi(G)\mathcal{K}) \subseteq \mathcal{K}$ implies that the closed G-invariant subspace $\mathcal{K}_G$ generated by $\mathcal{K}$ satisfies $\mathcal{K}_G^K = \mathcal{K}$. In particular, the set S acts irreducibly on $\mathcal{H}^K$ if $(\pi, \mathcal{H})$ is irreducible. Since S is commutative and $*$-invariant, in view of Schur's Lemma, this implies that $\dim \mathcal{H}^K = 1$, hence that $(\pi, \mathcal{H})$ is of class one.

Examples 5.3.15. If K is a topological group and $G := K \times K$, then $X := K$ is a homogeneous space of G by the action $(g_1, g_2).x := g_1 x g_2^{-1}$. Then the stabilizer of the element $x_0 := \mathbf{1}$ is the diagonal subgroup

$$\Delta_K := \{(k, k^{-1}) \colon k \in K\} \cong K,$$

so that $K \cong G/\Delta_K$. In this context a G-invariant kernel Q on K corresponds via $\varphi(k) := Q(k, \mathbf{1})$ to a conjugation invariant function on K. Such functions on a group are also called *central.*

One can show that $(K \times K, \Delta_K)$ always is a Gelfand pair if K is compact.

Typical examples of central functions on groups arise from finite dimensional unitary representations. For any such representation $(\pi, \mathcal{H})$ of K, the function

$$\chi_\pi(k) := \operatorname{tr}(\pi(k))$$

is a central positive definite function, called the *character of* π. In fact, the space $B(\mathcal{H})$ carries the Hilbert space structure given by the Hilbert–Schmidt scalar product

$$\langle A, B\rangle := \operatorname{tr}(AB^*) = \operatorname{tr}(B^*A),$$

which leads to a unitary representation of $K \times K$ on $B(\mathcal{H})$ via

$$\widetilde{\pi}(k_1, k_2)A := \pi(k_1) A \pi(k_2)^*,$$

and then

$$\chi_\pi(k) = \operatorname{tr}(\pi(k)) = \langle \widetilde{\pi}(k, \mathbf{1})\mathbf{1}, \mathbf{1}\rangle.$$

If $(\pi, \mathcal{H})$ is an irreducible representation of K, then one can derive from Proposition 4.3.9 that $(\widetilde{\pi}, B(\mathcal{H}))$ is also irreducible, so that $\chi_\pi(k_1 k_2^*)$ defines a spherical function on $K \times K$.

It is an important representation theoretic problem to classify for a given pair (G, K) the spherical functions. As the following example shows, this problem has tight links with problems in classical harmonic analysis.

Example 5.3.16. (Spherical functions on motion groups) Let V be a euclidean space and

$$G \cong V \rtimes \mathrm{O}(V)$$

be its group of surjective isometries (Exercise 5.3.4). We endow $\mathrm{O}(V)$ with the strong operator topology and recall that it acts continuously on V (Exercise 1.2.2), so that G is a topological group with respect to the product topology (Exercise 1.1.7). The group G acts transitively on V by $(v, g).x := g.x + v$ and the stabilizer of 0 is the group $K := \mathrm{O}(V)$ of linear surjective isometries.

We are interested in the spherical functions of the pair (G, K) and, more generally, in positive definite K-biinvariant functions on G. Since K acts transitively on the spheres in V (Exercise 5.3.9), every K-biinvariant continuous function $\varphi \colon G \to \mathbb{C}$ can be written as

$$\varphi(v, g) = f(\|v\|),$$

where $f \colon \mathbb{R}_+ \to \mathbb{C}$ is a continuous function. Now the main problem is to see for which functions f, the function φ is spherical, resp., positive definite.

We claim that the functions

$$\varphi_\lambda(v, k) := e^{-\lambda\|v\|^2} \tag{5.10}$$

are positive definite for $\lambda \geq 0$. To verify this claim, we note that the corresponding kernel is given by

$$\begin{aligned} Q((v,k),(w,h)) &= \varphi_\lambda((v,k)(w,h)^{-1}) = \varphi_\lambda((v,k)(-h^{-1}w, h^{-1})) \\ &= \varphi_\lambda(v - kh^{-1}w) = e^{-\lambda\|v - kh^{-1}w\|^2} = e^{-\lambda\|k^{-1}v - h^{-1}w\|^2} \\ &= e^{-\lambda\|v\|^2} e^{-\lambda\|w\|^2} e^{2\lambda\langle k^{-1}v, h^{-1}w\rangle} \end{aligned}$$

(cf. Exercise 5.3.1).

To see that this kernel is positive definite, we first use Remark 3.3.1(b) and Corollary 3.2.2 to see that it suffices to show that the kernel

$$((v,k),(w,h)) \mapsto \langle k^{-1}v, h^{-1}w\rangle$$

is positive definite, but this follows from Remark 3.3.1(a).

Now Proposition 3.2.1 implies immediately that any function of the form

$$\varphi(v,k) := \int_0^\infty \varphi_\lambda(v,k)\, d\mu(\lambda) = \int_0^\infty e^{-\lambda\|v\|^2}\, d\mu(\lambda)$$

for a finite Borel measure μ on $\mathbb{R}_+$ is positive definite and it is an important result of Schoenberg that, if V is infinite dimensional, all K-biinvariant positive definite functions on G are of this form ([Sch38]), i.e., $\varphi(v, k) = f(\|v\|)$, where

$$f(r) = \int_0^\infty e^{-\lambda r}\, d\mu(\lambda),$$

i.e., f is the Laplace transform of a finite Borel measure on $\mathbb{R}_+$. In particular, the functions φ_λ are the extreme points in the set $\mathcal{S}(G, K)$ of K-biinvariant positive definite normalized functions and therefore the corresponding representation $(\pi_{Q_\lambda}, \mathcal{H}_{Q_\lambda})$ on the reproducing kernel space associated to the G-invariant kernel Q_λ on V is irreducible. As we have seen above, this kernel is given by

$$Q_\lambda(v, w) = \varphi(v - w, 0) = e^{-\lambda\|v-w\|^2} = e^{-\lambda\|v\|^2} e^{-\lambda\|w\|^2} e^{2\lambda\langle v,w\rangle}.$$

For $Q'(v, w) := e^{2\lambda\langle v,w\rangle}$ we thus obtain from Exercise 5.1.2 that the map

$$\Phi\colon \mathcal{H}_{Q'} \to \mathcal{H}_{Q_\lambda}, \quad \Phi(f)(v) := e^{-\lambda\|v\|^2} f(v)$$

is unitary. Note that $\mathcal{H}_{Q'}$ is the Fock space $\mathcal{F}^{2\lambda}(V)$ on V, corresponding to the scaled inner product $2\lambda\langle v, w\rangle$.

On the space $\mathcal{H}_Q$, the unitary representation of the affine group G is simply given by composition

$$(\pi_\lambda(v, k)f)(x) = f((v, k)^{-1}.x) = f(k^{-1}(x - v)),$$

but the unitary representation π'_λ on the Fock space $\mathcal{H}_{Q'}$ transferred by Φ satisfies

$$(\pi'_\lambda(v, k)f)(x) = e^{-\lambda\|v\|^2+2\lambda\langle x,v\rangle} f(k^{-1}(x - v)),$$

which corresponds to the formula found in Remark 5.2.2 for the canonical action of the translation group on the Fock space.

For finite dimensional euclidean spaces $V \cong \mathbb{R}^n$, the spherical functions have a more complicated structure. They are the Fourier transforms of the invariant measures on spheres. We shall see in Chapter 6 below where this comes from.

Examples 5.3.17. (a) Let $(\pi, \mathcal{F}(\mathcal{H}))$ be the representation of the Heisenberg group $\mathrm{Heis}(\mathcal{H})$ on the Fock space

$$(\pi(t, v)f)(z) = e^{it+\langle z,v\rangle-\frac{1}{2}\langle v,v\rangle} f(z - v)$$

(Proposition 5.2.1). Recall that $\mathcal{F}(\mathcal{H}) = \mathcal{H}_Q$ for $Q(z, w) = e^{\langle z,w\rangle}$.

First we note that the constant function $1 = Q_0$ is a cyclic vector. Indeed, for $J((t, v), x) = e^{it+\langle x,v\rangle-\frac{1}{2}\langle v,v\rangle}$ we have

$$\pi(t, v)Q_x = \overline{J((t, v)^{-1}, x)} Q_{x+v} \in \mathbb{C}^\times Q_{x+v},$$

so that $Q_x \in \mathbb{C}^\times \pi(0, x)Q_1$, and since the set $\{Q_x : x \in \mathcal{H}\}$ is total, the vector 1 is cyclic. The corresponding positive definite function is

$$\varphi(t, v) = \langle \pi(t, v)1, 1\rangle = (\pi(t, v)1)(0) = J((t, v), 0) = e^{it-\frac{1}{2}\langle v,v\rangle}.$$

The affine space $\mathcal{H}$ is a homogeneous $\mathrm{Heis}(\mathcal{H})/K$ for $K = \mathbb{T} \times \{0\}$ and $\pi(t, 0) = e^{it}\mathbf{1}$ for all $t \in \mathbb{R}$. Therefore the character $\chi(t, 0) := e^{it}$ of K satisfies $\mathcal{F}(\mathcal{H})^{K,\chi} = \mathcal{F}(\mathcal{H})$, so that the irreducibility criterion Proposition 5.3.11

does not apply. However, one can show with tools based on complex analysis (Kobayashi's Theorem) that $(\pi, \mathcal{F}(\mathcal{H}))$ is irreducible.

(b) Let $(\pi, \mathcal{F}(\mathcal{H}))$ be the representation of the semidirect product group $G := \mathrm{Heis}(\mathcal{H}) \rtimes \mathrm{U}(\mathcal{H})$ on the Fock space

$$(\pi(t, v, g)f)(z) = e^{it+\langle z,v\rangle - \frac{1}{2}\langle v,v\rangle} f(g^{-1}(z - v))$$

(Remark 5.2.4). Since 1 is cyclic for $\mathrm{Heis}(\mathcal{H})$, it is in particular cyclic for G. For the action of G on $\mathcal{H}$ by $(t, v, g).z = gx + v$ we have $\mathcal{H} \cong G/K$ for $K = \mathbb{T} \times \{0\} \times \mathrm{U}(\mathcal{H})$, and 1 is a χ-eigenvector for K for the character $\chi(t, 0, g) := e^{it}$.

We claim that $\mathcal{F}(\mathcal{H})^{K,\chi} = \mathbb{C}1$. Considering the action of the subgroup $\mathbb{T}\mathbf{1} \subseteq \mathrm{U}(\mathcal{H})$, the decomposition $\mathcal{F}(\mathcal{H}) = \widehat{\oplus}_{m\in\mathbb{N}_0}\mathcal{F}_m(\mathcal{H})$ from Proposition 5.2.3 implies that $\mathcal{F}(\mathcal{H})^{K,\chi} \subseteq \mathcal{F}_0(\mathcal{H}) = \mathbb{C}1$. Therefore the irreducibility criterion Proposition 5.3.11 implies that the representation of G on $\mathcal{F}(\mathcal{H})$ is irreducible.

Note that the same conclusion applies for the *oscillator group* $\mathrm{Heis}(\mathcal{H}) \rtimes \mathbb{T}\mathbf{1}$.

(c) Let $(\pi_m, \mathcal{F}_m(\mathcal{H}))$ be the representation of $\mathrm{U}(\mathcal{H})$ on the subspace $\mathcal{F}_m(\mathcal{H})$ of homogeneous functions of degree m in $\mathcal{F}(\mathcal{H})$ by $(\pi_m(g)f)(z) = f(g^{-1}z)$. Its reproducing kernel is $Q(z, w) = \frac{\langle z,w\rangle^m}{m!}$ (Proposition 5.2.3). Since all functions in this space are homogeneous, they are uniquely determined by their restrictions to the unit sphere

$$\mathbb{S} := \{v \in \mathcal{H} \colon \|v\| = 1\}.$$

Therefore the set $\{Q_v \colon \|v\| = 1\}$ is total.

Let $(e_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}$, $j_0 \in J$ be a fixed element and $e_0 := e_{j_0}$. Then $\mathrm{U}(\mathcal{H})e_0 = \mathbb{S}$ and $\pi_m(g)Q_v = Q_{gv}$ imply that the element Q_{e_0} is a cyclic vector unit. The corresponding positive definite function is

$$\begin{aligned}\varphi(g) &= \langle \pi_m(g)Q_{e_0}, Q_{e_0}\rangle = (\pi_m(g)Q_{e_0})(e_0) = Q_{e_0}(g^{-1}e_0)\\ &= \frac{1}{m!}\langle g^{-1}e_0, e_0\rangle^m = \frac{1}{m!}\overline{\langle ge_0, e_0\rangle^m} = \frac{1}{m!}\overline{g_{00}}^m.\end{aligned}$$

Let $T \cong \mathbb{T}^J$ be the subgroup of diagonal operators in $\mathrm{U}(\mathcal{H})$ with respect to the orthonormal basis. Then Q_{e_0} is a T-eigenvector for some character χ of T, given by

$$\chi(t) = t_{j_0}^{-m},$$

and we have seen in Proposition 5.2.3 that any character $\chi_{\mathbf{m}}$ of T occurs at most with multiplicity 1 in $\mathcal{F}(\mathcal{H})$. Therefore

$$\mathcal{F}_m(\mathcal{H})^{T,\chi} = \mathbb{C}Q_{e_0}$$

is one-dimensional, and the irreducibility criterion Proposition 5.3.11 implies that $(\pi_m, \mathcal{F}_m(\mathcal{H}))$ is irreducible.

(d) We consider the representations $(\pi_m, \mathcal{H}_m)$ of the group $G = \mathrm{SU}_{1,1}(\mathbb{C})$ on the Hilbert space $\mathcal{H}_m \subseteq \mathcal{O}(\mathcal{D})$ on the open unit disc $\mathcal{D} \subseteq \mathbb{C}$ with kernel $Q(z, w) = (1 - z\overline{w})^{-m}$, where $m \in \mathbb{N}$ is fixed:

$$(\pi_m(g)f)(z) = J^m(g, z)f(g^{-1}.z) = (a - \overline{b}z)^{-m} f\left(\frac{\overline{a}z - b}{a - \overline{b}z}\right).$$

First we show that $1 = Q_0$ is a cyclic vector. Since the action of $\mathrm{SU}_{1,1}(\mathbb{C})$ on $\mathcal{D}$ is transitive, the relation

$$\pi_m(g)Q_0 = \overline{J^m(g^{-1}, 0)}Q_{g.0} = a^{-m}Q_{g.0}$$

implies that Q_0 is cyclic. This relation also implies that it is an eigenvector of the subgroup

$$T := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{T} \right\} \subseteq \mathrm{SU}_{1,1}(\mathbb{C})$$

of diagonal matrices, and the corresponding character is

$$\chi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^{-m}.$$

The other T-eigenvector in $\mathcal{H}_m$ are the monomials z^n, $n \in \mathbb{N}$, which form a total orthonormal system. For $g \in T$ we have $J^m(g, z) = a^{-m}$, so that we obtain for $f(z) = z^n$ the relation

$$(\pi_m(g)f)(z) = a^{-m} f(g^{-1}.z) = a^{-m-2n} f(z).$$

It follows in particular that $\mathcal{H}_m^{T,\chi} = \mathbb{C}1$, so that Proposition 5.3.11 implies that $(\pi_m, \mathcal{H}_m)$ is irreducible.

Example 5.3.18. (A non-type I factor representation) Let $\mathbf{1} \neq G$ be a discrete group such that all non-trivial conjugacy classes are infinite, such as $\mathrm{PGL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^\times \mathbf{1}$ (cf. Exercise 5.3.5).

We consider the representation $(\pi, \mathcal{H})$ of $G \times G$ on the Hilbert space $\mathcal{H} = \ell^2(G, \mathbb{C}) \subseteq \mathbb{C}^G$ given by

$$(\pi(g, h)f)(x) := f(g^{-1}xh).$$

First we show that this representation is irreducible. We consider the ONB $(\delta_g)_{g\in G}$, consisting of δ-functions satisfying

$$\pi(g, h)\delta_x = \delta_{gxh^{-1}}, \quad g, h, x \in G.$$

Clearly, the vector $\delta_{\mathbf{1}}$ is cyclic and invariant under the diagonal subgroup

$$K := \Delta_G := \{(g, g) \colon g \in G\}.$$

The K-invariance means for an element $f \in \ell^2(G, \mathbb{C})$ that it is constant on conjugacy classes, and since all non-trivial conjugacy classes are infinite, we have

$$\ell^2(G, \mathbb{C})^K = \mathbb{C}\delta_{\mathbf{1}}.$$

Therefore Proposition 5.3.11 implies that $(\pi, \ell^2(G, \mathbb{C}))$ is irreducible.

This implies that the right regular representation of G on $\ell^2(G, \mathbb{C})$, defined by $\pi_r(g) := \pi(\mathbf{1}, g)$, is a factor representation. We claim that this factor representation is not of type I, i.e., a multiple of an irreducible representation.

Assume the contrary, i.e., $\mathcal{H} \cong \mathcal{M}_\pi \widehat{\otimes} \mathcal{H}_\pi$ is isotypic with multiplicity space $\mathcal{M}_\pi$. Then $\pi_r(G)' \cong B(\mathcal{M}_\pi)$ by Lemma 4.3.8. We consider the map

$$T\colon \pi_r(G)' \to \mathbb{C}, \quad T(A) := \langle A\delta_{\mathbf{1}}, \delta_{\mathbf{1}} \rangle = (A\delta_{\mathbf{1}})(\mathbf{1}).$$

It clearly satisfies $T(\mathbf{1}) = 1$, and we claim that it is a trace, i.e.,

$$T(AB) = T(BA) \quad \text{for} \quad A, B \in \pi_r(G)'.$$

For each $A \in \pi_r(G)'$, the function $a\colon = A\delta_{\mathbf{1}}$ satisfies

$$A\delta_x = A\pi_r(x)^{-1}\delta_{\mathbf{1}} = \pi_r(x)^{-1}A\delta_{\mathbf{1}} = \pi_r(x)^{-1}a, \quad \text{i.e.,} \quad (A\delta_x)(y) = a(yx^{-1}).$$

This further leads to

$$(Af)(g) = \langle Af, \delta_g \rangle = \sum_{x \in G} f(x)\langle A\delta_x, \delta_g \rangle = \sum_{x \in G} a(gx^{-1})f(x).$$

For $b := B\delta_{\mathbf{1}}$, this leads to

$$\begin{aligned} T(AB) &= A(B\delta_{\mathbf{1}})(\mathbf{1}) = \sum_{x \in G} a(x^{-1})(B\delta_{\mathbf{1}})(x) \\ &= \sum_{x \in G} a(x^{-1})b(x) = \sum_{x \in G} b(x^{-1})a(x) = \ldots = T(BA). \end{aligned}$$

In view of Exercise 5.3.6, the existence of T implies that $\dim \mathcal{M}_\pi < \infty$, and hence that $\dim \operatorname{span} \pi_l(G) \le \dim B(\mathcal{M}_\pi) < \infty$. This in turn implies that $\pi_l(G)\delta_{\mathbf{1}} = \{\delta_g \colon g \in G\}$ is finite dimensional, which implies that G is finite, hence trivial. This contradicts our initial hypothesis.

Exercises for Section 5.3

Exercise 5.3.1. Let $G = N \rtimes_\alpha K$ be a semidirect product group and $\varphi \in \mathcal{P}(N)$ be a positive definite function on N which is K-invariant in the sense that

$$\varphi(k.n) = \varphi(n) \quad \text{for} \quad k \in K, n \in N.$$

Then

$$\psi\colon G \to \mathbb{C}, \quad \psi(n, k) := \varphi(n)$$

is a positive definite function on G. Hint: Show that the representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of N extends by $\pi_\varphi(k)f := f \circ \alpha(k)^{-1}$ to a unitary representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of G (Proposition 5.1.6) and consider $\pi_\varphi \in \mathcal{P}(G)$.

Exercise 5.3.2. Let $C \subseteq V$ be a convex cone in the real vector space V and $\alpha \in V^*$ with $\alpha(c) > 0$ for $0 \neq c \in C$. Show that

$$S := \{c \in C \colon \alpha(c) = 1\}$$

satisfies:

(a) $C = \mathbb{R}_+ S$.

(b) $x \in S$ is an extreme point of S if and only if $\mathbb{R}_+ x$ is an extremal ray of C.

Exercise 5.3.3. (Metric characterization of midpoints) Let $(X, \|\cdot\|)$ be a normed space and $x, y \in X$ distinct points. Let

$$M_0 := \{z \in X : \|z - x\| = \|z - y\| = \tfrac{1}{2}\|x - y\|\} \quad \text{and} \quad m := \frac{x + y}{2}.$$

For a subset $A \subseteq X$ we define its *diameter*

$$\delta(A) := \sup\{\|a - b\| : a, b \in A\}.$$

Show that:

(1) If X is a pre-Hilbert space (i.e., a vector space with a hermitian scalar product), then $M_0 = \{m\}$ is a one-element set.

(2) For $z \in M_0$ we have $\|z - m\| \leq \frac{1}{2}\delta(M_0) \leq \frac{1}{2}\|x - y\|$.

(3) For $n \in \mathbb{N}$ we define inductively:

$$M_n := \{p \in M_{n-1} : (\forall z \in M_{n-1})\ \|z - p\| \leq \tfrac{1}{2}\delta(M_{n-1})\}.$$

Then we have for each $n \in \mathbb{N}$:

(a) M_n is a convex set.

(b) M_n is invariant under the point reflection $s_m(a) := 2m - a$ in m.

(c) $m \in M_n$.

(d) $\delta(M_n) \leq \frac{1}{2}\delta(M_{n-1})$.

(4) $\bigcap_{n \in \mathbb{N}} M_n = \{m\}$.

Exercise 5.3.4. (Isometries of normed spaces are affine maps) Let $(X, \|\cdot\|)$ be a normed space endowed with the metric $d(x, y) := \|x - y\|$. Show that each isometry $\varphi\colon (X, d) \to (X, d)$ is an affine map by using the following steps:

(1) It suffices to assume that $\varphi(0) = 0$ and to show that this implies that φ is a linear map.

(2) $\varphi(\frac{x+y}{2}) = \frac{1}{2}(\varphi(x) + \varphi(y))$ for $x, y \in X$. Hint: Exercise 5.3.3.

(3) φ is continuous.

(4) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in 2^{\mathbb{Z}} \subseteq \mathbb{R}$.

(5) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for $x, y \in X$.

(6) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in \mathbb{R}$.

Exercise 5.3.5. Let X be an infinite set and $S_{(X)}$ be the group of all those permutations φ of X moving only finitely many points, i.e.,

$$|\{x \in X : \varphi(x) \neq x\}| < \infty.$$

Show that for each element $\varphi \neq \mathrm{id}_X$ in $S_{(X)}$ the conjugacy class

$$C_\varphi := \{\psi\varphi\psi^{-1} : \psi \in S_{(X)}\}$$

is infinite. Hint: Consider a description of φ in terms of cycles.

Exercise 5.3.6. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Show that every linear trace functional $T : B(\mathcal{H}) \to \mathbb{C}$ vanishes in $\mathbf{1}$, i.e.,

$$T(AB) = T(BA) \quad \text{for} \quad A, B \in B(\mathcal{H})$$

implies $T(\mathbf{1}) = 0$. Here are some steps to follow:

(a) T is conjugation invariant, i.e., $T(gAg^{-1}) = T(A)$ for $g \in \mathrm{GL}(\mathcal{H})$ and $A \in B(\mathcal{H})$.

(b) If P and Q are two orthogonal projections in $B(\mathcal{H})$ for which there are unitary isomorphisms $P(\mathcal{H}) \to Q(\mathcal{H})$ and $P(\mathcal{H})^\perp \to Q(\mathcal{H})^\perp$, then $T(P) = T(Q)$.

(c) For each $n \in \mathbb{N}$, there exists a unitary isomorphism $u_n : \mathcal{H} \to \mathcal{H}^n$, i.e.,

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \quad \text{with} \quad \mathcal{H}_j \cong \mathcal{H}.$$

Let $P_j^{(n)}$ denote the orthogonal projection onto $\mathcal{H}_j$.

(d) Show that $T(P_j^{(n)}) = \frac{1}{n}T(\mathbf{1})$ and use (b) to derive $T(P_1^{(2)}) = T(P_1^{(3)})$. Conclude that $T(\mathbf{1}) = 0$.

Exercise 5.3.7. Let G be a topological group, $O \subseteq G$ be an open subset and $S \subseteq G$ any subset. Then the subsets OS and SO of G are open. Hint: $OS = \bigcup_{s \in S} Os$.

Exercise 5.3.8. Let G be a topological group and $K \subseteq G$ be a closed subgroup. We endow G/K with the quotient topology, i.e., $O \subseteq G/K$ is open if and only if $q^{-1}(O) \subseteq G$ is open, where $q : G \to G/K$ is the quotient map. Show that:

(a) The quotient map $q : G \to G/K$ is open. Hint: Exercise 5.3.7.

(b) To see that G/K is Hausdorff, argue that for $y \notin xK$ there exists an open $\mathbf{1}$-neighborhood U in G with $U^{-1}Uy \cap xK = \emptyset$ and derive that $\pi(Uy) \cap \pi(Ux) = \emptyset$.

(c) The action $\sigma : G \times G/K \to G/K, (g, xK) \mapsto gxK$ is continuous. Hint: (a) and the openness of $\mathrm{id}_G \times q$.

(d) The map $q \times q\colon G \times G \to G/K \times G/K$ is an open map, i.e., $O \subseteq G/K \times G/K$ is open if and only if $(q \times q)^{-1}(O)$ is open in $G \times G$.

(e) Show that for every continuous K-biinvariant function $\varphi\colon G \to \mathbb{C}$, the kernel $Q(xK, yK) := \varphi(xy^{-1})$ on $G/K \times G/K$ is continuous.

Exercise 5.3.9. Show that for a euclidean space V, the group $\mathrm{O}(V)$ of linear surjective isometries acts transitively on the sphere

$$\mathbb{S}(V) = \{v \in V \colon \|v\| = 1\}.$$

Hint: For a unit vector $v \in \mathbb{S}(V)$ consider the map

$$\sigma_v(x) := x - 2\langle x, v\rangle v.$$

Show that $\sigma_v \in \mathrm{O}(V)$ and that for $x, y \in \mathbb{S}(V)$ there exists a $v \in \mathbb{S}$ with $\sigma_v(x) = y$.

Exercise 5.3.10. Let $(\pi, \mathcal{H})$ be a non-degenerate representation of the involutive semigroup $(S, *)$ and $v \in \mathcal{H}$. Show that the following assertions are equivalent:

(a) v is a cyclic vector for $\pi(S)$.

(b) v is a cyclic vector for the von Neumann algebra $\pi(S)''$.

(c) v separating for the von Neumann algebra $\pi(S)'$, i.e., the map $\pi(S)' \to \mathcal{H}, A \mapsto Av$ is injective.

Hint: To see that (c) implies (a), consider the projection P onto $(\pi(S)v)^\perp$, which is an element of $\pi(S)'$.

Exercise 5.3.11. Let $(\pi, \mathcal{H})$ be a non-degenerate multiplicity free representation of the involutive semigroup $(S, *)$ on the separable Hilbert space $\mathcal{H}$. Show that $(\pi, \mathcal{H})$ is cyclic. Hint: Write $\mathcal{H}$ as a direct sum of at most countably many cyclic representations $(\pi_j, \mathcal{H}_j, v_j)$ with cyclic unit vectors $(v_j)_{j \in J}$ and find $c_j > 0$ such that $v := \sum_{j \in J} c_j v_j$ converges in $\mathcal{H}$. Now show that v is a separating vector for $\pi(S)'$ and use Exercise 5.3.10. Note that the orthogonal projections P_j onto $\mathcal{H}_j$ are contained in $\pi(S)'$.

Exercise 5.3.12. Let G be a topological group, $K \subseteq G$ be a closed subgroup and $X := G/K$ the corresponding homogeneous space with base point $x_0 := \mathbf{1}K$. We fix a 1-cocycle $J\colon G \times X \to \mathbb{C}^\times$ and $0 \neq Q \in \mathcal{P}(X, \sigma, J)$, so that

$$(\pi(g)f)(x) := J(g, x)f(g^{-1}.x)$$

defines a unitary representation of G on $\mathcal{H}_Q \subseteq \mathbb{C}^X$ (Proposition 5.1.6). Show that:

(a) $\chi(k) := J(k, x_0)$ defines a character $\chi\colon K \to \mathbb{T}$.

(b) $\mathcal{H}_Q^{K,\chi} := \bigcap_{k \in K} \ker(\pi(k) - \chi(k)\mathbf{1}) \neq \{0\}$. It generates $\mathcal{H}_Q$ under the G-action.

(c) If $\mathcal{H}_Q^{K,\chi}$ is one dimensional, then the G-representation on $\mathcal{H}_Q$ is irreducible.

Hint: Proposition 5.3.11.

Exercise 5.3.13. Let $\sigma\colon G \times X \to X, (g,x) \mapsto g.x$ be a transitive continuous action of the topological group G on the topological space X. Fix $x_0 \in X$ and let $K := \{g \in G\colon g.x_0 = x_0\}$ be the stabilizer subgroup of x_0. Show that:

(a) We obtain a continuous bijective map $\eta\colon G/K \to X, gK \mapsto g.x_0$.

(b) Suppose that η has a continuous local section, i.e., x_0 has a neighborhood U for which there exists a continuous map $\tau\colon U \to G$ with $\tau(y).x_0 = y$ for $y \in U$. Then η is open, hence a homeomorphism.

(c) Let $G := \mathbb{R}_d$ be the group $(\mathbb{R}, +)$, endowed with the discrete topology and $X := \mathbb{R}$, endowed with the canonical topology. Then $\sigma(x,y) := x + y$ defines a continuous transitive action of G on X for which the orbit map η is continuous and bijective but not open.

Exercise 5.3.14. Let V be a euclidean space, $\mathbb{S} \subseteq V$ be its unit sphere, $G := \mathrm{O}(V)$ be its orthogonal group, endowed with the strong operator topology, $e_0 \in \mathbb{S}$ and $K \cong \mathrm{O}(e_0^\perp)$ be the stabilizer of e_0 in G. Show that the orbit map $\sigma^{e_0}\colon \mathrm{O}(V) \to \mathbb{S}, g \mapsto ge_0$ induces a homeomorphism

$$\eta\colon G/K = \mathrm{O}(V)/\mathrm{O}(e_0^\perp) \to \mathbb{S}, \quad gK \mapsto ge_0.$$

Hint: Show first that for $U := \mathbb{S} \setminus \{-e_0\}$ the map

$$\sigma\colon U \to \mathrm{O}(V), \quad \sigma(z)(v) := 2\frac{\langle v, e_0 + z\rangle}{\|e_0 + z\|^2}(e_0 + z) - v$$

is continuous and satisfies

$$\sigma(z)(e_0) = z.$$

Then apply Exercise 5.3.14.

Chapter 6

From the Group Algebra to Spectral Measures

The main goal of this chapter is to explain how representations of locally compact abelian groups G can be described in terms of spectral measures on their character group $\widehat{G}$, endowed with a suitable locally compact topology. This is of particular interest for the group $G = (\mathbb{R}, +)$ because it provides a description of all continuous one-parameter groups of the group $\mathrm{U}(\mathcal{H})_s$ in terms of spectral measures on $\mathbb{R}$.

To achieve this goal, we want to apply the Gelfand Representation Theorem to a suitable commutative Banach-$*$-algebra. Such a group algebra can be obtained for a locally compact group G from the convolution product on $L^1(G, \mu_G)$ and a suitable $*$-operation turning $L^1(G, \mu_G)$ into a Banach-$*$-algebra. The main point of this construction is that the continuous unitary representations of G are in one-to-one correspondence with the non-degenerate representations of this Banach-$*$-algebra.

Here we shall mainly exploit the applications of this algebra for abelian groups. We first discuss in Section 6.2 spectral measures and explore in Section 6.3 how this applies to representations of abelian locally compact groups. In particular, we obtain a rather complete description of the representations of the additive group $(\mathbb{R}, +)$.

6.1 The Group Algebra of a Locally Compact Group

6.1.1 The Convolution Product

Let G be a locally compact group and μ_G be a left Haar measure on G. For $f, g \in C_c(G)$ we define we define the *convolution product*

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)\, d\mu_G(y) = \int_G f(xy)g(y^{-1})\, d\mu_G(y). \tag{6.1}$$

This integral is defined because the first integrand is supported by the compact set $\mathrm{supp}(f)$. Using the modular factor Δ_G, we define an involution on $C_c(G)$ by

$$f^*(x) := \Delta_G(x)^{-1}\overline{f(x^{-1})}. \tag{6.2}$$

Lemma 6.1.1. *For $f, g \in C_c(G)$, we have*

(i) $f * g \in C_c(G)$ *with* $\mathrm{supp}(f * g) \subseteq \mathrm{supp}(f)\,\mathrm{supp}(g)$ *and convolution is associative.*

(ii) $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

(iii) $\|f^*\|_1 = \|f\|_1$.

(iv) $(f * g)^* = g^* * f^*$.

(v) *For $x \in G$ and $f \in C_c(G)$ we put $\lambda_x f := f \circ \lambda_x^{-1}$ and $\rho_x f := f \circ \rho_x$. Then*

  (a) $\lambda_x(f * g) = (\lambda_x f) * g$, $\rho_x(f * g) = f * \rho_x g$,

  (b) $\rho_x(f) * g = f * \Delta_G(x)^{-1}(\lambda_{x^{-1}} g)$.

  (c) $(\lambda_x f)^* = \Delta_G(x)\rho_x f^*$ *and* $(\rho_x f)^* = \Delta_G(x)^{-1}\lambda_x f^*$.

  (d) $\|\lambda_x f\|_1 = \|f\|_1$ *and* $\|\rho_x f\|_1 = \Delta_G(x)^{-1}\|f\|_1$.

(vi) *For $f \in C_c(G)$, the map $G \to L^1(G, \mu_G), x \mapsto \lambda_x(f)$ is continuous.*

Proof. (i) The continuity of $f*g$ follows from Lemma 2.3.7, applied to $\mathrm{supp}(f)\times G$. If $(f * g)(x) \neq 0$, then there exists a $y \in G$ with $f(y)g(y^{-1}x) \neq 0$. Then $y \in \mathrm{supp}(f)$ and $y^{-1}x \in \mathrm{supp}(g)$, so that $x \in \mathrm{supp}(f)\,\mathrm{supp}(g)$. In particular, $f * g$ has compact support if f and g have.

For the associativity of the convolution product, we calculate

$$\begin{aligned}
((f * g) * h)(x) &= \int_G \int_G f(z)g(z^{-1}y)h(y^{-1}x)\, d\mu_G(y)\, d\mu_G(z) \\
&= \int_G f(z) \int_G g(y)h(y^{-1}z^{-1}x)\, d\mu_G(y)\, d\mu_G(z) \\
&= \int_G f(z)(g * h)(z^{-1}z)\, d\mu_G(z) = (f * (g * h))(x).
\end{aligned}$$

(ii) We have

$$|f * g(x)| \leq \int_G |f(y)||g(y^{-1}x)| \; d\mu_G(y)$$

and therefore

$$\begin{aligned}
\|f * g\|_1 &\leq \int_G \int_G |f(y)||g(y^{-1}x)| \; d\mu_G(y) \; d\mu_G(x) \\
&\overset{\text{Fubini}}{=} \int_G |f(y)| \int_G |g(y^{-1}x)| \; d\mu_G(x) \; d\mu_G(y) \\
&= \int_G |f(y)| \int_G |g(x)| \; d\mu_G(x) \; d\mu_G(y) = \|g\|_1 \int_G |f(y)| \; d\mu_G(y) \\
&= \|g\|_1 \|f\|_1.
\end{aligned}$$

Here the application of Fubini's Theorem is justified by the fact that both integrals extend over sets of finite measure, so that the assumption of σ-finiteness is satisfied for the corresponding restricted measures.

(iii) In view of Lemma 2.4.8, we have

$$\|f^*\|_1 = \int_G \Delta_G(x^{-1})|f(x^{-1})| \; d\mu_G(x) = \int_G |f(x)| \; d\mu_G(x) = \|f\|_1.$$

(iv) As in (iii), we get

$$\begin{aligned}
(f * g)^*(x) &= \Delta_G(x)^{-1} \int_G \overline{f(y)g(y^{-1}x^{-1})} \; d\mu_G(y) \\
&= \Delta_G(x)^{-1} \int_G \overline{f(x^{-1}y)g(y^{-1})} \; d\mu_G(y) \\
&= \int_G g^*(y)\Delta_G(x^{-1}y)\overline{f(x^{-1}y)} \; d\mu_G(y) \\
&= \int_G g^*(y)f^*(y^{-1}x) \; d\mu_G(y) = (g^* * f^*)(x).
\end{aligned}$$

(v) (a) follows immediately from (6.1).

(b) follows from

$$\begin{aligned}
((\rho_x f) * g)(y) &= \int_G f(zx)g(z^{-1}y) \, d\mu_G(z) = \Delta_G(x)^{-1} \int_G f(z)g(xz^{-1}y) \, d\mu_G(z) \\
&= \Delta_G(x)^{-1}(f * \lambda_{x^{-1}} g)(y).
\end{aligned}$$

(c) follows from

$$\begin{aligned}
(\lambda_x f)^*(y) &= \Delta_G(y)^{-1}\overline{f(x^{-1}y^{-1})} = \Delta_G(x)\Delta_G(yx)^{-1}\overline{f(x^{-1}y^{-1})} \\
&= \Delta_G(x)f^*(yx) = \Delta_G(x)(\rho_x f^*)(y),
\end{aligned}$$

which in turn implies

$$(\rho_x f)^* = \Delta_G(x)^{-1}\lambda_x f^*.$$

(d) follows from the left invariance of μ_G.

(vi) In view of

$$\|\lambda_x f - \lambda_{x'} f\|_1 = \int_G |f(x^{-1}g) - f(x'^{-1}g)| \, d\mu_G(g),$$

the assertion follows from Lemma 2.3.7 for $f \in C_c(G)$. □

Let $L^1(G) := L^1(G, \mu_G)$ denote the completion of $C_c(G)$ with respect to $\|\cdot\|_1$. The preceding lemma implies that the convolution product and the involution extend to continuous maps on $L^1(G)$, turning it into a Banach-$*$-algebra. We also extend the left translations λ_g and the maps $\Delta_G(g)\rho_g$ to isometries of $L^1(G)$.

Proposition 6.1.2. *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the locally compact group G. For $f \in L^1(G)$ and $v, w \in \mathcal{H}$, we define by*

$$\langle \pi(f)v, w\rangle := \int_G f(g)\langle \pi(g)v, w\rangle \, d\mu_G(g) \tag{6.3}$$

an operator $\pi(f) \in B(\mathcal{H})$, which we also denote symbolically by

$$\pi(f) = \int_G f(g)\pi(g) \, d\mu_G(g).$$

It has the following properties:

(i) $\|\pi(f)\| \leq \|f\|_1$.

(ii) *The continuous linear extension $\pi\colon L^1(G) \to B(\mathcal{H})$ defines a representation of the Banach-$*$-algebra $L^1(G)$, i.e., it is a homomorphism of algebras satisfying $\pi(f)^* = \pi(f^*)$ for $f \in L^1(G)$.*

(iii) *For $x \in G$ and $f \in L^1(G)$ we have $\pi(x)\pi(f) = \pi(\lambda_x f)$ and $\pi(f)\pi(x) = \Delta_G(x)\pi(\rho_x f)$.*

(iv) *The representation π of $L^1(G)$ on $\mathcal{H}$ is non-degenerate.*

(v) *$\pi(L^1(G))' = \pi(G)'$ and, in particular, $\pi(G) \subseteq \pi(L^1(G))''$ and $\pi(L^1(G)) \subseteq \pi(G)''$.*

(vi) *The representations of G and $L^1(G)$ have the same closed invariant subspaces. In particular, one is irreducible if and only if the other has this property.*

Proof. (i) First we observe that the sesquilinear map

$$(v, w) \mapsto \int_G f(g)\langle \pi(g)v, w\rangle \, d\mu_G(g)$$

is continuous:

$$\Big|\int_G f(g)\langle\pi(g)v,w\rangle\ d\mu_G(g)\Big| \le \int_G |f(g)|\|\pi(g)v\|\|w\|\ d\mu_G(g) = \|f\|_1\|v\|\|w\|.$$

Hence there exists a unique operator $\pi(f)\in B(\mathcal{H})$ satisfying (i) and (6.3).

(ii) In view of (i), π defines a continuous linear map $L^1(G)\to B(\mathcal{H})$. It remains to verify

$$\pi(f*g)=\pi(f)\pi(g) \quad\text{and}\quad \pi(f)^*=\pi(f^*).$$

Since $C_c(G)$ is dense in $L^1(G)$, it suffices to verify these relations for $f,g\in C_c(G)$. For $v,w\in\mathcal{H}$, we have

$$\begin{aligned}
\langle\pi(f*g)v,w\rangle &= \int_G\int_G f(y)g(y^{-1}x)\langle\pi(x)v,w\rangle\ d\mu_G(y)\ d\mu_G(x)\\
&\overset{\text{Fubini}}{=} \int_G f(y)\int_G g(y^{-1}x)\langle\pi(x)v,w\rangle\ d\mu_G(x)\ d\mu_G(y)\\
&= \int_G f(y)\int_G g(x)\langle\pi(yx)v,w\rangle\ d\mu_G(x)\ d\mu_G(y)\\
&= \int_G f(y)\int_G g(x)\langle\pi(x)v,\pi(y^{-1})w\rangle\ d\mu_G(x)\ d\mu_G(y)\\
&= \int_G f(y)\langle\pi(g)v,\pi(y^{-1})w\rangle\ d\mu_G(y)\\
&= \int_G f(y)\langle\pi(y)\pi(g)v,w\rangle\ d\mu_G(y)\\
&= \langle\pi(f)\pi(g)v,w\rangle.
\end{aligned}$$

This proves that $\pi(f*g)=\pi(f)\pi(g)$. We further have

$$\begin{aligned}
\langle v,\pi(f^*)w\rangle &= \int_G \Delta_G(x)f(x^{-1})\langle v,\pi(x)w\rangle\ d\mu_G(x)\\
&= \int_G \Delta_G(x)f(x^{-1})\langle\pi(x^{-1})v,w\rangle\ d\mu_G(x)\\
&= \int_G f(x)\langle\pi(x)v,w\rangle\ d\mu_G(x)\\
&= \langle\pi(f)v,w\rangle,
\end{aligned}$$

which implies $\pi(f)^*=\pi(f^*)$.

(iii) Since λ_x defines an isometry of $L^1(G)$, it suffices to assume that $f\in C_c(G)$. For $v,w\in\mathcal{H}$, we have

$$\begin{aligned}
\langle\pi(x)\pi(f)v,w\rangle &= \langle\pi(f)v,\pi(x^{-1})w\rangle = \int_G f(y)\langle\pi(y)v,\pi(x^{-1})w\rangle\ d\mu_G(y)\\
&= \int_G f(y)\langle\pi(xy)v,w\rangle\ d\mu_G(y) = \int_G f(x^{-1}y)\langle\pi(y)v,w\rangle\ d\mu_G(y)\\
&= \langle\pi(\lambda_x f)v,w\rangle.
\end{aligned}$$

From this relation and (ii) we further derive

$$(\pi(f)\pi(x))^* = \pi(x^{-1})\pi(f^*) = \pi(\lambda_{x^{-1}}f^*) = \pi(\Delta_G(x)(\rho_x f)^*) = \Delta_G(x)\pi(\rho_x f)^*,$$

and this proves (iii).

(iv) To see that the representation of $L^1(G)$ is non-degenerate, we show that for every $0 \neq v \in \mathcal{H}$ there exists an $f \in C_c(G)$ with $\|\pi(f)v - v\| < \varepsilon$. To find such an f, let U be a $\mathbf{1}$-neighborhood in G with $\|\pi(g)v - v\| < \varepsilon$ for $g \in U$. Urysohn's Lemma implies the existence of $0 \neq f \in C_c(G)$ with $0 \leq f$ and $\mathrm{supp}(f) \subseteq U$. Then $\int_G f(g)\, d\mu_G(g) > 0$, and after multiplication with a suitable scalar, we may w.l.o.g. assume that $\int_G f(g)\, d\mu_G(g) = 1$. Then

$$\begin{aligned}
\|\pi(f)v - v\| &= \Big\| \int_G f(g)\pi(g)v - \int_G f(g)v\, d\mu_G(g)\Big\| \\
&= \Big\| \int_G f(g)(\pi(g)v - v)\, d\mu_G(g)\Big\| \\
&\leq \int_G |f(g)|\|\pi(g)v - v\|\, d\mu_G(g) \leq \varepsilon \int_G f(g)\, d\mu_G(g) = \varepsilon.
\end{aligned}$$

(v) First we show that $\pi(L^1(G)) \subseteq \pi(G)''$. So let $A \in \pi(G)'$. For $f \in L^1(G)$ and $v, w \in \mathcal{H}$ we then have

$$\begin{aligned}
&\langle A\pi(f)v, w\rangle = \langle \pi(f)v, A^*w\rangle = \int_G f(g)\langle \pi(g)v, A^*w\rangle\, d\mu_G(g) \\
&= \int_G f(g)\langle A\pi(g)v, w\rangle\, d\mu_G(g) = \int_G f(g)\langle \pi(g)Av, w\rangle\, d\mu_G(g) = \langle \pi(f)Av, w\rangle,
\end{aligned}$$

which implies that $A\pi(f) = \pi(f)A$.

Next we show that $\pi(G) \subseteq \pi(L^1(G))''$. If $A \in \pi(L^1(G))'$, then

$$\pi(g)A\pi(f) = \pi(g)\pi(f)A = \pi(\lambda_g f)A = A\pi(\lambda_g f) = A\pi(g)\pi(f)$$

for each $f \in L^1(G)$, and since the representation of $L^1(G)$ on $\mathcal{H}$ is non-degenerate, it follows that $\pi(g)A = A\pi(g)$.

From $\pi(L^1(G)) \subseteq \pi(G)''$, we now get $\pi(G)''' = \pi(G)' \subseteq \pi(L^1(G))'$, and likewise we derive from $\pi(G) \subseteq \pi(L^1(G))''$ that $\pi(L^1(G))' \subseteq \pi(G)'$, so that we have equality.

(vi) Since the closed invariant subspaces correspond to the orthogonal projections in the commutant (Lemma 1.3.1), this follows from (v). □

We have just seen how to "integrate" a continuous unitary representation of G to a representation of the Banach-$*$-algebra $L^1(G)$. Thinking of $\pi(f) = \int_G f(x)\pi(x)\, d\mu_G(x)$, this means a "smearing" of the unitary operators $\pi(x)$. If the group G is not discrete, then G is not contained in $L^1(G)$ (as δ-functions), so that it is not obvious how to recover the unitary representation of G from the corresponding representation of $L^1(G)$. However, since the representation of $L^1(G)$ is non-degenerate, the operators $\pi(x)$, $x \in G$, are uniquely determined by the relation $\pi(x)\pi(f) = \pi(\lambda_x f)$ for $f \in L^1(G)$. To make systematic use of such relations, we now introduce the concept of a multiplier of an involutive semigroup.

6.1.2 Unitary Multiplier Actions on Semigroups

Definition 6.1.3. Let $(S, *)$ be an involutive semigroup. A *multiplier* of S is a pair (λ, ρ) of maps $\lambda, \rho\colon S \to S$ satisfying the following conditions:

$$a\lambda(b) = \rho(a)b, \qquad \lambda(ab) = \lambda(a)b, \qquad \text{and} \qquad \rho(ab) = a\rho(b).$$

We write $M(S)$ for the set of all multipliers of S and turn it into an involutive semigroup by

$$(\lambda, \rho)(\lambda', \rho') := (\lambda \circ \lambda', \rho' \circ \rho) \quad \text{and} \quad (\lambda, \rho)^* := (\rho^*, \lambda^*),$$

where $\lambda^*(a) := \lambda(a^*)^*$ and $\rho^*(a) = \rho(a^*)^*$. We write

$$\mathrm{U}(M(S)) := \{(\lambda, \rho) \in M(S)\colon (\lambda, \rho)(\lambda, \rho)^* = (\lambda, \rho)^*(\lambda, \rho) = \mathbf{1}\}$$

for the *unitary group* of $M(S)$.

Remark 6.1.4. (a) The assignment $\eta_S\colon S \to M(S), a \mapsto (\lambda_a, \rho_a)$ defines a morphism of involutive semigroups which is surjective if and only if S has an identity. Its image is an involutive semigroup ideal in $M(S)$, i.e.,

$$M(S)\eta_S(S) \subseteq \eta_S(S) \quad \text{and} \quad \eta_S(S)M(S) \subseteq \eta_S(S).$$

(b) The map

$$M(S) \times S \to S, \quad ((\lambda, \rho), s) \mapsto \lambda(s)$$

defines a left action of the semigroup $M(S)$ on S, and

$$S \times M(S) \to S, \quad ((\lambda, \rho), s) \mapsto \rho(s)$$

defines a right action of $M(S)$ on S.

Example 6.1.5. (a) The C^*-algebra $(C^b(X), \|\cdot\|_\infty)$ of bounded continuous functions on a locally compact space acts via the multipliers

$$\lambda(f)h = \rho(f)h = fh$$

on the commutative C^*-algebra $C_0(X)$.

(b) Let $\mathcal{H}$ be a complex Hilbert space and $K(\mathcal{H})$ be the C^*-algebra of compact operators on $\mathcal{H}$. Then we obtain for each $A \in B(\mathcal{H})$ a multiplier (λ_A, ρ_A) on $K(\mathcal{H})$.

Lemma 6.1.6. *Let G be a locally compact group and $(L^1(G), *)$ be its convolution algebra. Then, for each $g \in G$, the pair*

$$m(g) := \big(\lambda_g, \Delta_G(g)^{-1}\rho_{g^{-1}}\big)$$

is a unitary multiplier of $L^1(G)$ and $m\colon G \to \mathrm{U}(M(L^1(G)))$ is a group homomorphism.

Proof. That each $m(g)$ is a multiplier of the involutive semigroup $L^1(G)$ follows from Lemma 6.1.1(v)(a),(b). We further obtain from Lemma 6.1.1(v)(c) that

$$\lambda_g^* = \Delta_G(g)\rho_g,$$

so that

$$m(g)^* = (\Delta_G(g^{-1})\rho_{g^{-1}}^*, \lambda_g^*) = (\lambda_{g^{-1}}, \Delta_G(g)\rho_g) = m(g^{-1}) = m(g)^{-1},$$

which shows that $m(g)$ is unitary. That m is multiplicative is an immediate consequence of the definitions. □

Proposition 6.1.7. *For each non-degenerate representation $(\pi, \mathcal{H})$ of S there exists a unique unitary representation $(\widetilde{\pi}, \mathcal{H})$ of $\mathrm{U}(M(S))$, determined by*

$$\widetilde{\pi}(g)\pi(s) = \pi(gs) \quad \textit{for} \quad g \in \mathrm{U}(M(S)), s \in S. \tag{6.4}$$

Proof. Every non-degenerate representation of S is a direct sum of cyclic ones (Exercise 1.3.10), which in turn are of the form $(\pi_\varphi, \mathcal{H}_\varphi)$ (Remark 5.3.2). We therefore may assume that $(\pi, \mathcal{H}) = (\pi_\varphi, \mathcal{H}_\varphi)$. The reproducing kernel K of $\mathcal{H}_\varphi$ is $K(s,t) := \varphi(st^*)$, and it is invariant under the right action of any $g = (\lambda_g, \rho_g) \in \mathrm{U}(M(S))$:

$$\begin{aligned} K(\rho_g(s), \rho_g(t)) &= \varphi(\rho_g(s)\rho_g(t)^*) = \varphi(\rho_g(s)\rho_g^*(t^*)) = \varphi(\rho_g(s)\lambda_g^{-1}(t^*)) \\ &= \varphi(s\lambda_g\lambda_g^{-1}(t^*)) = \varphi(st^*) = K(s,t). \end{aligned}$$

Hence $\widetilde{\pi}_\varphi(g)(f) := f \circ \rho_g$ defines a unitary representation $(\widetilde{\pi}_\varphi, \mathcal{H}_\varphi)$ of $\mathrm{U}(M(S))$ satisfying (6.4). That this condition determines $\widetilde{\pi}_\varphi$ uniquely follows from the non-degeneracy of the cyclic representation $(\pi_\varphi, \mathcal{H}_\varphi)$ (Exercise 1.3.10). □

6.1.3 Intermezzo on Banach Space-Valued Integrals

Let X be a compact space, μ a Radon probability measure on X, E a Banach space and $f\colon X \to E$ a continuous function. We want to define the E-valued integral $\int_X f(x)\, d\mu(x)$.

Lemma 6.1.8. *There exists at most one element $I \in E$ with*

$$\lambda(I) = \int_X \lambda(f(x))\, d\mu(x) \quad \textit{for each} \quad \lambda \in E'.$$

Proof. This is an immediate consequence of the fact that E' separates the points of E. □

We define a linear functional

$$\widetilde{I}\colon E' \to \mathbb{C}, \quad \widetilde{I}(\lambda) := \int_X \lambda(f(x))\, d\mu(x)$$

and observe that the integral exists because the integrand is a continuous function on X. We also observe that

$$|\widetilde{I}(\lambda)| \leq \int_X |\lambda(f(x))|\, d\mu(x) \leq \int_X \|\lambda\| \cdot \|f(x)\|\, d\mu(x) = \|\lambda\| \cdot \int_X \|f(x)\|\, d\mu(x),$$

so that $\widetilde{I} \in E''$ with

$$\|\widetilde{I}\| \leq \int_X \|f(x)\|\, d\mu(x).$$

We recall the isometric embedding

$$\eta_E \colon E \to E'', \quad \eta(v)(\lambda) = \lambda(v).$$

A Banach space E is said to be *reflexive* if η_E is surjective, but this is not the case for each Banach space. A typical examples is c_0 with $c_0'' = (\ell^1)' = \ell^\infty$. However, we want to show that $\widetilde{I} = \eta(I)$ for some $I \in E$, which means that I satisfies the condition in Lemma 6.1.8.

Let $K := \overline{\mathrm{conv}(f(X))}$ and recall from Exercise 4.2.6 that K is a compact subset of E with respect to the norm topology. Write E_w for the space E, endowed with the weak topology, i.e., the coarsest topology for which all elements $\lambda \in E'$ are continuous. Then the identity $E \to E_w$ is continuous and E_w is Hausdorff, which implies that K is also compact with respect to the weak topology. The embedding $\eta \colon E_w \to E''$ is clearly continuous with respect to the weak-$*$-topology on E'' (with respect to E') and the weak topology on E, because for each $\lambda \in E'$ the map $E_w \to \mathbb{C}, v \mapsto \eta(v)(\lambda) = \lambda(v)$ is continuous. Therefore the image $\eta(K) \subseteq E''$ is weak-$*$-compact.

Finally we show that $\widetilde{I} \in \eta(K)$. In fact, for each $\lambda \in E'$ we have

$$\widetilde{I}(\lambda) = \int_X \lambda(f(x))\, d\mu(x) \leq \big(\sup \lambda(K)\big)\mu(X) = \sup \lambda(K) = \sup \eta(K)(\lambda),$$

so that the Hahn–Banach Separation Theorem and the weak-$*$-closedness of $\eta(K)$ imply that

$$\widetilde{I} \in \eta(K).$$

This proves the following theorem:

Theorem 6.1.9. *Let X be a compact space, μ a Radon measure on X, E a Banach space and $f \colon X \to E$ a continuous function. Then there exists a unique element $I \in E$ with*

$$\lambda(I) = \int_X \lambda(f(x))\, d\mu(x) \quad \text{for} \quad \lambda \in E'.$$

Proof. It only remains to argue that the requirement that $\mu(X) = 1$ can be dropped. If $\mu(X) = 0$, we anyway have $I = 0$, and if $\mu(X) > 0$, then we simply replace μ by $\frac{1}{\mu(X)}\mu$ and f by $\mu(X)f$, and apply the preceding arguments. $\square$

We denote the element I also by

$$\int_X f(x)\, d\mu(x).$$

We have already seen that

$$\Big\| \int_X f(x)\, d\mu(x) \Big\| \leq \int_X \|f(x)\|\, d\mu(x). \tag{6.5}$$

Remark 6.1.10. If $A\colon E \to F$ is a continuous linear map between Banach spaces, then

$$A \int_X f(x)\, d\mu(x) = \int_X Af(x)\, d\mu(x).$$

For each $\lambda \in F'$ we have

$$\begin{aligned}\lambda\Big(A \int_X f(x)\, d\mu(x)\Big) &= (\lambda \circ A)\Big(\int_X f(x)\, d\mu(x)\Big) = \int_X (\lambda \circ A)(f(x))\, d\mu(x) \\ &= \lambda\Big(\int_X Af(x)\, d\mu(x)\Big),\end{aligned}$$

so that the assertion follows from Lemma 6.1.8.

6.1.4 Recovering the Representation of G

Proposition 6.1.11. *For $f, h \in C_c(G)$ we have*

$$f * h = \int_G f(x)\, \lambda_x(h)\, d\mu_G(x)$$

as an $L^1(G)$-valued integral.

Proof. Let $K \subseteq G$ be a compact subset containing

$$\mathrm{supp}(f) \cdot \mathrm{supp}(h) \supseteq \mathrm{supp}(f * h).$$

Since $\mathrm{supp}(f)$ is compact and the map

$$G \to C(K), \quad x \mapsto f(x)\lambda_x(h)|_K$$

is continuous (Lemma 2.3.7), Theorem 6.1.9 implies the existence of a $C(K)$-valued integral

$$I := \int_G f(x)\, \lambda_x(h)|_K\, d\mu_G(x).$$

If $A\colon C(K) \to L^1(G)$ is the canonical inclusion, defined by extending a function $f\colon K \to \mathbb{C}$ by 0 on $G \setminus K$, then

$$AI = \int_G f(x)\, \lambda_x(h)\, d\mu_G(x)$$

follows from the fact that $\lambda_x(h)$ vanishes outside of K. Since point evaluations on $C(K)$ are continuous, we have for each $y \in K$:

$$I(y) = \int_G f(x)\, \lambda_x(h)(y)\, d\mu_G(x) = (f * h)(y),$$

hence $I = (f * h)|_K$, and finally $AI = f * h$ follows from $\mathrm{supp}(f * h) \subseteq K$. ☐

Theorem 6.1.12. *Let G be a locally compact group. Then there exists for each non-degenerate representation $(\pi, \mathcal{H})$ of the Banach-$*$-algebra $L^1(G)$ a unique unitary representation $(\pi_G, \mathcal{H})$ with the property that*

$$\pi_G(g)\pi(f) = \pi(\lambda_g f) \quad \text{for} \quad g \in G, f \in L^1(G).$$

The representation $(\pi_G, \mathcal{H})$ is continuous, and for $f \in L^1(G)$ we have

$$\pi(f) = \int_G f(x)\pi_G(x)\, d\mu_G(x),$$

so that π coincides with the representation of $L^1(G)$ defined by π_G.

Proof. Since we have the homomorphism

$$m\colon G \to \mathrm{U}(M(L^1(G))), \quad g \mapsto \big(\lambda_g, \Delta_G(g)^{-1}\rho_{g^{-1}}\big)$$

from Example 6.1.5, the existence of π_G follows from Proposition 6.1.7.

To see that π_G is continuous, let $v \in \mathcal{H}$ and $f \in C_c(G)$. Then the map

$$G \to \mathcal{H}, g \mapsto \pi_G(g)\pi(f)v = \pi(\lambda_g f)v$$

is continuous because the map $G \to L^1(G), g \mapsto \lambda_g f$ is continuous (Lemma 6.1.1(vi)). Since the elements of the form $\pi(f)v$ span a dense subspace, the continuity of π_G follows from Lemma 1.2.6.

To see that integration of π_G yields the given representation π, it suffices to show that for $f, h \in C_c(G)$ and $v \in \mathcal{H}$ we have

$$\pi_G(f)\pi(h)v = \pi(f)\pi(h)v$$

because the elements of the form $\pi(h)v$, $h \in C_c(G)$, $v \in \mathcal{H}$, form a dense subset of $\mathcal{H}$. For $v, w \in \mathcal{H}$ we obtain a continuous linear functional

$$\omega\colon L^1(G) \to \mathbb{C}, \quad f \mapsto \langle \pi(f)v, w\rangle.$$

Applying, Proposition 6.1.11 to this functional, we get with Proposition 6.1.2(iii):

$$\begin{aligned} &\langle \pi(f)\pi(h)v, w\rangle = \langle \pi(f * h)v, w\rangle = \omega(f * h) = \int_G f(x)\omega(\lambda_x h)\, d\mu_G(x) \\ &= \int_G f(x)\langle \pi(\lambda_x h)v, w\rangle\, d\mu_G(x) = \int_G f(x)\langle \pi_G(x)\pi(h)v, w\rangle\, d\mu_G(x) \\ &= \langle \pi_G(f)\pi(h)v, w\rangle. \end{aligned}$$

This proves that $\pi(f)\pi(h) = \pi_G(f)\pi(h)$. ☐

Exercises for Section 6.1

Exercise 6.1.1. Let G be a locally compact group. Show that the convolution product on $C_c(G)$ satisfies

$$\|f * h\|_\infty \leq \|f\|_1 \cdot \|h\|_\infty.$$

Conclude that convolution extends to a continuous bilinear map

$$L^1(G, \mu_G) \times C_0(G) \to C_0(G).$$

Conclude that for $f \in L^1(G, \mu_G)$ and $h \in C_c(G)$, the convolution product $f * h$ can be represented by a continuous function in $C_0(G)$.

Exercise 6.1.2. Let G be a compact group. Show that every left or right invariant closed subspace of $L^2(G)$ consists of continuous functions. Hint: Use Exercise 6.1.1 and express the integrated representation of $L^1(G)$ on $L^2(G)$ in terms of the convolution product.

Exercise 6.1.3. Let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a unitary representation of the locally compact group G on $\mathcal{H}$ which is norm-continuous, i.e., continuous with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. Show that there exists an $f \in C_c(G)$ for which the operator $\pi(f)$ is invertible.

6.2 Projection-valued Measures

We have already seen in Chapter 1 that forming direct sums of Hilbert spaces and decomposing a given Hilbert space as an orthogonal direct sum of closed subspaces is an important technique in representation theory. However, this technique only leads to a complete understanding of those representations which are direct sums of irreducible ones, i.e., $\mathcal{H} = \mathcal{H}_d$ in the notation of Chapter 4. Up to this point, we know that even the group $(\mathbb{R}, +)$ has natural representations not containing any irreducible subrepresentation. In this section we develop the concept of a projection valued measure, which provides a continuous analog of direct sum decompositions of Hilbert spaces. In particular, it can be used to study the structure of representations without irreducible subrepresentations. The general idea is that a representation may be composed from irreducible ones in the same way as a measure space is composed from its points, which need not have positive measure.

6.2.1 Spectral Measures

Definition 6.2.1. Let $\mathcal{H}$ be a Hilbert space and

$$\mathcal{P}_\mathcal{H} := \{P \in B(\mathcal{H})\colon P = P^2 = P^*\}$$

be the set of all orthogonal projections on $\mathcal{H}$. Further, let $(X, \mathfrak{S})$ be a measurable space. A map $P\colon \mathfrak{S} \to \mathcal{P}_\mathcal{H}$ is called a *spectral measure* or a *projection valued measure* if

(SM1) $P(X) = \mathbf{1}$, and

(SM2) If $(E_j)_{j\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{S}$, then

$$P(\cup_{j=1}^\infty E_j)v = \sum_{j=1}^\infty P(E_j)v \quad \text{for each} \quad v \in \mathcal{H}.$$

In this sense we have

$$P\Big(\bigcup_{j=1}^\infty E_j\Big) = \sum_{j=1}^\infty P(E_j)$$

in the strong operator topology.

Remark 6.2.2. For each spectral measure P on $(X, \mathfrak{S})$ and each $v \in \mathcal{H}$,

$$P^v(E) := \langle P(E)v, v\rangle = \|P(E)v\|^2$$

defines a positive measure on $(X, \mathfrak{S})$ with total mass $\|v\|^2$. In particular, it is a probability measure if v is a unit vector.

Definition 6.2.3. If X is a locally compact space and $\mathfrak{S} := \mathfrak{B}(X)$ the σ-algebra of Borel subsets of X, then a spectral measures $P\colon \mathfrak{B}(X) \to \mathcal{P}_\mathcal{H}$ is said to be *regular* if all measures P^v, $v \in \mathcal{H}$, are regular.

Remark 6.2.4. If $(X, \mathfrak{S})$ is a measurable space, then $\mathfrak{S}$ is an abelian involutive semigroup with respect to the operations

$$A \cdot B := A \cap B \quad \text{and} \quad A^* := A.$$

In Proposition 6.2.6 below we shall see that every spectral measure is in particular a representation of this involutive semigroup $(\mathfrak{S}, *)$.

The following lemma describes a typical situation where spectral measures arise.

Lemma 6.2.5. *Let $(X, \mathfrak{S}, \mu)$ be a measure space and $\mathcal{H} := L^2(X, \mu)$. For $E \in \mathfrak{S}$ we define an operator on $\mathcal{H}$ by $P(E)f := \chi_E f$. Then P defines a projection valued measure on $\mathcal{H}$.*

Proof. From Lemma 2.1.5 we recall the homomorphism of C^*-algebras

$$\rho\colon L^\infty(X, \mu) \to B(L^2(X, \mu)), \quad \rho(h)f = hf.$$

For each $E \in \mathfrak{S}$, the characteristic function χ_E satisfies $\chi_E = \chi_E^* = \chi_E^2$, so that $P(E) = \rho(\chi_E) \in \mathcal{P}_\mathcal{H}$.

Clearly, $P(X) = \mathbf{1}$. Now let $(E_j)_{j\in\mathbb{N}}$ be a disjoint sequence in X and $f \in \mathcal{H}$. We put $F_k := \bigcup_{j=1}^k E_j$ and $F := \bigcup_{j=1}^\infty E_j$. Then

$$\|P(F)f - P(F_k)f\|^2 = \|P(F \setminus F_k)f\|^2 = \int_{F\setminus F_k} |f(x)|^2 \, d\mu(x) \to 0$$

by Lebesgue's Dominated Convergence Theorem. Therefore

$$P(F)f = \lim_{k\to\infty} P(F_k)f = \sum_{n=1}^{\infty} P(E_n)f,$$

and this completes the proof. $\square$

Proposition 6.2.6. *For a spectral measure $P\colon \mathfrak{S} \to \mathcal{P}_{\mathcal{H}}$, the following assertions hold:*

(i) $P(\emptyset) = \mathbf{0}$.

(ii) $P(E \cup F) + P(E \cap F) = P(E) + P(F)$.

(iii) $P(E \cap F) = P(E)P(F) = P(F)P(E)$. *In particular, the set $P(\mathfrak{S})$ is commutative and $P\colon (\mathfrak{S}, \cap) \to B(H)$ is a representation of the involutive semigroup $(\mathfrak{S}, *)$.*

(iv) *If $E_j \subseteq E_{j+1}$ for all $j \in \mathbb{N}$, then we have in the strong operator topology*

$$P\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \lim_{j\to\infty} P(E_j).$$

(v) *If $E_j \supseteq E_{j+1}$ for all $j \in \mathbb{N}$, then we have in the strong operator topology*

$$P\Big(\bigcap_{j=1}^{\infty} E_j\Big) = \lim_{j\to\infty} P(E_j).$$

Proof. (i) We consider the disjoint sequence defined by $E_1 := X$ and $E_n := \emptyset$ for $n > 1$. Then (SM2) implies

$$\mathbf{1} = P(X) = \mathbf{1} + \sum_{n=2}^{\infty} P(\emptyset),$$

which leads to $\mathbf{1} = \mathbf{1} + P(\emptyset)$, and therefore to $P(\emptyset) = \mathbf{0}$.

(ii) We write

$$F \cup E = (E \cap F) \cup (F \setminus E) \cup (E \setminus F)$$

and $F = (F \cap E) \cup (F \setminus E)$. Then $P(F) = P(E \cap F) + P(F \setminus E)$, and therefore

$$\begin{aligned} P(E \cup F) &= P(E \cap F) + \big(P(F) - P(E \cap F)\big) + \big(P(E) - P(E \cap F)\big) \\ &= P(E) + P(F) - P(E \cap F). \end{aligned}$$

(iii) We first assume that E and F are disjoint. Then

$$\begin{aligned} P(E) + P(F) = P(E \cup F) &= P(E \cup F)^2 = (P(E) + P(F))^2 \\ &= P(E)^2 + P(F)^2 + P(E)P(F) + P(F)P(E) \\ &= P(E) + P(F) + P(E)P(F) + P(F)P(E), \end{aligned}$$

so that

$$P(E)P(F) = -P(F)P(E). \tag{6.6}$$

Multiplying (6.6) from the left with $P(E)$, we obtain

$$P(E)P(F) = P(E)^2P(F) = -P(E)P(F)P(E) = P(F)P(E)^2 = P(F)P(E).$$

With (6.6) we thus obtain

$$P(E)P(F) = P(F)P(E) = \mathbf{0}.$$

Now let $E, F \in \mathfrak{S}$ be arbitrary. With the preceding discussion we obtain

$$\begin{aligned} P(E)P(F) &= (P(E \cap F) + P(E \setminus F))(P(F \cap E) + P(F \setminus E)) \\ &= P(E \cap F)P(F \cap E) = P(E \cap F). \end{aligned}$$

(iv), (v) are easy consequences of (SM2). For (iv), we apply (SM2) to the disjoint sequence $F_k := E_k \setminus E_{k-1}$, $k \in \mathbb{N}$, with $E_0 := \emptyset$, to obtain

$$P\Big(\bigcup_{n=1}^\infty E_n\Big) = P\Big(\bigcup_{n=1}^\infty F_n\Big) = \sum_{n=1}^\infty P(F_n) = \lim_{m\to\infty} \sum_{n=1}^m P(F_n) = \lim_{m\to\infty} P(E_m).$$

For (v), we apply (iv) to the sets $F_n := E_1 \setminus E_n$ to get

$$\begin{aligned} P(E_1) - P(E_n) = P(F_n) \to P\Big(\bigcup_{n\in\mathbb{N}} E_1 \setminus E_n\Big) &= P\Big(E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n\Big) \\ &= P(E_1) - P\Big(\bigcap_{n\in\mathbb{N}} E_n\Big). \end{aligned}$$

□

Example 6.2.7. (Discrete spectral measures)

(a) Let $\mathcal{H} = \widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ be a direct sum of Hilbert spaces (Lemma 1.3.6). For each subset $I \subseteq J$, we define $P(I) \in \mathcal{P}_\mathcal{H}$ as the orthogonal projection onto the subspace $\overline{\sum_{j\in I}\mathcal{H}_j}$. Then $P\colon \mathbb{P}(J) \to \mathcal{P}_\mathcal{H}$ is a spectral measure. Since each element $v \in \mathcal{H}$ is a countable orthogonal sum $v = \sum_{j\in J} v_j$, we have for $I \subseteq J$:

$$P^v(I) = \sum_{j\in I} P^v(\{j\}) = \sum_{j\in I} \|v_j\|^2,$$

where the sum is countable. Therefore every measure P^v is a countable linear combination of point measures with the mass $\|v_j\|^2$ in the point j.

(b) Conversely, let X be a set and $P\colon \mathbb{P}(X) \to \mathcal{P}_\mathcal{H}$ be a spectral measure with

$$\|v\|^2 = P^v(X) = \sum_{x\in X} P^v(\{x\}) \quad \text{for} \quad v \in \mathcal{H}. \tag{6.7}$$

For each $x \in X$, we then obtain a closed subspace $\mathcal{H}_x := P(\{x\})\mathcal{H}$, and, in view of $P(\{x\})P(X \setminus \{x\}) = \mathbf{0}$ (Proposition 6.2.6(iii)), these subspaces are mutually orthogonal. Now our assumptions imply that for each $v \in \mathcal{H}$:

$$\|v\|^2 = \sum_{x \in X} P^v(\{x\}) = \sum_{x \in X} \|P(\{x\})v\|^2.$$

and therefore $v \in \overline{\sum_{x \in X} \mathcal{H}_x}$. This implies that $\sum_{x \in X} \mathcal{H}_x$ is dense in $\mathcal{H}$, so that $\mathcal{H}$ is isomorphic to the Hilbert direct sum $\widehat{\bigoplus}_{x \in X} \mathcal{H}_x$ (Exercise 1.3.5). We therefore have exactly the same situation as in (a) because (6.7) implies $P(I) = \sum_{x \in I} P(\{x\})$ for each subset $I \subseteq X$, where the right hand side is the projection onto the subspace $\overline{\sum_{x \in I} \mathcal{H}_x}$.

Proposition 6.2.8. (The Spectral Integral) *Let $P \colon \mathfrak{S} \to \mathcal{P}_{\mathcal{H}}$ be a spectral measure on $(X, \mathfrak{S})$. Then there exists a unique linear map*

$$P \colon L^\infty(X, \mathfrak{S}) \to B(\mathcal{H}),$$

called the spectral integral, *with $P(\chi_E) = P(E)$ for $E \in \mathfrak{S}$. This map satisfies*

$$P(f)^* = P(\overline{f}), \quad P(fg) = P(f)P(g) \quad \textit{and} \quad \|P(f)\| \leq \|f\|_\infty, \tag{6.8}$$

so that $(P, \mathcal{H})$ is a representation of the commutative C^*-algebra $L^\infty(X, \mathfrak{S})$ of all bounded measurable functions on $(X, \mathfrak{S})$.

Proof. (1) If $f \colon X \to \mathbb{C}$ is a measurable step function, then we write $f = \sum_j \alpha_j \chi_{A_j}$ with $A_j = f^{-1}(\alpha_j)$ and pairwise different $\alpha_j \in \mathbb{C}$. Then the set A_j form a measurable partition of X and we define

$$P(f) := \sum_j \alpha_j P(A_j).$$

Let $\mathcal{S}(X, \mathfrak{S})$ be the space of measurable step function. Then the definition above yields a map

$$P \colon \mathcal{S}(X, \mathfrak{S}) \to B(\mathcal{H}).$$

(2) If $f = \sum_k \beta_k \chi_{B_k}$ is another representation of f as a linear combination of characteristic functions corresponding to pairwise disjoint measurable subsets, then

$$f = \sum_{k,j} \beta_k \chi_{B_k \cap A_j}$$

and restriction to A_j leads to $\beta_k = \alpha_j$ if $B_k \cap A_j \neq \emptyset$. Now we obtain

$$\sum_k \beta_k P(B_k) = \sum_{k,j} \beta_k P(B_k \cap A_j) = \sum_{k,j} \alpha_j P(B_k \cap A_j) = \sum_j \alpha_j P(A_j).$$

Therefore the integral does not depend on the representation of f.

(3) The advantage of (2) consists in the fact that for two step function f and g we find a finite measurable partition of X by $A_1, \ldots, A_n$ such that both can be written as $f = \sum_j \alpha_j \chi_{A_j}$ and $g = \sum_j \beta_j \chi_{A_j}$. This leads immediately to

$$P(f+g) = P(f) + P(g).$$

Since $P(\lambda f) = \lambda P(f)$ follows directly from (1), the map $P\colon \mathcal{S}(X, \mathfrak{S}) \to B(\mathcal{H})$ is linear. We further have

$$P(f)^* = \sum_j \overline{\alpha_j} P(A_j)^* = \sum_j \overline{\alpha_j} P(A_j) = P(\overline{f})$$

and Proposition 6.2.6 implies

$$\begin{aligned} P(f)P(g) &= \sum_j \alpha_j P(A_j) \sum_k \beta_k P(A_k) = \sum_{j,k} \alpha_j \beta_k P(A_j) P(A_k) \\ &= \sum_{j,k} \alpha_j \beta_k P(A_j \cap A_k) = \sum_j \alpha_j \beta_j P(A_j) = P(fg). \end{aligned}$$

(4) (Continuity of P) Let $v \in \mathcal{H}$ and let $A_1, \ldots, A_n$ in $\mathfrak{S}$ be disjoint subsets in $\mathfrak{S}$. Then the vectors $P(A_j)v$ are pairwise orthogonal (Proposition 6.2.6(ii)), so that

$$\|v\|^2 \geq \|P(\bigcup_{j=1}^n A_j)v\|^2 = \sum_{j=1}^n \|P(A_j)v\|^2,$$

and we obtain for any function of the form $f = \sum_{j=1}^n \alpha_j \chi_{A_j}$ the relation

$$\begin{aligned} \|P(f)v\|^2 &= \|\sum_j \alpha_j P(A_j)v\|^2 = \sum_j |\alpha_j|^2 \|P(A_j)v\|^2 \\ &\leq \|f\|_\infty^2 \sum_j \|P(A_j)v\|^2 \leq \|f\|_\infty^2 \|v\|^2. \end{aligned}$$

This proves that $\|P(f)\| \leq \|f\|_\infty$, and the density of $\mathcal{S}(X, \mu)$ in $L^\infty(X, \mathfrak{S})$ implies that P extends to a continuous linear map $\widehat{P}\colon L^\infty(X, \mathfrak{S}) \to B(\mathcal{H})$. Continuity implies that (6.8) also holds for $\widehat{P}$. $\square$

Remark 6.2.9. (a) If P is a spectral measure on $(X, \mathfrak{S})$, then we obtain for each $v \in \mathcal{H}$ a measure P^v on $(X, \mathfrak{S})$. For each measurable function $f \in L^\infty(X, \mathfrak{S})$, we then obtain the relation

$$\langle P(f)v, v\rangle = \int_X f(x)\, dP^v(x) \quad \text{and} \quad \|P(f)v\|^2 = \int_X |f(x)|^2\, dP^v(x). \tag{6.9}$$

between usual integrals with respect to the measure P^v and the spectral integrals with respect to P.

If $f = \chi_E$ is a characteristic function, then (6.9) reproduces simply the definition of the measure P^v, which implies the first relation for step functions because both sides of the equation are linear in f. Since both sides define continuous linear functionals on $L^\infty(X, \mathfrak{S})$

$$\Big| \int_X f(x)\, dP^v(x) \Big| \le \int_X |f(x)|\, dP^v(x) \le \|f\|_\infty P^v(X) = \|f\|_\infty \|v\|^2$$

(cf. (6.8) in Proposition 6.2.8), step functions form a dense subspace, they coincide on all of $L^\infty(X, \mathfrak{S})$. The second relation now follows from

$$\|P(f)v\|^2 = \langle P(f)^* P(f)v, v\rangle = \langle P(|f|^2)v, v\rangle = \int_X |f(x)|^2\, dP^v(x).$$

(b) From (6.9) we further get

$$\begin{aligned} P^{P(f)v}(E) &= \langle P(E)P(f)v, P(f)v\rangle = \langle P(\overline{f})P(E)P(f)v, v\rangle \\ &= \langle P(|f|^2\chi_E)v, v\rangle = \int_E |f|^2\, dP^v, \end{aligned}$$

which in turn leads to

$$P^{P(f)v} = |f|^2 P^v \tag{6.10}$$

in the sense of measures with densities.

Example 6.2.10. Let $(X, \mathfrak{S}, \mu)$ be a measure space and $\mathcal{H} = L^2(X, \mu)$ be the corresponding L^2-space. In Lemma 6.2.5 we have see that $P(E)f = \chi_E f$ defines a spectral measure $P\colon \mathfrak{S} \to B(\mathcal{H})$. In view of $\|hf\|_2 \le \|h\|_\infty \|f\|_2$, the multiplication map

$$L^\infty(X, \mu) \times L^2(X, \mu) \to L^2(X, \mu), \quad h \cdot f := h \cdot f$$

is continuous, so that we obtain $P(h)f = h \cdot f$ for all $h \in L^\infty(X, \mu)$ because this relation holds for all measurable step functions. Therefore the spectral integral of a function is a multiplication operator, and we recover the representation of $L^\infty(X, \mu)$ from Lemma 2.1.5.

For the proof of the spectral theorem in the following subsection, we need the following lemma.

Lemma 6.2.11. *Let $\mathcal{H} \cong \widehat{\bigoplus}_{j\in J} \mathcal{H}_j$ be a direct sum of Hilbert spaces and $(X, \mathfrak{S})$ be a measurable space. Suppose that we are given for each $j \in J$ a spectral measure $P_j\colon \mathfrak{S} \to P_{\mathcal{H}_j}$. Then*

$$P(E)v := \big(P_j(E)v_j\big) \quad \textit{for} \quad v = (v_j)_{j\in J} \in \mathcal{H}$$

defines a spectral measure $P\colon \mathfrak{S} \to P_{\mathcal{H}}$.

Proof. Clearly, $P(E)^* = P(E) = P(E)^2$, so that each $P(E)$ is indeed an orthogonal projection. Further, $P(X) = \mathbf{1}$, and for any disjoint sequence $(E_n)_{n\in\mathbb{N}}$

of Borel subsets of X, we obtain $P(E_n)P(E_m) = 0$ for $n \neq m$. Therefore the sequence $\big(P(E_n)v\big)_{n\in\mathbb{N}}$ is orthogonal for each $v \in \mathcal{H}$. Further,

$$\begin{aligned}\|P(E)v\|^2 &= \sum_{j\in J} \|P_j(E)v_j\|^2 = \sum_{j\in J}\sum_{n\in\mathbb{N}} \|P_j(E_n)v_j\|^2 \\ &= \sum_{n\in\mathbb{N}}\sum_{j\in J} \|P_j(E_n)v_j\|^2 = \sum_{n\in\mathbb{N}} \|P(E_n)v\|^2,\end{aligned}$$

i.e.,

$$\lim_{n\to\infty} \|\sum_{m=1}^{n} P(E_m)v\|^2 = \|P(E)v\|^2$$

and

$$\begin{aligned}\langle P(E)v, w\rangle &= \sum_{j\in J}\langle P_j(E)v_j, w_j\rangle = \sum_{j\in J}\sum_{n=1}^{\infty}\langle P_j(E_n)v_j, w_j\rangle \\ &= \sum_{n=1}^{\infty}\sum_{j\in J}\langle P_j(E_n)v_j, w_j\rangle = \sum_{n=1}^{\infty}\langle P(E_n)v, w\rangle,\end{aligned}$$

follows from absolute summability. With Exercise 6.2.1 we now obtain

$$\lim_{n\to\infty} \sum_{m=1}^{n} P(E_m)v = P(E)v,$$

i.e., P is a spectral measure. □

6.2.2 The Spectral Theorem for Commutative Banach-$*$-Algebras

In this section we shall prove one of the central tools in representation theory, namely the Spectral Theorem for commutative Banach-$*$-algebras. Basically, all other spectral theorems can be derived from this one by specialization.

Theorem 6.2.12. (Spectral Theorem for commutative Banach-$*$-algebras) *Let $\mathcal{A}$ be a commutative Banach-$*$-algebra. Then the following assertions hold:*

(i) *If $P\colon \mathfrak{B}(\widehat{\mathcal{A}}) \to P_{\mathcal{H}}$ is a regular Borel spectral measure on $\widehat{\mathcal{A}}$, then $\pi_P(a) := P(\widehat{a})$ defines a non-degenerate representation of $\mathcal{A}$ on $\mathcal{H}$.*

(ii) *If $(\pi, \mathcal{H})$ is a non-degenerate representation of $\mathcal{A}$, then there exists a unique regular spectral measure P on $\widehat{\mathcal{A}}$ with $\pi = \pi_P$.*

Proof. (i) Since the Gelfand transform $\mathcal{G}\colon \mathcal{A} \to C_0(\widehat{\mathcal{A}}), a \mapsto \widehat{a}$ is a homomorphism of Banach-$*$-algebras (Section 4.1) and the same holds for the homomorphism

$$P\colon L^\infty(\widehat{\mathcal{A}}) \to B(\mathcal{H}), \quad f \mapsto P(f)$$

obtained as the spectral integral (Proposition 6.2.8), the composition

$$\pi_P := P \circ \mathcal{G} \colon \mathcal{A} \to B(\mathcal{H}), \quad a \mapsto P(\widehat{a})$$

is a representation of the Banach-$*$-algebra $\mathcal{A}$.

To see that the representation $(\pi, \mathcal{H})$ is non-degenerate, let $v \in \mathcal{H}$ be a unit vector. Then the positive Borel measure P^v on X is regular, so that there exists for each $\varepsilon > 0$ a compact subset $C \subseteq \widehat{\mathcal{A}}$ with $P^v(C) > 1 - \varepsilon$. For any continuous function $f \in C_c(\widehat{\mathcal{A}})$ with $0 \leq f \leq 1$ and $f|_C = 1$ we then have

$$1 - \varepsilon < P^v(C) = \int_C dP^v(\chi) \leq \int_{\widehat{\mathcal{A}}} f(\chi)\, dP^v(\chi) = \langle P(f)v, v\rangle,$$

and $\|P(f)v\|^2 = \langle P(f^2)v, v\rangle \leq 1$, so that

$$\begin{aligned}\|P(f)v - v\|^2 &= \|P(f)v\|^2 + \|v\|^2 - 2\langle P(f)v, v\rangle \leq 2 - 2\langle P(f)v, v\rangle \\ &\leq 2 - 2(1-\varepsilon) = 2\varepsilon.\end{aligned}$$

Therefore $P(C_c(\widehat{\mathcal{A}}))\mathcal{H}$ is dense in $\mathcal{H}$. Since $\mathcal{G}(\mathcal{A})$ is dense in $C_0(\widehat{\mathcal{A}})$ by the Stone–Weierstraß Theorem (Remark 4.1.2), it follows that $\pi_P(\mathcal{A})\mathcal{H}$ is also dense in $\mathcal{H}$, and therefore the representation $(\pi, \mathcal{H})$ is non-degenerate.

(ii) First we show that we may assume that $\mathcal{A} = C_0(X)$ for some locally compact space X. So let $\mathcal{B} := \overline{\pi(\mathcal{A})}$. Then $\mathcal{B}$ is a commutative C^*-algebra and $\pi \colon \mathcal{A} \to \mathcal{B}$ is a morphism of Banach-$*$-algebras with dense range. In view of the Gelfand Representation Theorem (Theorem 4.1.1), $\mathcal{B} \cong C_0(Y)$ for some locally compact space Y. For $y \in Y$ and $\delta_y(f) := f(y)$, we have $\pi^*(\delta_y) := \delta_y \circ \pi \in \widehat{\mathcal{A}}$ because $\pi^*(\delta_y) \neq 0$ follows from the fact that π has dense range. The so obtained map $\pi^* \colon Y \to \widehat{\mathcal{A}}$ is continuous because for each $a \in \mathcal{A}$, the function $y \mapsto \pi^*(\delta_y)(a) = \pi(a)(y)$ is continuous. Moreover, it extends to an injective continuous map

$$\pi^* \colon Y \cup \{0\} = \mathrm{Hom}(\mathcal{B}, \mathbb{C}) \to \widehat{\mathcal{A}} \cup \{0\} = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$$

of compact spaces which therefore is a topological embedding. This implies that $\pi^*(Y) \cup \{0\}$ is a compact subset of $\widehat{\mathcal{A}} \cup \{0\}$ so that $Y \cong \pi^*(Y)$ is a closed subset of $\widehat{\mathcal{A}}$. We may therefore assume that Y is a closed subset of $\widehat{\mathcal{A}}$, so that π obtains the simple form $\pi(a) = \widehat{a}|_Y$. If P_B is a regular Borel spectral measure on Y with $P_B(f) = f$ for $f \in \mathcal{B} \subseteq B(\mathcal{H})$, then $P_A(E) := P_B(E \cap Y)$ is a regular Borel spectral measure on $\widehat{\mathcal{A}}$ (cf. Exercise 6.2.3(d)), and for $a \in \mathcal{A}$ we have

$$P_A(\widehat{a}) = P_B(\widehat{a}|_Y) = P_B(\pi(a)) = \pi(a).$$

Replacing $\mathcal{A}$ by $\mathcal{B}$, we may thus assume that $\mathcal{A} = C_0(X)$ holds for some locally compact space X.

Next we show the uniqueness of the spectral measure. Let P and $\widetilde{P}$ be spectral measures with the desired properties. For $v \in \mathcal{H}$ we then obtain two positive measures P^v and $\widetilde{P}^v$ on $\widehat{\mathcal{A}}$ with

$$\int_{\widehat{\mathcal{A}}} \widehat{a}(\chi)\; dP^v(\chi) = \langle P(\widehat{a})v, v\rangle = \langle \pi(a)v, v\rangle = \int_{\widehat{\mathcal{A}}} \widehat{a}(\chi)\; d\widetilde{P}^v(\chi)$$

for $a \in \mathcal{A}$. In view of the Riesz Representation Theorem 2.3.2, the regularity assumption implies $P^v = \widetilde{P}^v$. Since each $P(E)$ is uniquely determined by the numbers $P^v(E) = \langle P(E)v, v\rangle$, $v \in \mathcal{H}$, the uniqueness of P follows.

Now we prove the existence. To this end, we decompose the representation $(\pi, \mathcal{H})$ into cyclic representations $(\pi_j, \mathcal{H}_j)$, $j \in J$ (Exercise 1.3.10). If we have for each $j \in J$ a spectral measure P^j with values on $B(\mathcal{H}_j)$ and the desired properties, then Lemma 6.2.11 implies that

$$P(E)v := \big(P_j(E)v_j\big) \quad \text{for} \quad v = (v_j)_{j\in J} \in \mathcal{H}$$

defines a spectral measure. We may thus assume that the representation of $\mathcal{A} = C_0(X)$ is cyclic. Let $v \in \mathcal{H}$ be a cyclic vector, so that $\pi(\mathcal{A})v$ is dense in $\mathcal{H}$. Then

$$\pi^v \colon C_0(X) \to \mathbb{C}, \quad f \mapsto \langle \pi(f)v, v\rangle$$

is a positive functional, and the Riesz Representation Theorem 2.3.2 provides a unique regular Borel measure P^v on X with

$$\pi^v(f) = \int_X f(\chi)\, dP^v(\chi) \quad \text{for} \quad f \in C_0(X).$$

Next we show that $\mathcal{H} \cong L^2(X, P^v)$. To this end, we consider the map $\widetilde{\Phi} \colon C_0(X) \to \mathcal{H}, a \mapsto \pi(a)v$. Then

$$\begin{aligned}\langle \widetilde{\Phi}(a), \widetilde{\Phi}(b)\rangle &= \langle \pi(a)v, \pi(b)v\rangle = \langle \pi(ab^*)v, v\rangle \\ &= \pi^v(ab^*) = \int_X a(x)\overline{b(x)}\, dP^v(x) = \langle a, b\rangle_{L^2(X,P^v)}.\end{aligned}$$

Hence the map $L^2(X, P^v) \ni a \to \pi(a)v \in \mathcal{H}$ is well defined and, since $C_0(X)$ is dense in $L^2(X, P^v)$ (Proposition 2.3.6), it extends to an isometric embedding

$$\Phi \colon L^2(X, P^v) \to \mathcal{H},$$

which is surjective because $\pi(\mathcal{A})v$ is dense.

Let $\rho \colon L^\infty(X) \to B\big(L^2(X, P^v)\big)$ denote the representation from Lemma 2.1.5. For $a, b \in \mathcal{A}$ we then have

$$\pi(a)\Phi(b) = \pi(a)\pi(b)v = \pi(ab)v = \Phi(ab) = \Phi(\rho(a)b).$$

In view of the density of $C_0(X)$ in $L^2(X, P^v)$, Φ is an intertwining operator for the representations ρ and π of $\mathcal{A}$. We may thus assume that $\mathcal{H} = L^2(X, P^v)$.

Finally, let $P(E)f = \chi_E f$ denote the spectral measure on $L^2(X, P^v)$ from Example 6.2.10. For $a \in \mathcal{A}$ we now have $P(a) = \rho(a)$. It remains to show that, for $f \in L^2(X, P^v)$, the measures

$$E \mapsto P^f(E) = \langle P(E)f, f\rangle = \langle \chi_E f, f\rangle = \int_E |f(x)|^2\, dP^v(x)$$

are regular, but this is a consequence of the following Lemma 6.2.13. $\square$

Lemma 6.2.13. *If μ is a regular Borel measure on the locally compact space X and $f \in \mathcal{L}^2(X,\mu)$, then the finite measure $\mu_f(E) := \int_E |f(x)|^2 \, d\mu(x)$ is also regular.*

Proof. Let $E \subseteq X$ be a Borel set. We have to show that E is outer regular. We may assume $\mu(E) < \infty$ because otherwise there is nothing to show. Let $\varepsilon > 0$. For $n \in \mathbb{N}$ we consider the sets $F_n := \{x \in X : |f(x)| \geq n\}$. Then $\mu_f(X \setminus F_n) \to \mu_f(X) = \|f\|_2^2$ implies that $\mu_f(F_n) \leq \varepsilon$ for $n \geq N_\varepsilon$. If $V \supseteq E$ is an open subset with $\mu(V) \leq \mu(E) + \frac{\varepsilon}{N_\varepsilon^2}$, then we obtain for $n = N_\varepsilon$:

$$\begin{aligned}\mu_f(V \setminus E) &= \mu_f\big((V \cap F_n) \setminus E\big) + \mu_f\big((V \setminus F_n) \setminus E\big) \\ &\leq \mu_f(F_n) + \mu_f\big((V \setminus F_n) \setminus E\big) \leq \varepsilon + \frac{\varepsilon}{n^2} n^2 = 2\varepsilon.\end{aligned}$$

This proves the outer regularity of E.

To see that each open subset $U \subseteq X$ is inner regular, we argue similarly. □

The preceding proof even implies the following:

Corollary 6.2.14. *A representation $(\pi, \mathcal{H})$ of a commutative Banach-$*$-algebra is cyclic with cyclic vector v if and only if there exists a finite Radon measure μ on $\widehat{\mathcal{A}}$ such that $(\pi, \mathcal{H}, v)$ is unitarily equivalent to the representation $\big(\pi_\mu, L^2(\widehat{\mathcal{A}}, \mu), 1\big)$ with $\pi(a)f = \widehat{a} \cdot f$.*

Proof. First we show that the representation π_μ on $L^2(\widehat{\mathcal{A}}, \mu)$ is cyclic. Since μ is finite, $1 \in L^2(X, \mu)$. Further, $\pi_\mu(\mathcal{A})1 = \{\widehat{a} : a \in \mathcal{A}\}$. According to the Stone-Weierstraß Theorem, $\mathcal{G}(\mathcal{A})$ is dense in $C_0(\widehat{\mathcal{A}})$, so that the density of $C_c(\widehat{\mathcal{A}}) \subseteq C_0(\widehat{\mathcal{A}})$ in $L^2(\widehat{\mathcal{A}}, \mu)$ (Proposition 2.3.6) implies that 1 is a cyclic vector.

If, conversely, $(\pi, \mathcal{H}, v)$ is a cyclic representation of $\mathcal{A}$ with cyclic vector v, then the argument in the proof of Theorem 6.2.12 implies that it is equivalent to a representation $(\pi_\mu, L^2(\widehat{\mathcal{A}}, \mu), 1)$. □

Lemma 6.2.15. *Let $P : \mathfrak{S} \to P_{\mathcal{H}}$ be a spectral measure and $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty(X, \mathfrak{S})$ converging pointwise to some $f : X \to \mathbb{C}$. Then $P(f_n)v \to P(f)v$ holds for each $v \in \mathcal{H}$.*

Proof. From the Dominated Convergence Theorem we obtain

$$\langle P(f_n)v, v\rangle = \int_X f_n(x)\, dP^v(x) \to \int_X f(x)\, dP^v(x) = \langle P(f)v, v\rangle,$$

which also implies

$$\|P(f_n)v\|^2 = \langle P(|f_n|^2)v, v\rangle \to \langle P(|f|^2)v, v\rangle = \|P(f)v\|^2.$$

By polarization (Exercise 1.3.1(i)), we also obtain

$$\langle P(f_n)v, w\rangle \to \langle P(f)v, w\rangle, \quad v, w \in \mathcal{H}.$$

Now the assertion follows from Exercise 6.2.1. □

Remark 6.2.16. (a) In Mackey's books (cf. [Ma76, p. 93]) one finds the notion of a *projection-valued measure.* If $(X, \mathfrak{S})$ is a measurable space and $\mathcal{H}$ a Hilbert space, then a map $P\colon \mathfrak{S} \to \mathcal{P}_{\mathcal{H}}$ is called a projection-valued measure if:

(1) $\mathcal{H}$ is separable.

(2) $P(\emptyset) = \mathbf{0}$ and $P(X) = \mathbf{1}$.

(3) $P(E \cap F) = P(E)P(F)$ for $E, F \in \mathfrak{S}$.

(4) $P(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} P(E_j)$ pointwise for each disjoint sequence $(E_j)_{j\in\mathbb{N}}$ in $\mathfrak{S}$.

It is clear that projection-valued measures define spectral measures, and with Proposition 6.2.6 we also see that every spectral measure on a separable Hilbert space is a projection-valued measure.

(b) In Rudin's Functional Analysis [Ru73] one finds the notion of a *resolution of the identity* $P\colon \mathfrak{S} \to \mathcal{P}_{\mathcal{H}}$, where the following properties are required:

(1) $P(\emptyset) = \mathbf{0}$ and $P(X) = \mathbf{1}$.

(2) $P(E \cap F) = P(E)P(F)$ for $E, F \in \mathfrak{S}$.

(3) If $E \cap F = \emptyset$, then $P(E \cup F) = P(E) + P(F)$.

(4) The functions $P^v\colon E \mapsto \langle P(E)v, v\rangle$, $v \in \mathcal{H}$, are measures on $(X, \mathfrak{S})$.

We have already seen that all these properties are satisfied by spectral measures. Conversely, it is shown in [Ru73, Prop. 12.18] that every resolution of the identity is a spectral measure.

6.2.3 An Application to von Neumann Algebras

Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a von Neumann algebra, i.e., $\mathcal{A} = \mathcal{A}''$ (cf. Definition 4.2.4). We want to show that $\mathcal{A}$ is generated, as a von Neumann algebra, by its projections, i.e., that the set

$$P_{\mathcal{A}} := \{P \in \mathcal{A}\colon P = P^* = P^2\}$$

satisfies $P_{\mathcal{A}}'' = \mathcal{A}$.

Lemma 6.2.17. *Let $(\pi, \mathcal{H})$ be a non-degenerate representation of the Banach-$*$-algebra $\mathcal{A}$ and $P\colon \mathfrak{S} = \mathfrak{B}(\widehat{\mathcal{A}}) \to P_{\mathcal{H}}$ the corresponding regular Borel spectral measure with $P(\widehat{a}) = \pi(a)$ for $a \in \mathcal{A}$. Then*

$$\pi(\mathcal{A})'' = P(\mathfrak{S})''.$$

Proof. Clearly $P(\mathfrak{S})''$ is a von Neumann algebra in $B(\mathcal{H})$, hence in particular norm closed. Since it contains all operators $P(E)$, $E \in \mathfrak{S}$, it contains the operators $P(f)$ for all measurable step functions $f\colon \widehat{\mathcal{A}} \to \mathbb{C}$. As these form a dense subspace of $L^\infty(\widehat{\mathcal{A}})$, we see that

$$\pi(\mathcal{A}) = P(\{\widehat{a}\colon a \in \mathcal{A}\}) \subseteq P(C_0(\widehat{\mathcal{A}})) \subseteq P(L^\infty(\widehat{\mathcal{A}})) \subseteq P(\mathfrak{S})''.$$

To prove the converse inclusion, we have to show that each $P(E)$ is contained in $\pi(\mathcal{A})''$, i.e., that it commutes with $\pi(\mathcal{A})'$. In view of Exercise 4.1.5, the unital C^*-algebra $\pi(\mathcal{A})'$ is spanned by its unitary elements, so that it suffices to show that $P(E)$ commutes with all unitary elements $u \in \pi(\mathcal{A})'$. For any such unitary element

$$P_u(E) := uP(E)u^{-1}$$

defines a regular spectral measure with the property that for $a \in \mathcal{A}$ we have

$$P_u(\widehat{a}) = uP(\widehat{a})u^{-1} = u\pi(a)u^{-1} = \pi(a).$$

Therefore the uniqueness of the spectral measure representing π implies that $P_u = P$, i.e., that each $P(E)$ commutes with u. $\square$

The following proposition is a useful tool in representation theory. It implies in particular that the commutant of any unitary representation is completely determined by its orthogonal projections, hence by the closed invariant subspaces.

Proposition 6.2.18. *For any von Neumann algebra $\mathcal{A} \subseteq B(\mathcal{H})$ we have $P_{\mathcal{A}}'' = \mathcal{A}$, where $P_{\mathcal{A}}$ is the set of orthogonal projections contained in $\mathcal{A}$.*

Proof. From $P_{\mathcal{A}} \subseteq \mathcal{A} = \mathcal{A}''$ we immediately derive that $P_{\mathcal{A}}'' \subseteq \mathcal{A}'' = \mathcal{A}$.

To see that we actually have equality, let $a \in \mathcal{A}$ be a hermitian element and $\mathcal{B} \subseteq \mathcal{A}$ be the C^*-algebra generated by a. Then there exists a spectral measure $P\colon \mathfrak{S} = \mathfrak{B}(\widehat{\mathcal{B}}) \to P_{\mathcal{H}}$ with $P(\widehat{b}) = b$ for each $b \in \mathcal{B}$. From Lemma 6.2.17 we now derive that

$$a \in \mathcal{B} \subseteq \mathcal{B}'' = P(\mathfrak{S})''.$$

Further, $P(\mathfrak{S}) \subseteq P(\mathfrak{S})'' = \mathcal{B}'' \subseteq \mathcal{A}'' = \mathcal{A}$ implies $P(\mathfrak{S}) \subseteq P_{\mathcal{A}}$, so that we arrive at $a \in P(\mathfrak{S})'' \subseteq P_{\mathcal{A}}''$. Since a was arbitrary and each element of $\mathcal{A}$ is a linear combination of two hermitian elements, we obtain $\mathcal{A} \subseteq P_{\mathcal{A}}''$. $\square$

Corollary 6.2.19. *Let $(X, \mathfrak{S}, \mu)$ be a σ-finite measure space and $S \subseteq L^\infty(X, \mu)$ be a $*$-subsemigroup with the property that $\mathfrak{S}$ is the smallest σ-algebra on X for which all elements of S are measurable. We identify $L^\infty(X, \mu)$ with the corresponding subalgebra of $B(L^2(X, \mu))$ acting by multiplication operators. Then $S'' = L^\infty(X, \mu)$ holds in $B(L^2(X, \mu))$ and if μ is finite, then 1 is a cyclic vector for the representation of S on $L^2(X, \mu)$.*

Proof. Let $\mathcal{A} := S'' \subseteq L^\infty(X, \mu)$ (Exercise 4.2.1) be the von Neumann algebra generated by S and $P_{\mathcal{A}}$ its set of projections. We claim that

$$\mathfrak{A} := \{E \subseteq X\colon \chi_E \in \mathcal{A}\}$$

is a σ-algebra. Clearly $0 \in \mathcal{A}$ implies $\emptyset \in \mathfrak{A}$, and since $\mathbf{1} \in \mathfrak{A}$, we also have $\chi_{E^c} = \mathbf{1} - \chi_E \in \mathcal{A}$ for each $E \in \mathfrak{A}$. From $\chi_E \cdot \chi_F = \chi_{E\cap F}$ we also derive that $\mathfrak{A}$ is closed under intersections. Further

$$\chi_{E\cup F} = \chi_E + \chi_F - \chi_{E\cap F}$$

implies that $\mathfrak{A}$ is closed under finite unions.

Now let $(E_n)_{n\in\mathbb{N}}$ be a sequence of elements in $\mathfrak{A}$. It remains to show that $E := \bigcup_{n\in\mathbb{N}} E_n \in \mathfrak{A}$. Let $F_n := E_1 \cup \cdots \cup E_n$. Then $F_n \in \mathfrak{A}$ implies $\chi_{F_n} \in \mathcal{A}$. In view of Proposition 6.2.6(iv) $\chi_{F_n} \to \chi_E$ holds in the strong operator topology because the map $E \mapsto \chi_E$ is a spectral measure (Lemma 6.2.5). As $\mathcal{A}$ is closed in this topology (Remark 4.2.5(a)), $\chi_E \in \mathcal{A}$.

We may w.l.o.g. assume that S is a $*$-algebra. If $f = f^* \in S$, the commutative C^*-algebra generated by f is contained in $\mathcal{A}$ and contains all functions $h \circ f$, where $h\colon \mathbb{C} \to \mathbb{C}$ is continuous (this is an easy consequence of the Gelfand Representation Theorem). In particular, it contains for each $c \in \mathbb{R}$ the function $\max(f, c)$. Now $e^{-n(\max(f,c)-c)} \in \mathcal{A}$ converges pointwise to the characteristic function $\chi_{f\leq c}$. Therefore Lemma 6.2.15 implies that $\chi_{f\leq c} \in \mathcal{A}$, so that $\{f \leq c\} \in \mathfrak{A}$. This proves that each real-valued function $f \in S$ is $\mathfrak{A}$-measurable, and hence that every $f \in S$ is $\mathfrak{A}$-measurable. By assumption, we now find that $\mathfrak{A} = \mathfrak{S}$, which implies $\mathcal{A} = L^\infty(X, \mu)$ because $L^\infty(X, \mu)$ is a von Neumann algebra in $B(L^2(X, \mu))$ (Exercise 4.2.1).

Clearly, 1 is a cyclic vector for $L^\infty(X, \mu) = S''$, and therefore Exercise 5.3.10 implies that it is also cyclic for S. $\square$

Remark 6.2.20. (Central decomposition) If G is a topological group and $(\pi, \mathcal{H})$ a continuous unitary representation, then $\mathcal{Z} := Z(\pi(G)')$ is a commutative von Neumann algebra and one can apply the spectral theorem to this algebra to obtain a spectral measure on $\widehat{\mathcal{Z}}$. Then all spectral projections $P(E)$ are contained in $\mathcal{Z}$, and for any such projection we obtain a G-invariant decomposition

$$\mathcal{H} = P(E)\mathcal{H} \oplus P(E^c)\mathcal{H}.$$

Since $P(E)$ also commutes with $\pi(G)'$, the commutant preserves these two subspaces, which implies that

$$B_G(P(E)\mathcal{H}, P(E^c)\mathcal{H}) = \{0\},$$

which means that both pieces are "disjoint" as G-representations. Refining these techniques leads to a so-called "direct integral decomposition" of $\mathcal{H}$ into factor representations. This is an important generalization of the isotypical decomposition of a unitary representation with $\mathcal{H} = \mathcal{H}_d$ to the general case, when there is a non-zero continuous part.

Example 6.2.21. (a) Let A be an abelian topological group, H a topological group, and $\alpha\colon H \to \mathrm{Aut}(A)$ be a homomorphism defining a continuous action of H on A. We want to construct irreducible unitary representations of the semidirect product group $G := A \rtimes_\alpha H$.

On the character group $\widehat{A}$, we consider the smallest σ-algebra $\mathfrak{S}$ for which all functions $\widehat{a}\colon \widehat{A} \to \mathbb{T}, \chi \mapsto \chi(a)$, are measurable. If $\mu\colon \mathfrak{S} \to [0,1]$ is a probability measure on $\widehat{A}$ invariant under the right action of H on $\widehat{A}$, given by $\widehat{\alpha}(h)(\chi) := \chi \circ \alpha(h)$, we obtain a unitary representation of G by

$$\pi(a,h)f := \widehat{a} \cdot (f \circ \widehat{\alpha}(h)).$$

In view of Corollary 6.2.19, we then have

$$\pi(A)'' = L^\infty(\widehat{A}, \mu),$$

and therefore

$$\pi(A)' = \pi(A)''' = L^\infty(\widehat{A}, \mu)' = L^\infty(\widehat{A}, \mu)$$

by Exercise 4.2.1. This implies that

$$\pi(G)' = \pi(A)' \cap \pi(H)' = L^\infty(\widehat{A}, \mu)^H := \{f \in L^\infty(\widehat{A}, \mu)\colon (\forall h \in H)\, f \circ \widehat{\alpha}(h) = f\}.$$

Note that the relation $f \circ \widehat{\alpha}(h) = f$ only means an identity of elements of $L^\infty(\widehat{A}, \mu)$, i.e., that these functions coincide up to a set of μ-measure zero.

We call the measure μ *strongly ergodic* if

$$L^\infty(\widehat{A}, \mu)^H = \mathbb{C}\mathbf{1}.$$

Now the preceding argument implies that μ is strongly ergodic if and only if $\pi(G)' = \mathbb{C}\mathbf{1}$, which, by Schur's Lemma, is equivalent to the irreducibility of the representation $(\pi, L^2(\widehat{A}, \mu))$.

(b) To see a more concrete example, let $A = \mathbb{R}^n$ and $H = \mathrm{O}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R}) \cong \mathrm{Aut}(A)$. We identify $\widehat{A}$ with $\mathbb{R}^n$ via $\chi_y(x) := e^{i\langle x,y\rangle}$. Then the surface measure μ_r on the sphere

$$\mathbb{S}(r) := \{v \in \mathbb{R}^n\colon \|v\| = r\}$$

is H-invariant, and by identifying $\mathbb{S}(r)$ with the coset space $\mathrm{O}_n(\mathbb{R})/\mathrm{O}_{n-1}(\mathbb{R})$, we derive from Exercise 6.2.8 that this measure is strongly ergodic. In particular, we obtain an irreducible unitary representation of the motion group $G = \mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R})$ on $L^2(\mathbb{S}(r), \mu_r)$ by

$$(\pi_r(x,g)f)(y) := e^{i\langle x,y\rangle} f(g^{-1}y), \quad 0 \leq r < \infty.$$

For more general examples and results in this context, we refer to [Fa08] (cf. also the discussion in Example 5.3.16).

Remark 6.2.22. The construction in the preceding example can be generalized as follows. Let A be an abelian topological group and X a set. Consider a homomorphism $\beta\colon A \to \mathbb{T}^X$ (which is the same as a map $X \to \widehat{A}$), and let $\mathfrak{S}$ be the smallest σ-algebra on X for which all maps $\beta(a)$, $a \in A$, are measurable.

Suppose, in addition, that H is a topological group acting continuously on A by $\alpha\colon H \to \mathrm{Aut}(A)$ and that it acts on X by $\sigma_X\colon H \to S_X$ with $\beta(a) \circ \sigma_X^{-1} =$

$\beta(\alpha(h)a)$ for $a \in A$, $h \in H$ (this means that the map $X \to \widehat{A}$ is H-equivariant). Then H acts by automorphism on the measurable space $(X, \mathfrak{S})$ and every H-strongly ergodic measure μ on $(X, \mathfrak{S})$ leads to an irreducible representation of G on $L^2(X, \mu)$ by

$$(\pi(a, h)f)(x) := \beta(a) \cdot (f \circ \sigma_X(h)^{-1}).$$

Exercises for Section 6.2

Exercise 6.2.1. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert space $\mathcal{H}$ which converges weakly to v and satisfies $\|v_n\| \to \|v\|$. Then we have $v_n \to v$.

Exercise 6.2.2. Let $\alpha\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of commutative Banach-$*$-algebras which is *non-degenerate* in the sense that no (non-zero) character of $\mathcal{B}$ vanishes on $\alpha(\mathcal{A})$. Show that

$$\widehat{\alpha}\colon \widehat{\mathcal{B}} \to \widehat{\mathcal{A}}, \quad \chi \mapsto \chi \circ \alpha$$

is a continuous map which is proper, i.e., inverse images of compact subsets of $\widehat{\mathcal{A}}$ are compact. Hint: Extend $\widehat{\alpha}$ to a continuous map $\mathrm{Hom}(\mathcal{B}, \mathbb{C}) \to \mathrm{Hom}(\mathcal{A}, \mathbb{C})$.

Exercise 6.2.3. Let $f\colon X \to Y$ be a continuous proper map between locally compact spaces. Show that

(a) f is a closed map, i.e., maps closed subsets to closed subsets.

(b) If f is injective, then it is a topological embedding onto a closed subset.

(c) There is a well-defined homomorphism $f^*\colon C_0(Y) \to C_0(X)$ of C^*-algebras, defined by $f^*h := h \circ f$. Identifying X with $C_0(X)\widehat{}$ and Y with $C_0(Y)\widehat{}$, we have $\widehat{f^*} = f$.

(d) For each regular Borel measure μ on X, the push-forward measure $f_*\mu$ on Y, defined by $(f_*\mu)(E) := \mu(f^{-1}(E))$ is regular. Hint: To verify outer regularity, pick an open $O \supseteq f^{-1}(E)$ with $\mu(O \setminus f^{-1}(E)) < \varepsilon$. Then $U := f(O^c)^c$ is an open subset of Y containing E and $\widetilde{O} := f^{-1}(U)$ satisfies $f^{-1}(E) \subseteq \widetilde{O} \subseteq O$, which leads to $(f_*\mu)(U \setminus E) < \varepsilon$.

Exercise 6.2.4. (Cyclic spectral measures) Let $P\colon \mathfrak{S} \to P_{\mathcal{H}}$ be a spectral measure on $(X, \mathfrak{S})$ with a cyclic vector v. Find a unitary isomorphism $\Phi\colon L^2(X, P^v) \to \mathcal{H}$ with $\Phi(\chi_E f) = P(E)\Phi(f)$ for $f \in L^2(X, P^v)$.

Exercise 6.2.5. (Unitary one-parameter groups) Let $P\colon \mathfrak{S} \to P_{\mathcal{H}}$ be a spectral measure on $(X, \mathfrak{S})$ and $f\colon X \to \mathbb{R}$ be a measurable function. Show that

(a) $\pi\colon \mathbb{R} \to \mathrm{U}(\mathcal{H}), \pi(t) := P(e^{itf})$ defines a continuous unitary representation of $\mathbb{R}$ on $\mathcal{H}$.

(b) If f is bounded, then π is norm continuous.

(c) If f is norm-continuous, then f is essentially bounded.

Exercise 6.2.6. (Strongly ergodic measures) Let $(X,\mathfrak{S})$ be a measurable space and
$\sigma\colon G\times X\to X$ an action of a group G on X by measurable maps. Show that for a finite G-invariant measure μ on $(X,\mathfrak{S})$, the following are equivalent

(a) $L^2(X,\mu)^G=\mathbb{C}\mathbf{1}$, i.e., the only elements of $L^2(X,\mu)$ invariant under the representation $(\pi(g)f)(x):=f(g^{-1}x)$ are the constants.

(b) $L^\infty(X,\mu)^G=\mathbb{C}\mathbf{1}$.

Then the measure μ is called *strongly* G-ergodic.

Exercise 6.2.7. Let G be a compact group and μ_G be normalized Haar measure of G. Show μ_G is strongly ergodic for the multiplication action $\sigma(g,h):=gh$ of G on itself is strongly ergodic. Hint: Exercise 6.1.2.

Exercise 6.2.8. Let G be a compact group, μ_G be normalized Haar measure of G, H a closed subgroup of G, $q\colon G\to G/H$ the quotient map and $\mu:=q_*\mu_G$. Show that μ is strongly ergodic with respect to the left translation action $\sigma(g,xH):=gxH$ of G on the quotient space G/H of left cosets of H. Hint: Exercise 6.2.7

Exercise 6.2.9. Let H be a compact group, $G\subseteq H$ be a dense subgroup and μ_H normalized Haar measure of H. Show that μ_H is strongly ergodic with respect to the multiplication action $\sigma(g,h):=gh$ of G on H.

Exercise 6.2.10. Show that the Haar measure on $\mathbb{T}$ is ergodic for the action of $\mathbb{Z}$ on $\mathbb{T}$ by $n.e^{it}:=e^{i(t+n\theta)}$, where θ is an irrational number.

Exercise 6.2.11. Let $\mathcal{A}\subseteq B(\mathcal{H})$ be a commutative von Neumann algebra, where $\mathcal{H}$ is separable. Show that the following assertions are equivalent

(a) $\mathcal{A}$ is maximal commutative, i.e., $\mathcal{A}'=\mathcal{A}$.

(b) The representation of $\mathcal{A}$ on $\mathcal{H}$ is multiplicity free, i.e., $\mathcal{A}'$ is commutative.

(c) The representation of $\mathcal{A}$ on $\mathcal{H}$ is cyclic.

Hint: Use Exercise 5.3.11 for (b) $\Rightarrow$ (c) and for (b) $\Rightarrow$ (a) observe that $\mathcal{A}'$ is commutative if and only if $\mathcal{A}'\subseteq\mathcal{A}''=\mathcal{A}$. For (c) $\Rightarrow$ (b) use Corollary 6.2.14 to identify the cyclic representations as some $L^2(\widehat{\mathcal{A}},\mu)$, and then Lemma 6.2.17 to see that in this case the commutant is the commutative algebra $L^\infty(\widehat{\mathcal{A}},\mu)'=L^\infty(\widehat{\mathcal{A}},\mu)\subseteq B(L^2(\widehat{\mathcal{A}},\mu))$ (Exercise 4.2.1).

6.3 Representations of Abelian Locally Compact Groups

Proposition 6.3.1. *For an abelian locally compact group G, the following assertions hold:*

(a) *$L^1(G)$ is a commutative Banach-$*$-algebra.*

(b) *The map*

$$\eta\colon \widehat{G} \to L^1(G)', \quad \eta(\chi)(f) := \int_G f(x)\chi(x)\, d\mu_G(x)$$

maps the character group $\widehat{G}$ bijectively onto $L^1(G)\widehat{\ }$.

Proof. (a) It suffices to show that the convolution product is commutative on the dense subalgebra $C_c(G)$. Since the modular factor Δ_G of G is trivial (Proposition 2.4.7), we have

$$\begin{aligned}(f * h)(y) &= \int_G f(x)h(x^{-1}y)\, d\mu_G(x) = \int_G f(x^{-1})h(yx)\, d\mu_G(x)\\ &= \int_G f(x^{-1}y)h(x)\, d\mu_G(x) = (h * f)(y).\end{aligned}$$

(b) Since each character $\chi \in \widehat{G}$ is a bounded measurable function, it defines an element in $L^1(G)'$. If $\pi_\chi(g) = \chi(g)\mathbf{1}$ is the one-dimensional irreducible representation of G, defined by the character χ, then the corresponding integrated representation of $L^1(G)$ is given by

$$\pi_\chi(f) = \int_G f(x)\chi(x)\mathbf{1}\, d\mu_G(x) = \eta(\chi)(f)\mathbf{1},$$

so that $\eta(\chi)\colon L^1(G) \to \mathbb{C}$ defines a non-zero algebra homomorphism because it is a non-degenerate representation.

If, conversely, $\gamma\colon L^1(G) \to \mathbb{C}$ is a non-zero continuous homomorphism of Banach-$*$-algebras, then $\pi(f) := \gamma(f)\mathbf{1}$ defines a one dimensional non-degenerate representation of $L^1(G)$, and the corresponding representation of G is given by a continuous character χ with $\pi_\chi = \pi$. This implies that $\gamma = \eta(\chi)$. $\square$

In the following we endow the character group $\widehat{G}$ always with the locally compact topology for which η is a homeomorphism. This is the coarsest topology for which all functions

$$\widehat{f}\colon \widehat{G} \to \mathbb{C}, \quad \chi \mapsto \int_G f(x)\chi(x)\, d\mu_G(x)$$

are continuous, and, by definition, all these functions vanish at infinity, i.e., $\widehat{f} \in C_0(\widehat{G})$. The function $\widehat{f}$ is called the *Fourier transform* of f.

Example 6.3.2. For $G = \mathbb{R}^n$ we have already seen that each element of $\widehat{G}$ is of the form $\chi_x(y) = e^{i\langle x,y\rangle}$ for some $x \in \mathbb{R}^n$. Therefore the Fourier transform can be written as

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{i\langle x,y\rangle}\,dx.$$

From Lebesgue's Dominated Convergence Theorem it follows immediately that all the functions $\widehat{f}$ are continuous with respect to the standard topology of $\mathbb{R}^n$. Therefore the bijection

$$\iota\colon \mathbb{R}^n \to \widehat{\mathbb{R}}^n, \quad x \mapsto \chi_x$$

is continuous. Further, the Riemann–Lebesgue Lemma (Proposition 6.3.3 below) implies that all functions $\widehat{f}$ vanish at infinity, and this implies that ι extends to a continuous map

$$\iota_\omega\colon (\mathbb{R}^n)_\omega \to (\widehat{\mathbb{R}}^n)_\omega \cong \mathrm{Hom}(L^1(\mathbb{R}^n), \mathbb{C})$$

of the one-point compactifications. As ι_ω is a bijection and $\mathbb{R}^n_\omega$ is compact, it follows that ι_ω, and hence also ι, is a homeomorphism.

Proposition 6.3.3. (Riemann–Lebesgue) *For each $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\widehat{f}$ vanishes at infinity, i.e., $\widehat{f} \in C_0(\mathbb{R}^n)$.*

Proof. For each $0 \neq x \in \mathbb{R}^n$ we obtain with $e^{-i\pi} = -1$ and the translation invariance of Lebesgue measure the relation

$$\begin{aligned}
\widehat{f}(x) &= 2\frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x,y\rangle} f(y)\,dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x,y\rangle} f(y)\,dy - \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x,y-\frac{\pi}{\|x\|^2}x\rangle} f(y)\,dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x,y\rangle} f(y)\,dy - \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x,y\rangle} f\Big(y + \frac{\pi}{\|x\|^2}x\Big)\,dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x,y\rangle}\Big[f(y) - f\Big(y + \frac{\pi}{\|x\|^2}x\Big)\Big]\,dy.
\end{aligned}$$

This implies that

$$|\widehat{f}(x)| \leq \frac{1}{2}\int_{\mathbb{R}^n}\Big|f(y) - f\Big(y + \frac{\pi}{\|x\|^2}x\Big)\Big|\,dy.$$

Now the assertion follows from the continuity of the map

$$\mathbb{R}^n \to L^1(\mathbb{R}^n), \quad x \mapsto \lambda_x f$$

in 0 (Lemma 6.1.1) and $\lim_{x\to\infty} \frac{x}{\|x\|^2} = 0$. $\square$

Theorem 6.3.4. (Spectral Theorem for locally compact abelian groups) *Let G be a locally compact abelian group and $\widehat{G} \cong L^1(G)\widehat{}$ be its character group. Then, for each regular spectral measure P on the locally compact space $\widehat{G}$, the unitary representation*

$$\pi_P \colon G \to \mathrm{U}(\mathcal{H}), \quad \pi_P(g) := P(\widehat{g}), \quad \widehat{g}(\chi) = \chi(g),$$

is continuous. If, conversely, $(\pi, \mathcal{H})$ is a continuous unitary representation of G on $\mathcal{H}$, then there exists a unique regular spectral measure P with $\pi = \pi_P$.

Proof. First we use the spectral measure P to define a non-degenerate representation π_P of $L^1(G)$ by $\pi_P(f) := P(\widehat{f})$ (Theorem 6.2.12). Then we clearly have

$$\pi_P(g)\pi_P(f) = P(\widehat{g})P(\widehat{f}) = P(\widehat{g}\widehat{f}), \quad g \in G, f \in L^1(G).$$

Next we observe that for each character $\chi \in \widehat{G}$ we have

$$\begin{aligned}(\lambda_g f)\widehat{}(\chi) &= \int_G (\lambda_g f)(x)\chi(x)\, d\mu_G(x) = \int_G f(g^{-1}x)\chi(x)\, d\mu_G(x) \\ &= \int_G f(x)\chi(gx)\, d\mu_G(x) = \chi(g)\int_G f(x)\chi(x)\, d\mu_G(x) = \widehat{g}(\chi)\widehat{f}(\chi).\end{aligned}$$

We thus obtain

$$\pi_P(g)\pi_P(f) = P(\widehat{g}\widehat{f}) = P((\lambda_g f)\widehat{}) = \pi_P(\lambda_g f),$$

so that $\pi_P \colon G \to \mathrm{U}(\mathcal{H})$ is the unique continuous unitary representation of G on $\mathcal{H}$ corresponding to the representation of $L^1(G)$ (Theorem 6.1.12). In particular, π_P is continuous. It follows in particular that π_P is continuous.

If, conversely, $(\pi, \mathcal{H})$ is a continuous unitary representation of G and $\pi \colon L^1(G) \to B(\mathcal{H})$ the corresponding non-degenerate representation of $L^1(G)$, then we use Theorem 6.2.12 to obtain a regular spectral measure P on $\widehat{G} \cong L^1(G)\widehat{}$ with $\pi(f) = P(\widehat{f})$, $f \in L^1(G)$. Then

$$\pi(g)\pi(f) = \pi(\lambda_g f) = P(\widehat{g}\widehat{f}) = P(\widehat{g})P(\widehat{f}) = P(\widehat{g})\pi(f)$$

implies that $P(\widehat{g}) = \pi(g)$ holds for each $g \in G$ (Theorem 6.1.12). □

Definition 6.3.5. Let P be a regular spectral measure on $\widehat{G}$. If $(U_i)_{i\in I}$ are open subset of $\widehat{G}$ with $P(U_i) = \mathbf{0}$, then the same holds for $U := \bigcup_{i\in I} U_i$. In fact, since P is inner regular, it suffices to observe that for each compact subset $C \subseteq U$ we have $P(C) = \mathbf{0}$, but this follows from the fact that C is covered by finitely many U_i. We conclude that there exists a maximal open subset $U \subseteq \widehat{G}$ with $P(U) = \mathbf{0}$, and its complement

$$\mathrm{supp}(P) := U^c$$

is called the *support of* P, resp., the support of the corresponding representation. It is the smallest closed subset A of $\widehat{G}$ with $P(A) = \mathbf{1}$.

Combining Corollary 6.2.14 with the arguments in the preceding proof, we immediately obtain a description of the cyclic representations:

Corollary 6.3.6. *A representation $(\pi, \mathcal{H})$ of an abelian locally compact group G is cyclic if and only if there exists a finite Radon measure μ on $\widehat{G}$ such that $(\pi, \mathcal{H})$ is equivalent to the cyclic representation $\big(\pi_\mu, L^2(\widehat{G}, \mu), 1\big)$, given by $\pi_\mu(g)f = \widehat{g}\cdot f$. In particular, all these representations are continuous and cyclic.*

Theorem 6.3.7. (Bochner's Theorem) *A continuous function φ on the locally compact abelian group G is positive definite if and only if there exists a finite Radon measure μ on $\widehat{G}$ with $\varphi = \widehat{\mu}$. Then μ is uniquely determined by φ.*

Proof. We have already seen in Proposition 5.3.3 that φ is positive definite if and only if $\varphi = \pi^v$ holds for a continuous cyclic unitary representation $(\pi, \mathcal{H}, v)$. In view of Corollary 6.3.6, any such representation is equivalent to a representation of the form $(\pi_\mu, \widehat{G}, 1)$, where μ is a regular Borel measure on $\widehat{G}$ and $\pi_\mu(g)f = \widehat{g}f$. Now the assertion follows from

$$\langle \pi_\mu(g)1, 1\rangle = \langle \widehat{g}, 1\rangle = \int_G \widehat{g}(\chi)\, d\mu(\chi) = \widehat{\mu}(g).$$

To see that μ is unique, we note that it can be written as

$$\mu(E) = \langle P(E)1, 1\rangle = P^1(E)$$

in terms of the spectral measure P. Therefore its uniqueness follows from the uniqueness of the spectral measure. $\square$

Remark 6.3.8. An important but subtle point of the theory of non-discrete spectral measures is the measurement of multiplicities. For a discrete spectral measure $P\colon \mathfrak{S} \to \mathcal{P}_\mathcal{H}$ the multiplicity of $x \in X$ can simply be measured by $\dim P(\{x\})\mathcal{H}$, but for the continuous part of $\mathfrak{S}$ this is more complicated. One way to deal with this problem is to decompose $\mathcal{H}$ into cyclic subspaces

$$\mathcal{H}_v := \overline{\operatorname{span}\{P(E)v\colon E \in \mathfrak{S}\}}$$

with respect to P. From

$$\langle P(E)v, P(F)v\rangle = \langle P(E\cap F)v, v\rangle = P^v(E \cap F)$$

one easily derives that

$$\mathcal{H}_v \cong L^2(X, P^v)$$

(cf. Examples 3.3.6(d)). On this space the representation of the involutive semigroup $(\mathfrak{S}, \mathrm{id}_\mathfrak{S})$ is multiplicity free because $(P(\mathfrak{S})|_{\mathcal{H}_v})' \cong L^\infty(X, P^v)$ (Exercise 4.2.1) is commutative. Now $\mathcal{H}$ is a direct sum of such spaces $\mathcal{H}_v$, so that one may count multiplicities by comparing the measures P^v. This problem is studied systematically in Halmos' nice book [Ha57] which is still the best reference for these issues.

Exercises for Section 6.3

Exercise 6.3.1. Show that every cyclic representation of an abelian involutive semigroup $(S, *)$ is multiplicity free. Hint: Since $\pi(S)' = \pi(S)'''$, one may assume that $S = \mathcal{A}$ is a commutative C^*-algebra. In this case we know the cyclic representations and the corresponding commutants (Exercise 4.2.1). Combined with Exercise 5.3.11, this proves that a non-degenerate representation of an abelian involutive semigroup on a separable Hilbert space is cyclic if and only if it is multiplicity free.

6.4 Unitary Representations of $\mathbb{R}$

The following theorem shows how spectral measures lead to unitary representations of the group $(\mathbb{R}, +)$ and vice versa. It may be considered as a classification of unitary one-parameter groups in terms of spectral measures on $\mathbb{R}$ which provides important structural information.

Theorem 6.4.1. *Let $P\colon \mathfrak{S} \to \mathcal{P}_{\mathcal{H}}$ be a spectral measure and $f\colon X \to \mathbb{R}$ be a measurable function. Then*

$$\pi(t) := P(e^{itf})$$

defines a continuous unitary representation $\pi\colon \mathbb{R} \to \mathrm{U}(\mathcal{H})$.

Conversely, every continuous unitary representation of $\mathbb{R}$ has this form for $X = \mathbb{R}$ and $f(x) = x$.

Proof. Since $\mathbb{R} \to \mathrm{U}(L^\infty(X, \mathbb{C})), t \mapsto e^{itf}$ is a homomorphism into the unitary group of the C^*-algebra $L^\infty(X, \mathbb{C})$, it follows from Proposition 6.2.8 that $\pi(t) := P(e^{itf})$ defines a unitary representation of $\mathbb{R}$.

Now let $v \in \mathcal{H}$. Then Remark 6.2.9 implies that

$$\pi^v(t) = \langle \pi(t)v, v\rangle = \langle P(e^{itf})v, v\rangle = \int_X e^{itf(x)}\, dP^v(x),$$

and the continuity of this function follows from Lebesgue's Theorem on Dominated Convergence.

If, conversely, $(\pi, \mathcal{H})$ is a continuous unitary representation of $\mathbb{R}$, then Theorem 6.3.4 implies the existence of a spectral measure P on $\widehat{\mathbb{R}}$ with $\pi(t) = P(\widehat{t})$ for $t \in \mathbb{R}$. Identifying the locally compact space $\widehat{\mathbb{R}}$ with $\mathbb{R}$ in such a way that $\widehat{t}(x) = e^{itx}$ (Example 6.3.2), the assertion follows. $\square$

Remark 6.4.2. If π is defined as above, then all operators $P(E)$, $E \in \mathfrak{S}$, commute with $\pi(\mathbb{R})$.

Example 6.4.3. Note that Theorem 6.4.1 applies in particular for $X = \mathbb{R}$ and $\mathfrak{S} = \mathfrak{B}(\mathbb{R})$ (the σ-algebra of Borel sets) and $f(x) = x$.

If, f.i., μ is a Borel measure on $\mathbb{R}$ and $\mathcal{H} = L^2(\mathbb{R}, \mu)$ as in Lemma 6.2.5, then we obtain a continuous unitary representation by $\pi(t)f = e^{itx}f$ (Exercise).

Remark 6.4.4. We have seen above that every unitary one-parameter group is of the form $\pi(t) = P(e^{it\,\mathrm{id}_{\mathbb{R}}})$ for some spectral measure P on $\mathbb{R}$. If the spectral measure P is supported by a bounded subset of $\mathbb{R}$, i.e., there exist $a, b \in \mathbb{R}$ with $P([a,b]) = \mathbf{1}$, then the function $\mathrm{id}_{\mathbb{R}}$ is essentially bounded and $A := P(\mathrm{id}_{\mathbb{R}}) \in B(\mathcal{H})$ is a bounded hermitian operator satisfying

$$\pi(t) = P(e^{it\,\mathrm{id}_{\mathbb{R}}}) = \sum_{k=0}^{\infty} \frac{1}{k!} P(it\,\mathrm{id}_{\mathbb{R}})^k = \sum_{k=0}^{\infty} \frac{1}{k!} (itA)^k = e^{itA}.$$

Taking derivatives in t, we obtain

$$A = \lim_{t\to 0} \frac{1}{it}\big(\pi(t) - \mathbf{1}\big)$$

in the operator norm.

For a general unitary one-parameter group the corresponding spectral measure P on $\mathbb{R}$ may have unbounded supported, which is reflected in the fact that the limit

$$Av := \lim_{t\to 0} \frac{1}{it}\big(\pi(t)v - v\big)$$

may not exist for every $v \in \mathcal{H}$. Write $\mathcal{D}(A)$ for the linear subspace of $\mathcal{H}$ consisting of all elements for which this limit exists. Then

$$A \colon \mathcal{D}(A) \to \mathcal{H}$$

is called an *unbounded operator*. A closer analysis of this situation leads to the theory of unbounded selfadjoint operators on Hilbert spaces.

Here we only note that for every bounded subset $B \subseteq \mathbb{R}$ the closed subspace $\mathcal{H}_B := P(B)\mathcal{H}$ is invariant, and on this subspace we have

$$\pi(t)|_{\mathcal{H}_B} = P(e^{it\,\mathrm{id}_{\mathbb{R}}})|_{\mathcal{H}_B} = P(e^{it\,\mathrm{id}_{\mathbb{R}}\,\chi_B})|_{\mathcal{H}_B} = e^{itP(\mathrm{id}_{\mathbb{R}}\,\chi_B)}|_{\mathcal{H}_B},$$

and since the operator $P(\mathrm{id}_{\mathbb{R}}\,\chi_B)$ is bounded, we obtain in particular that $\mathcal{H}_B \subseteq \mathcal{D}(A)$ with

$$A|_{\mathcal{H}_B} = P(\mathrm{id}_{\mathbb{R}}\,\chi_B)|_{\mathcal{H}_B}.$$

In view of $P([-n,n]) \to P(\mathbb{R}) = \mathbf{1}$ in the strong operator topology, the union $\bigcup_{n\in\mathbb{N}} \mathcal{H}_{[-n,n]}$ is dense in $\mathcal{H}$, which implies that $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$. We say that A is *densely defined*.

This suggests that A should be something like $P(\mathrm{id}_{\mathbb{R}})$, and to make sense out of that, one has to extend the spectral integral to unbounded measurable functions.

Exercises for Section 6.4

Exercise 6.4.1. (One-parameter groups of $\mathrm{U}(\mathcal{H})$)

(1) Let $A = A^* \in B(\mathcal{H})$ be a bounded hermitian operator. Then $\gamma_A(t) := e^{itA}$ defines a norm-continuous unitary representation of $(\mathbb{R}, +)$.

(2) Let $P\colon (X, \mathfrak{S}) \to B(\mathcal{H})$ be a spectral measure and $f\colon X \to \mathbb{R}$ a measurable function. Then $\gamma_f(t) := P(e^{itf})$ is a continuous unitary representation of $(\mathbb{R}, +)$. Show that γ_f is norm-continuous if and only of f is essentially bounded.

Chapter 7

Closed Subgroups of Banach Algebras

In this chapter we study one of the central tools in Lie theory: the exponential function of a Banach algebra. This function has various applications in the structure theory of Lie groups. First of all, it is naturally linked to the one-parameter subgroups, and it turns out that the local group structure of $\mathcal{A}^\times$ for a unital Banach algebra $\mathcal{A}$ in a neighborhood of the identity is determined by its one-parameter subgroups.

In Section 7.1, we discuss some basic properties of the exponential function of a unital Banach algebra $\mathcal{A}$. In Section 7.2, we then use the exponential function to associate to each closed subgroup $G \subseteq \mathcal{A}^\times$ a Banach–Lie algebra $\mathbf{L}(G)$, called the *Lie algebra of* G. We then show that the elements of $\mathbf{L}(G)$ are in one-to-one correspondence with the one-parameter groups of G and study some functorial properties of the assignment $\mathbf{L}\colon G \mapsto \mathbf{L}(G)$. The last section of this chapter is devoted to some tools to calculate the Lie algebras of closed subgroups of $\mathcal{A}$.

7.1 Elementary Properties of the Exponential Function

Let $\mathcal{A}$ be a unital Banach algebra. For $x \in \mathcal{A}$ we define

$$e^x :\, = \sum_{k=0}^{\infty} \frac{1}{k!} x^k. \tag{7.1}$$

The absolute convergence of the series on the right follows directly from the estimate

$$\sum_{k=0}^{\infty} \frac{1}{k!} \|x^k\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|x\|^k = e^{\|x\|}$$

and the Comparison Test for absolute convergence of a series in a Banach space. We define the *exponential function of* $\mathcal{A}$ by

$$\exp\colon \mathcal{A} \to \mathcal{A}, \quad \exp(x) := e^x.$$

Lemma 7.1.1. *Let* $x, y \in \mathcal{A}$.

(i) *If* $xy = yx$, *then* $\exp(x+y) = \exp x \exp y$.

(ii) $\exp(\mathcal{A}) \subseteq \mathcal{A}^\times$, $\exp(0) = \mathbf{1}$, *and* $(\exp x)^{-1} = \exp(-x)$.

(iii) *For* $g \in \mathcal{A}^\times$ *we have* $ge^xg^{-1} = e^{gxg^{-1}}$.

(iv) $\exp$ *is differentiable in* 0 *with* $\mathtt{d}\exp(\mathbf{0})x = x$ *for* $x \in \mathcal{A}$.

Proof. (i) Using the general form of the Cauchy Product Formula (Exercise 7.1.3), we obtain

$$\begin{aligned}\exp(x+y) &= \sum_{k=0}^\infty \frac{(x+y)^k}{k!} = \sum_{k=0}^\infty \frac{1}{k!}\sum_{\ell=0}^k \binom{k}{\ell} x^\ell y^{k-\ell} \\ &= \sum_{k=0}^\infty \sum_{\ell=0}^k \frac{x^\ell}{\ell!}\frac{y^{k-\ell}}{(k-\ell)!} = \Big(\sum_{p=0}^\infty \frac{x^p}{p!}\Big)\Big(\sum_{\ell=0}^\infty \frac{y^\ell}{\ell!}\Big).\end{aligned}$$

(ii) From (i) we derive in particular $\exp x \exp(-x) = \exp 0 = \mathbf{1}$, which implies (ii).

(iii) is a consequence of $gx^ng^{-1} = (gxg^{-1})^n$ and the continuity of the conjugation map $c_g(x) := gxg^{-1}$ on $\mathcal{A}$.

(iv) For the exponential series we have the estimate

$$\|e^x - \mathbf{1} - x\| = \|\sum_{n\geq 2}\frac{x^n}{n!}\| \leq \sum_{n\geq 2}\frac{\|x\|^n}{n!} = e^{\|x\|} - \mathbf{1} - \|x\|,$$

showing that

$$\lim_{x\to 0}\frac{\|e^x - \mathbf{1} - x\|}{\|x\|} \leq \lim_{x\to 0}\frac{e^{\|x\|} - \mathbf{1} - \|x\|}{\|x\|} = 0,$$

and hence that exp is differentiable in 0 with $\mathtt{d}\exp(\mathbf{0})x = x$ for $x \in \mathcal{A}$. □

Remark 7.1.2. (a) For $n = 1$, the exponential function

$$\exp\colon \mathbb{R} \cong M_n(\mathbb{R}) \to \mathbb{R}^\times \cong \mathrm{GL}_n(\mathbb{R}), \quad x \mapsto e^x$$

is injective, but this is not the case for $n > 1$. In fact,

$$\exp\begin{pmatrix} 0 & -2\pi \\ 2\pi & 0\end{pmatrix} = \mathbf{1}$$

follows from

$$\exp\begin{pmatrix} 0 & -t \\ t & 0\end{pmatrix} = \begin{pmatrix}\cos t & -\sin t \\ \sin t & \cos t\end{pmatrix}, \quad t \in \mathbb{R}.$$

This example is the real picture of the relation $e^{2\pi i} = 1$.

Definition 7.1.3. A *one-parameter (sub)group* of a group G is a group homomorphism $\gamma\colon (\mathbb{R},+) \to G$. The following result describes the differentiable one-parameter subgroups of $\mathcal{A}^\times$.

Theorem 7.1.4. (One-parameter Group Theorem) *For each $x \in \mathcal{A}$, the map*

$$\gamma_x\colon (\mathbb{R},+) \to \mathcal{A}, \quad t \mapsto \exp(tx)$$

is a smooth group homomorphism solving the initial value problem

$$\gamma_x(0) = \mathbf{1} \quad \text{and} \quad \gamma_x'(t) = \gamma_x(t)x \quad \text{for } t \in \mathbb{R}.$$

Conversely, every continuous one-parameter group $\gamma\colon \mathbb{R} \to \mathcal{A}^\times$ is of this form.

Proof. In view of Lemma 7.1.1(i) and the differentiability of exp in 0, we have

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h}\big(\gamma_x(t+h) - \gamma_x(t)\big) &= \lim_{h\to 0} \frac{1}{h}\big(\gamma_x(t)\gamma_x(h) - \gamma_x(t)\big) \\ &= \gamma_x(t) \lim_{h\to 0} \frac{1}{h}\big(e^{hx} - \mathbf{1}\big) = \gamma_x(t)x.\end{aligned}$$

Hence γ_x is differentiable with $\gamma_x'(t) = x\gamma_x(t) = \gamma_x(t)x$. From that it immediately follows that γ_x is smooth with $\gamma_x^{(n)}(t) = x^n\gamma_x(t)$ for each $n \in \mathbb{N}$.

We first show that each one-parameter group $\gamma\colon \mathbb{R} \to \mathcal{A}^\times$ which is differentiable in 0 has the required form. For $x := \gamma'(0)$, the calculation

$$\gamma'(t) = \lim_{s\to 0} \frac{\gamma(t+s) - \gamma(t)}{s} = \lim_{s\to 0} \gamma(t)\frac{\gamma(s) - \gamma(0)}{s} = \gamma(t)\gamma'(0) = \gamma(t)x$$

implies that γ is continuously differentiable. Therefore

$$\frac{d}{dt}(e^{-tx}\gamma(t)) = -e^{-tx}x\gamma(t) + e^{-tx}\gamma'(t) = 0$$

implies that $e^{-tx}\gamma(t) = \gamma(0) = \mathbf{1}$ for each $t \in \mathbb{R}$, so that $\gamma(t) = e^{tx}$.

Eventually we consider the general case, where $\gamma\colon \mathbb{R} \to \mathcal{A}^\times$ is only assumed to be continuous. The idea is to construct a differentiable function $\widetilde{\gamma}$ by applying a smoothing procedure to γ and to show that the smoothness of $\widetilde{\gamma}$ implies that of γ. So let $f\colon \mathbb{R} \to \mathbb{R}_+$ be a twice continuously differentiable function with $f(t) = 0$ for $|t| > \varepsilon$ and $\int_{\mathbb{R}} f(t)\,dt = 1$, where ε is chosen such that $\|\gamma(t) - \mathbf{1}\| < \frac{1}{2}$ holds for $|t| \leq \varepsilon$.

We define

$$\widetilde{\gamma}(t) := \int_{\mathbb{R}} f(s)\gamma(t-s)\;ds = \gamma(t)\int_{\mathbb{R}} f(s)\gamma(-s)\;ds = \gamma(t)\int_{-\varepsilon}^{\varepsilon} f(s)\gamma(-s)\,ds.$$

Here we use the existence of Riemann integrals of continuous curves with values in Banach spaces, which follows from Theorem 6.1.9. Change of Variables leads to

$$\widetilde{\gamma}(t) = \int_{\mathbb{R}} f(t-s)\gamma(s)\,ds,$$

which is differentiable because

$$\frac{\widetilde{\gamma}(t+h)-\widetilde{\gamma}(t)}{h} = \int_{\mathbb{R}} \frac{f(t+h-s)-f(t-s)}{h}\gamma(s)\, ds$$

and the functions $f_h(t) := \frac{f(t+h)-f(t)}{h}$ converge uniformly for $h \to 0$ to f' (this is a consequence of the Mean Value Theorem). We further have

$$\begin{aligned}\Big\|\int_{-\varepsilon}^{\varepsilon} f(s)\gamma(-s)\, ds - \mathbf{1}\Big\| &= \Big\|\int_{-\varepsilon}^{\varepsilon} f(s)\big(\gamma(-s)-\mathbf{1}\big)\, ds\Big\| \\ &\leq \int_{-\varepsilon}^{\varepsilon} f(s)\|\gamma(-s)-\mathbf{1}\|\, ds \leq \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} f(s)\, ds = \frac{1}{2},\end{aligned}$$

because of the inequality $\|\int h(s)\, ds\| \leq \int \|h(s)\|\, ds$ (cf. Subsection 6.1.3).

Let $g := \int_{-\varepsilon}^{\varepsilon} f(s)\gamma(-s)\, ds$. In view of $\|g-\mathbf{1}\| \leq \frac{1}{2}$ we have $g \in \mathcal{A}^\times$ (see the proof of Proposition 1.1.10) and therefore $\gamma(t) = \widetilde{\gamma}(t)g^{-1}$. Now the differentiability of $\widetilde{\gamma}$ implies that γ is differentiable, and one can argue as above. $\square$

Product and Commutator Formula

We have seen in Lemma 7.1.1 that the exponential image of a sum $x+y$ can be computed easily if x and y commute. In this case we also have for the commutator $[x,y] := xy - yx = 0$ the formula $\exp[x,y] = \mathbf{1}$. The following proposition gives a formula for $\exp(x+y)$ and $\exp([x,y])$ in the general case.

If g,h are elements of a group G, then $(g,h) := ghg^{-1}h^{-1}$ is called their *commutator*. On the other hand, we call for two element $a,b \in \mathcal{A}$ the expression

$$[a,b] := ab - ba$$

their *commutator bracket.*

Proposition 7.1.5. *For $x,y \in \mathcal{A}$, the following assertions hold:*

(i) $\lim_{k\to\infty}\left(e^{\frac{1}{k}x}e^{\frac{1}{k}y}\right)^k = e^{x+y}$ (Trotter Product Formula).

(ii) $\lim_{k\to\infty}\left(e^{\frac{1}{k}x}e^{\frac{1}{k}y}e^{-\frac{1}{k}x}e^{-\frac{1}{k}y}\right)^{k^2} = e^{xy-yx}$ (Commutator Formula).

Proof. We start with a general consideration. We shall have to estimate an expression of the form $A^k - B^k$. To this end we write

$$\begin{aligned}A^k - B^k &= (A^k - A^{k-1}B) + (A^{k-1}B - A^{k-2}B^2) + \ldots + (AB^{k-1} - B^k) \\ &= A^{k-1}(A-B) + A^{k-2}(A-B)B + \ldots + (A-B)B^{k-1} \\ &= \sum_{j=0}^{k-1} A^j(A-B)B^{k-1-j}.\end{aligned}$$

Let us assume that there exists a constant $C > 0$ with

$$\|A^j\|, \|B^j\| \leq C \quad \text{for} \quad 1 \leq j \leq k. \tag{7.2}$$

Then we get the estimate

$$\|A^k - B^k\| \leq \sum_{j=0}^{k-1} \|A^j\| \cdot \|B^{k-1-j}\| \cdot \|A - B\| \leq kC^2\|A - B\|.$$

(i) We apply the estimate from above to some special situations. First we consider $k = n$, $A = e^{\frac{1}{n}x}e^{\frac{1}{n}y}$ and $B = e^{\frac{1}{n}(x+y)}$. Then

$$A^n - B^n = \left(e^{\frac{1}{n}x}e^{\frac{1}{n}y}\right)^n - e^{(x+y)},$$

and we have to show that this expression tends to zero. We therefore check the assumptions from above. We have

$$\|A\| \leq \|e^{\frac{1}{n}x}\| \, \|e^{\frac{1}{n}y}\| \leq e^{\frac{1}{n}\|x\|}e^{\frac{1}{n}\|y\|} = e^{\frac{1}{n}(\|x\|+\|y\|)},$$

and therefore $\|A^j\| \leq e^{\|x\|+\|y\|}$ for $j \leq n$. We likewise obtain $\|B^j\| \leq e^{\|x\|+\|y\|}$ for $j \leq n$, and hence

$$\|A^n - B^n\| \leq e^{2(\|x\|+\|y\|)}n\|A - B\|.$$

We further have

$$\begin{aligned} n(A-B) &= \frac{e^{\frac{1}{n}x}e^{\frac{1}{n}y} - e^{\frac{1}{n}(x+y)}}{\frac{1}{n}} \\ &= \frac{e^{\frac{1}{n}x}(e^{\frac{1}{n}y} - \mathbf{1}) + (e^{\frac{1}{n}x} - \mathbf{1}) + \mathbf{1} - e^{\frac{1}{n}(x+y)}}{\frac{1}{n}} \\ &\to e^0 \cdot \mathtt{d}\exp(\mathbf{0})y + \mathtt{d}\exp(\mathbf{0})x - \mathtt{d}\exp(\mathbf{0})(x+y) \\ &= y + x - (x+y) = \mathbf{0}. \end{aligned}$$

This implies $A^n - B^n \to 0$ and hence the Trotter Formula.

(ii) Now let $k = n^2$, $A = e^{\frac{1}{n}x}e^{\frac{1}{n}y}e^{-\frac{1}{n}x}e^{-\frac{1}{n}y}$ and $B = e^{\frac{1}{n^2}(xy-yx)}$. Then

$$A^{n^2} - B^{n^2} = \left(e^{\frac{1}{n}x}e^{\frac{1}{n}y}e^{-\frac{1}{n}x}e^{-\frac{1}{n}y}\right)^{n^2} - e^{xy-yx},$$

and we have to show that this expression tends to zero. Again, we verify (7.2). In view of

$$\|B\| \leq e^{\frac{1}{n^2}\|xy-yx\|} \leq e^{\frac{1}{n^2}(2\|x\|\cdot\|y\|)},$$

we have $\|B^j\| \leq e^{2\|x\|\cdot\|y\|}$ for $j \leq n^2$. To estimate the A-part, let us write $O(t^k)$ for a function for which $t^{-k}O(t^k)$ is bounded for $t \to 0$. We likewise write $O(n^k)$, $k \in \mathbb{Z}$, for a function for which $n^{-k}O(n^k)$ is bounded for $n \to \infty$. We consider the smooth curve

$$\gamma\colon \mathbb{R} \to \mathcal{A}^\times, \quad t \mapsto e^{tx}e^{ty}e^{-tx}e^{-ty}.$$

Then $e^{tx} = \mathbf{1} + tx + \frac{t^2}{2}x^2 + O(t^3)$ leads to

$$\begin{aligned}\gamma(t) &= \big(\mathbf{1} + tx + \frac{t^2}{2}x^2 + O(t^3)\big)\big(\mathbf{1} + ty + \frac{t^2}{2}y^2 + O(t^3)\big)\\ &\quad \cdot \big(\mathbf{1} - tx + \frac{t^2}{2}x^2 + O(t^3)\big)\big(\mathbf{1} - ty + \frac{t^2}{2}y^2 + O(t^3)\big)\\ &= \mathbf{1} + t(x + y - x - y) + t^2(x^2 + y^2 + xy - x^2 - xy - yx - y^2 + xy) + O(t^3)\\ &= \mathbf{1} + t^2(xy - yx) + O(t^3).\end{aligned}$$

This implies that $\gamma'(0) = 0$ and $\gamma''(0) = 2(xy - yx)$. Moreover, for $j \leq n^2$ we have for each $\varepsilon > 0$

$$\begin{aligned}\|A^j\| = \|\gamma(\tfrac{1}{n})^j\| \leq \|\gamma(\tfrac{1}{n})\|^j &\leq \big(1 + \tfrac{1}{n^2}\|xy - yx\| + O(n^{-3})\big)^j\\ &\leq \big(1 + \frac{1}{n^2}\|xy - yx\| + O(n^{-3})\big)^{n^2}.\end{aligned}$$

For sufficiently large n, we thus obtain for all $j \leq n^2$:

$$\|A^j\| \leq \big(1 + \frac{1}{n^2}(\|xy - yx\| + 1)\big)^{n^2} \leq e^{\|xy-yx\|+1}.$$

This proves the existence of a constant $C > 0$ (independent of n) with

$$\|A^{n^2} - B^{n^2}\| \leq Cn^2\|A - B\| \quad \text{for all} \quad n \in \mathbb{N}.$$

We further have

$$\begin{aligned}n^2(A - B) &= \frac{\gamma(\frac{1}{n}) - e^{\frac{1}{n^2}(xy-yx)}}{\frac{1}{n^2}} = \frac{\gamma(\frac{1}{n}) - \mathbf{1} + \mathbf{1} - e^{\frac{1}{n^2}(xy-yx)}}{\frac{1}{n^2}}\\ &\to \tfrac{1}{2}\gamma''(0) - (xy - yx) = (xy - yx) - (xy - yx) = 0.\end{aligned}$$

This proves (ii). □

Exercises for Section 7.1

Exercise 7.1.1. Let $X_1, \ldots, X_n$ be Banach spaces and $\beta\colon X_1 \times \ldots \times X_n \to Y$ a continuous n-linear map.

(a) Show that there exists a constant $C \geq 0$ with

$$\|\beta(x_1, \ldots, x_n)\| \leq C\|x_1\| \cdots \|x_n\| \quad \text{for} \quad x_i \in X_i.$$

(b) Show that β is differentiable with

$$\mathtt{d}\beta(x_1, \ldots, x_n)(h_1, \ldots, h_n) = \sum_{j=1}^{n} \beta(x_1, \ldots, x_{j-1}, h_j, x_{j+1}, \ldots, x_n).$$

Exercise 7.1.2. Let Y be a Banach space and $a_{n,m}$, $n,m \in \mathbb{N}$, elements in Y with

$$\sum_{n,m} \|a_{n,m}\| := \sup_{N \in \mathbb{N}} \sum_{n,m \leq N} \|a_{n,m}\| < \infty.$$

(a) Show that

$$A := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{n,m}$$

and that both iterated sums exist.

(b) Show that for each sequence $(S_n)_{n \in \mathbb{N}}$ of finite subsets $S_n \subseteq \mathbb{N} \times \mathbb{N}$, $n \in \mathbb{N}$, with $S_n \subseteq S_{n+1}$ and $\bigcup_n S_n = \mathbb{N} \times \mathbb{N}$ we have

$$A = \lim_{n \to \infty} \sum_{(j,k) \in S_n} a_{j,k}.$$

Exercise 7.1.3. (Cauchy Product Formula) Let X, Y, Z be Banach spaces and $\beta \colon X \times Y \to Z$ a continuous bilinear map. Suppose that if $x := \sum_{n=0}^{\infty} x_n$ is absolutely convergent in X and if $y := \sum_{n=0}^{\infty} y_n$ is absolutely convergent in Y, then

$$\beta(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta(x_k, y_{n-k}).$$

Hint: Use Exercise 7.1.2(b).

Exercise 7.1.4. The function

$$\Phi : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} e^{-\frac{1}{t}}, & \text{for } t > 0 \\ 0, & \text{for } t \leq 0 \end{cases}$$

is smooth. Hint: The higher derivatives of $e^{-\frac{1}{t}}$ are of the form $P(t^{-1})e^{-\frac{1}{t}}$, where P is a polynomial.

(b) For $\lambda > 0$ the function $\Psi(t) := \Phi(t)\Phi(\lambda - t)$ is a non-negative smooth function with $\mathrm{supp}(\Psi) = [0, \lambda]$.

Exercise 7.1.5. (A smoothing procedure) Let $f \in C_c^1(\mathbb{R})$ be a C^1-function with compact support and $\gamma \in C(\mathbb{R}, E)$, where E is a Banach space. Then the convolution

$$h := f * \gamma \colon \mathbb{R} \to E, \quad t \mapsto \int_{\mathbb{R}} f(s)\gamma(t - s)\, ds = \int_{\mathbb{R}} f(t - s)\gamma(s)\, ds$$

of f and γ is continuously differentiable with $h' = f' * \gamma$. Hint:

$$\int_{\mathbb{R}} f(t - s)\gamma(s)\, ds = \int_{t - \mathrm{supp}(f)} f(t - s)\gamma(s)\, ds.$$

Exercise 7.1.6. Show that for $X = -X^* \in M_n(\mathbb{C})$ the matrix e^X is unitary and that the exponential function

$$\exp : \mathrm{Aherm}_n(\mathbb{C}) := \{X \in M_n(\mathbb{C}) \colon X^* = -X\} \to \mathrm{U}_n(\mathbb{C}), \quad X \mapsto e^X$$

is surjective.

Exercise 7.1.7. Show that for $X^\top = -X \in M_n(\mathbb{R})$ the matrix e^X is orthogonal and that the exponential function

$$\exp : \mathrm{Skew}_n(\mathbb{R}) := \{X \in M_n(\mathbb{R}) \colon X^\top = -X\} \to \mathrm{O}_n(\mathbb{R})$$

is not surjective. Can you determine which orthogonal matrices are contained in the image? Can you interprete the result geometrically in terns of the geometry of the flow $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, (t, v) \mapsto e^{tX}v$.

Exercise 7.1.8. Show that for $\mathcal{A} := C(\mathbb{S}^1)$ the exponential function

$$\exp : \mathrm{Aherm}(\mathcal{A}) := \{a \in \mathcal{A} \colon a^* = -a\} \to \mathrm{U}(\mathcal{A}) = C(\mathbb{S}^1, \mathbb{T}), \quad a \mapsto e^a$$

is not surjective. It requires some covering theory to determine which elements $f \in C(\mathbb{S}^1, \mathbb{T})$ lie in its image. Hint: Use the winding number with respect to 0.

Exercise 7.1.9. Show that for any measure space $(X, \mathfrak{S})$ and the C^*-algebra $\mathcal{A} := L^\infty(X, \mathfrak{S})$, the exponential function

$$\exp : \mathrm{Aherm}(\mathcal{A}) \to \mathrm{U}(\mathcal{A}), \quad a \mapsto e^a$$

is surjective.

Exercise 7.1.10. Show that for every von Neumann algebra $\mathcal{A}$, the exponential function

$$\exp : \mathrm{Aherm}(\mathcal{A}) \to \mathrm{U}(\mathcal{A}), \quad a \mapsto e^a$$

is surjective. This applies in particular to $\mathcal{A} = B(\mathcal{H})$, so that for every complex Hilbert space $\mathcal{H}$, the exponential function $\exp : \mathrm{Aherm}(\mathcal{H}) \to \mathrm{U}(\mathcal{H}), a \mapsto e^a$ is surjective.

Exercise 7.1.11. (a) Calculate e^{tN} for $t \in \mathbb{K}$ and the matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \cdot & 0 & 1 & 0 & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 1 \\ 0 & & \dots & & 0 \end{pmatrix} \in M_n(\mathbb{K}).$$

(b) If A is a block diagonal matrix $\mathrm{diag}(A_1, \dots, A_k)$, then e^A is the block diagonal matrix $\mathrm{diag}(e^{A_1}, \dots, e^{A_k})$.

(c) Calculate e^{tA} for a matrix $A \in M_n(\mathbb{C})$ given in Jordan normal form. Hint: Use (a) and (b).

Exercise 7.1.12. Let $a, b \in M_n(\mathbb{C})$ be commuting elements.

(a) If a and b are nilpotent, then $a + b$ is nilpotent.

(b) If a and b are diagonalizable, then $a + b$ and ab are diagonalizable.

(c) If a and b are unipotent, then ab is unipotent.

Exercise 7.1.13. For $A \in M_n(\mathbb{C})$ we have $e^A = \mathbf{1}$ if and only if A is diagonalizable with all eigenvalues contained in $2\pi i\mathbb{Z}$. Hint: Exercise 2.2.10.

7.2 Closed Subgroups of Banach Algebras

We call a closed subgroup $G \subseteq \mathcal{A}^\times$ of a unital Banach algebra $\mathcal{A}$ a *linear group*. In this section we shall use the exponential function to assign to each linear group G a vector space

$$\mathbf{L}(G) := \{x \in \mathcal{A} \colon \exp(\mathbb{R}x) \subseteq G\},$$

called the *Lie algebra of* G. This subspace carries an additional algebraic structure because for $x, y \in \mathbf{L}(G)$ the commutator $[x, y] = xy - yx$ is contained in $\mathbf{L}(G)$, so that $[\cdot, \cdot]$ defines a skew-symmetric bilinear operation on $\mathbf{L}(G)$. As a first step, we shall see how to calculate $\mathbf{L}(G)$ for concrete groups.

7.2.1 The Lie Algebra of a Linear Group

We start with the introduction of the concept of a Lie algebra.

Definition 7.2.1. (a) Let $\mathbb{K}$ be a field and L a $\mathbb{K}$-vector space. A bilinear map $[\cdot, \cdot] \colon L \times L \to L$ is called a *Lie bracket* if

(L1) $[x, x] = 0$ for $x \in L$ and

(L2) $\big[x, [y, z]\big] = \big[[x, y], z\big] + \big[y, [x, z]\big]$ for $x, y, z \in L$ (Jacobi identity).[1]

A *Lie algebra*[2] (over $\mathbb{K}$) is a $\mathbb{K}$-vector space L endowed with a Lie bracket. A subspace $E \subseteq L$ of a Lie algebra is called a *subalgebra* if $[E, E] \subseteq E$. A *homomorphism* $\varphi \colon L_1 \to L_2$ of Lie algebras is a linear map with $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for $x, y \in L_1$. A Lie algebra is said to be *abelian* if $[x, y] = 0$ holds for all $x, y \in L$.

A *Banach–Lie algebra* is a Banach space L, endowed with a Lie algebra structure for which the bracket $[\cdot, \cdot]$ is continuous, i.e., there exists a $C > 0$ with

$$\|[x, y]\| \leq C\|x\| \cdot \|y\| \quad \text{for} \quad x, y \in L.$$

The following lemma shows that each associative algebra also carries a natural Lie algebra structure.

[1] Carl Gustav Jacob Jacobi (1804–1851), mathematician in Berlin and Königsberg (Kaliningrad). He found his famous identity about 1830 in the context of Poisson brackets, which are related to Hamiltonian Mechanics and Symplectic Geometry.

[2] The notion of a Lie algebra was coined in the 1920s by Hermann Weyl.

Lemma 7.2.2. *Each associative algebra $\mathcal{A}$ is a Lie algebra $\mathcal{A}_L$ with respect to the commutator bracket*

$$[a,b] := ab - ba.$$

If $\mathcal{A}$ is Banach algebra, then $\mathcal{A}_L$ is a Banach–Lie algebra.

Proof. (L1) is obvious. For (L2) we calculate

$$[a,bc] = abc - bca = (ab - ba)c + b(ac - ca) = [a,b]c + b[a,c],$$

and this implies

$$[a,[b,c]] = [a,b]c + b[a,c] - [a,c]b - c[a,b] = [[a,b],c] + [b,[a,c]].$$

If, in addition, $\mathcal{A}$ is a Banach algebra, then the norm on $\mathcal{A}$ is submultiplicative, and this leads to

$$\|[x,y]\| = \|xy - yx\| \leq \|x\|\|y\| + \|y\|\|x\| = 2\|x\|\|y\|.$$

□

Definition 7.2.3. A closed subgroup $G \subseteq \mathcal{A}$ is called a *linear group*. For each subgroup $G \subseteq \mathcal{A}$ we define the set

$$\mathbf{L}(G) := \{x \in \mathcal{A} \colon \exp(\mathbb{R}x) \subseteq G\}$$

and observe that $\mathbb{R}\,\mathbf{L}(G) \subseteq \mathbf{L}(G)$ follows immediately from the definition.

The next proposition assigns a Lie algebra to each linear group.

Proposition 7.2.4. *If $G \subseteq \mathcal{A}^\times$ is a closed subgroup, then $\mathbf{L}(G)$ is a closed real Lie subalgebra of $\mathcal{A}_L$ and we obtain a map*

$$\exp_G \colon \mathbf{L}(G) \to G, \quad x \mapsto e^x.$$

We call $\mathbf{L}(G)$ the *Lie algebra of G* and the map $\exp_G$ the *exponential function of G*. In particular we have

$$\mathbf{L}(\mathcal{A}^\times) = \mathcal{A}_L.$$

Proof. Let $x, y \in \mathbf{L}(G)$. For $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we have $\exp \frac{t}{k}x, \exp \frac{t}{k}y \in G$ and with the Trotter Formula (Proposition 7.1.5), we get for all $t \in \mathbb{R}$:

$$\exp(t(x+y)) = \lim_{k\to\infty} \left(\exp\frac{tx}{k}\exp\frac{ty}{k}\right)^k \in G$$

because G is closed. Therefore $x + y \in \mathbf{L}(G)$.

Similarly we use the Commutator Formula to get

$$\exp t[x,y] = \lim_{k\to\infty} \left(\exp\frac{tx}{k}\exp\frac{y}{k}\exp-\frac{tx}{k}\exp-\frac{y}{k}\right)^{k^2} \in G,$$

hence $[x,y] \in \mathbf{L}(G)$. □

Remark 7.2.5. If G is an abelian subgroup of $\mathcal{A}^\times$, then $\mathbf{L}(G)$ is also abelian.

Lemma 7.2.6. *Let $G \subseteq \mathcal{A}^\times$ be a subgroup. If* $\mathrm{Hom}(\mathbb{R}, G)$*, denotes the set of all continuous group homomorphisms* $(\mathbb{R}, +) \to G$*, then the map*

$$\Gamma\colon \mathbf{L}(G) \to \mathrm{Hom}(\mathbb{R}, G), \quad x \mapsto \gamma_x, \quad \gamma_x(t) = \exp(tx)$$

is a bijection.

Proof. For each $x \in \mathbf{L}(G)$, the map γ_x is a continuous group homomorphism (Theorem 7.1.4), and since $x = \gamma_x'(0)$, the map Γ is injective. To see that it is surjective, let $\gamma\colon \mathbb{R} \to G$ be a continuous group homomorphism and $\iota\colon G \to \mathcal{A}^\times$ the natural embedding. Then $\iota \circ \gamma\colon \mathbb{R} \to \mathcal{A}^\times$ is a continuous group homomorphism, so that there exists an $x \in \mathcal{A}$ with $\gamma(t) = \iota(\gamma(t)) = e^{tx}$ for all $t \in \mathbb{R}$ (Theorem 7.1.4). This implies that $x \in \mathbf{L}(G)$, and therefore that $\gamma_x = \gamma$. □

Remark 7.2.7. The preceding lemma implies in particular that for a linear group, the set $\mathbf{L}(G)$ can also be defined in terms of the topological group structure on G as $\mathcal{L}(G) := \mathrm{Hom}(\mathbb{R}, G)$, the set of continuous one-parameter groups. From the Trotter Formula and the Commutator Formula we also know that the Lie algebra structure on $\mathcal{L}(G)$ can be defined intrinsically by

$$(\lambda\gamma)(t) := \gamma(\lambda t),$$

$$(\gamma_1 + \gamma_2)(t) := \lim_{n\to\infty} \Big(\gamma_1(\tfrac{t}{n})\gamma_2(\tfrac{t}{n})\Big)^{\frac{1}{n}}$$

and

$$[\gamma_1, \gamma_2](t) := \lim_{n\to\infty} \Big(\gamma_1(\tfrac{t}{n})\gamma_2(\tfrac{1}{n})\gamma_1(-\tfrac{t}{n})\gamma_2(-\tfrac{1}{n})\Big)^{\frac{1}{n^2}}.$$

This shows that the Lie algebra $\mathbf{L}(G)$ does not depend on the special realization of G as a group of matrices.

Examples 7.2.8. Let X be a Banach space.

(a) Then $B(X)$ is a unital Banach algebra. We write $\mathrm{GL}(X) := B(X)^\times$ for its unit group and $\mathfrak{gl}(X) := (B(X), [\cdot,\cdot])$ of its Lie algebra.

(b) Let $\widetilde{X} := X \times \mathbb{R}$. We consider the homomorphism

$$\Phi\colon X \to \mathrm{GL}(\widetilde{X}), \quad x \mapsto \begin{pmatrix} \mathbf{1} & x \\ 0 & 1 \end{pmatrix}$$

and observe that Φ is an isomorphism of the topological group $(X, +)$ onto a linear group.

The continuous one-parameter groups $\gamma\colon \mathbb{R} \to X$ are easily determined because $\gamma(nt) = n\gamma(t)$ for all $n \in \mathbb{Z}$, $t \in \mathbb{R}$, implies further $\gamma(q) = q\gamma(1)$ for all $q \in \mathbb{Q}$ and hence, by continuity, $\gamma(t) = t\gamma(1)$ for all $t \in \mathbb{R}$. Since $(X, +)$ is abelian, the Lie bracket on the Lie algebra $\mathbf{L}(X, +)$ vanishes, and we obtain

$$\mathbf{L}(X, +) = (X, 0) \cong \mathbf{L}(\Phi(X)) = \Big\{ \begin{pmatrix} \mathbf{0} & x \\ 0 & 0 \end{pmatrix} : x \in X \Big\}$$

(Exercise).

Definition 7.2.9. A *linear Lie group* is a closed subgroup G of the unit group $\mathcal{A}^\times$ of a unital Banach algebra $\mathcal{A}$ for which the exponential function

$$\exp_G\colon \mathbf{L}(G) \to G$$

is a local homeomorphism in 0, i.e., it maps some open 0-neighborhood U in $\mathbf{L}(G)$ homeomorphically onto an open $\mathbf{1}$-neighborhood in G.

7.2.2 Functorial Properties of the Lie Algebra

So far we have assigned to each linear group G its Lie algebra $\mathbf{L}(G)$. We shall also see that this assignment can be "extended" to continuous homomorphisms between linear groups in the sense that we assign to each such homomorphism $\varphi\colon G_1 \to G_2$ a homomorphism $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ of Lie algebras, and this assignment satisfies

$$\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)} \quad \text{and} \quad \mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1)$$

for a composition $\varphi_1 \circ \varphi_2$ of two continuous homomorphisms $\varphi_1\colon G_2 \to G_1$ and $\varphi_2\colon G_3 \to G_2$. In the language of category theory, this means that $\mathbf{L}$ defines a functor from the category of linear groups (where the morphisms are the continuous group homomorphisms) to the category of real Banach–Lie algebras.

Proposition 7.2.10. *Let $\varphi\colon G_1 \to G_2$ be a continuous group homomorphism of linear groups. Then the derivative*

$$\mathbf{L}(\varphi)(x) := \frac{d}{dt}\Big|_{t=0} \varphi(\exp_{G_1}(tx))$$

exists for each $x \in \mathbf{L}(G_1)$ and defines a homomorphism of Lie algebras $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ with

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1}, \tag{7.3}$$

i.e., the following diagram commutes

$$\begin{tikzcd} G_1 \arrow[r, "\varphi"] & G_2 \\ \mathbf{L}(G_1) \arrow[u, "\exp_{G_1}"] \arrow[r, "\mathbf{L}(\varphi)"] & \mathbf{L}(G_2). \arrow[u, "\exp_{G_2}"'] \end{tikzcd}$$

Then $\mathbf{L}(\varphi)$ is the uniquely determined linear map satisfying (7.3)*.*

If, in addition, H is a linear Lie group, then $\mathbf{L}(\varphi)$ is continuous.

Proof. For $x \in \mathbf{L}(G_1)$ we consider the homomorphism $\gamma_x \in \mathrm{Hom}(\mathbb{R}, G_1)$ given by $\gamma_x(t) = e^{tx}$. According to Lemma 7.2.6, we have

$$\varphi \circ \gamma_x(t) = \exp_{G_2}(ty)$$

for some $y \in \mathbf{L}(G_2)$, because $\varphi \circ \gamma_x : \mathbb{R} \to G_2$ is a continuous group homomorphism. Then clearly $y = (\varphi \circ \gamma_x)'(0) = \mathbf{L}(\varphi)x$. For $t = 1$ we obtain in particular

$$\exp_{G_2}(\mathbf{L}(\varphi)x) = \varphi(\exp_{G_1}(x)),$$

which is (7.3).

Conversely, every linear map $\psi \colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ with

$$\exp_{G_2} \circ \psi = \varphi \circ \exp_{G_1}$$

satisfies

$$\varphi \circ \exp_{G_1}(tx) = \exp_{G_2}(\psi(tx)) = \exp_{G_2}(t\psi(x)),$$

and therefore

$$\mathbf{L}(\varphi)x = \frac{d}{dt}\Big|_{t=0} \exp_{G_2}(t\psi(x)) = \psi(x).$$

Next we show that $\mathbf{L}(\varphi)$ is a homomorphism of Lie algebras. From the definition of $\mathbf{L}(\varphi)$ we immediately get for $x \in \mathbf{L}(G_1)$:

$$\exp_{G_2}(s\,\mathbf{L}(\varphi)(tx)) = \varphi(\exp_{G_1}(stx)) = \exp_{G_2}(ts\,\mathbf{L}(\varphi)(x)), \quad s, t \in \mathbb{R},$$

which leads to $\mathbf{L}(\varphi)(tx) = t\,\mathbf{L}(\varphi)(x)$.

Since φ is continuous, the Trotter Formula implies that

$$\begin{aligned}
&\exp_{G_2}(\mathbf{L}(\varphi)(x+y)) = \varphi\big(\exp_{G_1}(x+y)\big) \\
&= \lim_{k\to\infty} \varphi\Big(\exp_{G_1} \frac{1}{k}x \exp_{G_1} \frac{1}{k}y\Big)^k = \lim_{k\to\infty} \Big(\varphi\Big(\exp_{G_1} \frac{1}{k}x\Big)\varphi\Big(\exp_{G_1} \frac{1}{k}y\Big)\Big)^k \\
&= \lim_{k\to\infty} \Big(\exp_{G_2} \frac{1}{k}\mathbf{L}(\varphi)(x) \exp_{G_2} \frac{1}{k}\mathbf{L}(\varphi)(y)\Big)^k \\
&= \exp_{G_2}\big(\mathbf{L}(\varphi)(x) + \mathbf{L}(\varphi)(y)\big)
\end{aligned}$$

for all $x, y \in \mathbf{L}(G_1)$. Therefore $\mathbf{L}(\varphi)(x+y) = \mathbf{L}(\varphi)(x) + \mathbf{L}(\varphi)(y)$ because the same formula holds with tx and ty instead of x and y. Hence $\mathbf{L}(\varphi)$ is additive and therefore linear.

We likewise obtain with the Commutator Formula

$$\varphi(\exp[x,y]) = \exp[\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$$

and thus $\mathbf{L}(\varphi)([x,y]) = [\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$.

If, in addition, H is a linear Lie group, then $\exp_H$ is a local homeomorphism in 0, so that the relation $\varphi \circ \exp_G = \exp_H \circ \mathbf{L}(\varphi)$ implies that $\mathbf{L}(\varphi)$ is continuous on some 0-neighborhood, and since it is a linear map, it is continuous. □

Note that we did not show at this point that $\mathbf{L}(\varphi)$ is continuous. To verify this claim, we have to restrict to the class of those groups for which the exponential function is a local homeomorphism around 0, and this is precisely the class of linear Banach–Lie groups.

Corollary 7.2.11. *If $\varphi_1 : G_1 \to G_2$ and $\varphi_2 \colon G_2 \to G_3$ are continuous homomorphisms of linear Lie groups, then*

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1).$$

Moreover, $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$ *.*

Proof. We have the relations

$$\varphi_1 \circ \exp_{G_1} = \exp_{G_2} \circ \mathbf{L}(\varphi_1) \quad \text{and} \quad \varphi_2 \circ \exp_{G_2} = \exp_{G_3} \circ \mathbf{L}(\varphi_2),$$

which immediately lead to

$$(\varphi_2 \circ \varphi_1) \circ \exp_{G_1} = \varphi_2 \circ \exp_{G_2} \circ \mathbf{L}(\varphi_1) = \exp_{G_3} \circ (\mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1)),$$

and the uniqueness assertion of Proposition 7.2.10 implies that

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1).$$

Clearly $\mathrm{id}_{\mathbf{L}(G)}$ is a linear map satisfying $\exp_G \circ \mathrm{id}_{\mathbf{L}(G)} = \mathrm{id}_G \circ \exp_G$, so that the uniqueness assertion of Proposition 7.2.10 implies $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$. □

Corollary 7.2.12. *If $\varphi \colon G_1 \to G_2$ is an isomorphism of linear Lie groups, then $\mathbf{L}(\varphi)$ is an isomorphism of Lie algebras.*

Proof. Since φ is an isomorphism of linear Lie groups, it is bijective and $\psi := \varphi^{-1}$ also is a continuous homomorphism. We then obtain with Corollary 7.2.11 the relations $\mathrm{id}_{\mathbf{L}(G_2)} = \mathbf{L}(\mathrm{id}_{G_2}) = \mathbf{L}(\varphi \circ \psi) = \mathbf{L}(\varphi) \circ \mathbf{L}(\psi)$ and likewise

$$\mathrm{id}_{\mathbf{L}(G_1)} = \mathbf{L}(\psi) \circ \mathbf{L}(\varphi).$$

Hence $\mathbf{L}(\varphi)$ is an isomorphism with $\mathbf{L}(\varphi)^{-1} = \mathbf{L}(\psi)$. □

Definition 7.2.13. If V is a vector space and G a group, then a homomorphism $\varphi \colon G \to \mathrm{GL}(V)$ is called a *representation of G on V*. If $\mathfrak{g}$ is a Lie algebra, then a homomorphism of Lie algebras $\varphi \colon \mathfrak{g} \to \mathfrak{gl}(V)$ is called a *representation of $\mathfrak{g}$ on V*.

As a consequence of Proposition 7.2.10, we obtain

Corollary 7.2.14. *If $\varphi \colon G \to \mathrm{GL}(V)$ is a continuous representation of the linear group G on the Banach space V, then $\mathbf{L}(\varphi) \colon \mathbf{L}(G) \to \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathbf{L}(G)$.*

Definition 7.2.15. The representation $\mathbf{L}(\varphi)$ obtained in Corollary 7.2.14 from the group representation φ is called the *derived representation.* This is motivated by the fact that for each $x \in \mathbf{L}(G)$ we have

$$\mathbf{L}(\varphi)x = \frac{d}{dt}\Big|_{t=0} e^{t\,\mathbf{L}(\varphi)x} = \frac{d}{dt}\Big|_{t=0} \varphi(\exp tx).$$

7.2.3 The Adjoint Representation

Let $G \subseteq \mathrm{GL}(V)$ be a linear Lie group and $\mathbf{L}(G) \subseteq \mathfrak{gl}(V)$ the corresponding Lie algebra. For $g \in G$ we define the conjugation automorphism $c_g \in \mathrm{Aut}(G)$ by $c_g(x) := gxg^{-1}$. Then

$$\begin{aligned}\mathbf{L}(c_g)(x) &= \frac{d}{dt}\Big|_{t=0} c_g(\exp tx) = \frac{d}{dt}\Big|_{t=0} g(\exp tx)g^{-1} \\ &= \frac{d}{dt}\Big|_{t=0} \exp(tgxg^{-1}) = gxg^{-1}\end{aligned}$$

(Lemma 7.1.1), and therefore $\mathbf{L}(c_g) = c_g|_{\mathbf{L}(G)}$. We define the *adjoint representation of G on* $\mathbf{L}(G)$ by

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{L}(G)), \quad \mathrm{Ad}(g)(x) := \mathbf{L}(c_g)x = gxg^{-1}.$$

(That this is a representation follows immediately from the explicit formula).

For each $x \in \mathbf{L}(G)$, the map $G \to \mathbf{L}(G), g \mapsto \mathrm{Ad}(g)(x) = gxg^{-1}$ is continuous and each $\mathrm{Ad}(g)$ is an automorphism of the Lie algebra $\mathbf{L}(G)$. Therefore Ad is a continuous homomorphism from the linear group G to the linear group $\mathrm{Aut}(\mathbf{L}(G)) \subseteq \mathrm{GL}(\mathbf{L}(G))$. The derived representation

$$\mathbf{L}(\mathrm{Ad})\colon \mathbf{L}(G) \to \mathfrak{gl}(\mathbf{L}(G))$$

is a representation of $\mathbf{L}(G)$ on $\mathbf{L}(G)$. The following lemma gives a formula for this representation. First we define for $x \in \mathbf{L}(G)$:

$$\mathrm{ad}(x)\colon \mathbf{L}(G) \to \mathbf{L}(G), \quad \mathrm{ad}\, x(y) := [x, y] = xy - yx.$$

Lemma 7.2.16. *For each $x \in \mathcal{A}$ we have*

$$\mathrm{Ad}(\exp x) = \exp(\mathrm{ad}\, x). \tag{7.4}$$

Proof. We define the linear maps

$$L_x\colon \mathcal{A} \to \mathcal{A}, \quad y \mapsto xy, \quad R_x\colon \mathcal{A} \to \mathcal{A}, \quad y \mapsto yx.$$

Then $L_xR_x = R_xL_x$ and $\mathrm{ad}\, x = L_x - R_x$, so that Lemma 7.1.1(ii) leads to

$$\mathrm{Ad}(\exp x)y = e^x y e^{-x} = e^{L_x} e^{-R_x} y = e^{L_x - R_x} y = e^{\mathrm{ad}\, x} y.$$

This proves (7.4). □

Lemma 7.2.17. $\mathbf{L}(\mathrm{Ad}) = \mathrm{ad}$.

Proof. In view of Proposition 7.2.10, this is an immediate consequence of the relation $\mathrm{Ad}(\exp x) = e^{\mathrm{ad}\, x}$ (Lemma 7.2.16). □

Exercises for Section 7.2

Exercise 7.2.1. (a) If $(G_j)_{j\in J}$ is a family of linear groups in $\mathcal{A}^\times$, then their intersection $G := \bigcap_{j\in J} G_j$ also is a linear group.

(b) If $(G_j)_{j\in J}$ is a family of subgroups of $\mathcal{A}^\times$, then

$$\mathbf{L}\Big(\bigcap_{j\in J} G_j\Big) = \bigcap_{j\in J} \mathbf{L}(G_j).$$

Exercise 7.2.2. Let $G := \mathrm{GL}_n(\mathbb{K})$ and $V := P_k(\mathbb{K}^n)$ be the space of homogeneous polynomials of degree k in $x_1, \ldots, x_n$, considered as functions $\mathbb{K}^n \to \mathbb{K}$. Show that:

(1) $\dim V = \binom{k+n-1}{n-1}$.

(2) We obtain a continuous representation $\rho\colon G \to \mathrm{GL}(V)$ of G on V by $(\rho(g)f)(x) := f(g^{-1}x)$.

(3) For the elementary matrix $E_{ij} = (\delta_{ij})$ we have $\mathbf{L}(\rho)(E_{ij}) = -x_j \frac{\partial}{\partial x_i}$. Hint: $(\mathbf{1} + tE_{ij})^{-1} = \mathbf{1} - tE_{ij}$.

Exercise 7.2.3. If $X \in \mathrm{End}(V)$ is nilpotent, then $\mathrm{ad}\, X \in \mathrm{End}(\mathrm{End}(V))$ is also nilpotent. Hint: $\mathrm{ad}\, X = L_X - R_X$ and both summands commute.

Exercise 7.2.4. If $(V, \cdot)$ is an associative algebra, then we have $\mathrm{Aut}(V, \cdot) \subseteq \mathrm{Aut}(V, [\cdot, \cdot])$.

Exercise 7.2.5. (a) For each linear group G, $\mathrm{Ad} : G \to \mathrm{Aut}(\mathbf{L}(G))$ is a group homomorphism.

(b) For each Lie algebra $\mathfrak{g}$, the operators $\mathrm{ad}\, x(y) := [x, y]$ are derivations and the map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism of Lie algebras.

Exercise 7.2.6. Let V and W be vector spaces and $q : V \times V \to W$ a skew-symmetric bilinear map. Then

$$[(v, w), (v', w')] := \big(0, q(v, v')\big)$$

is a Lie bracket on $\mathfrak{g} := V \times W$. For $x, y, z \in \mathfrak{g}$ we have $\big[x, [y, z]\big] = 0$.

Exercise 7.2.7. Let $\mathfrak{g}$ be a Lie algebra with $\big[x, [y, z]\big] = 0$ for $x, y, z \in \mathfrak{g}$. Then

$$x * y := x + y + \frac{1}{2}[x, y]$$

defines a group structure on $\mathfrak{g}$. An example for such a Lie algebra is the three-dimensional *Heisenberg algebra*

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{K} \right\}.$$

7.3 Calculating Lie Algebras of Linear Groups

In this section we shall see various techniques to determine the Lie algebra of a linear Lie group.

Example 7.3.1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then the group $G := \mathrm{SL}_n(\mathbb{K}) = \det^{-1}(1) = \ker\det$ is a linear group. To determine its Lie algebra, we first claim that

$$\det(e^x) = e^{\mathrm{Tr}\, x} \tag{7.5}$$

holds for $x \in M_n(\mathbb{K})$. To verify this claim, we consider

$$\det\colon M_n(\mathbb{K}) \cong (\mathbb{K}^n)^n \to \mathbb{K}$$

as a multilinear map, where each matrix x is considered as an n-tuple of its column vectors $x_1, \ldots, x_n$. Then Exercise 7.1.1 implies that

$$\begin{aligned}&(\mathtt{d}\det)(\mathbf{1})(x) = (\mathtt{d}\det)(e_1, \ldots, e_n)(x_1, \ldots, x_n)\\ &= \det(x_1, e_2, \ldots, e_n) + \ldots + \det(e_1, \ldots, e_{n-1}, x_n) = x_{11} + \ldots + x_{nn} = \mathrm{Tr}\, x.\end{aligned}$$

Now we consider the curve $\gamma\colon \mathbb{R} \to \mathbb{K}^\times \cong \mathrm{GL}_1(\mathbb{K}), t \mapsto \det(e^{tx})$. Then γ is a continuous group homomorphism, hence of the form $\gamma(t) = e^{at}$ for $a = \gamma'(0)$ (Theorem 7.1.4). On the other hand the Chain Rule implies

$$a = \gamma'(0) = \mathtt{d}\det(\mathbf{1})\big(\mathtt{d}\exp(\mathbf{0})(x)\big) = \mathrm{Tr}(x),$$

and this implies (7.5). We conclude that

$$\begin{aligned}\mathfrak{sl}_n(\mathbb{K}) := \mathbf{L}(\mathrm{SL}_n(\mathbb{K})) &= \{x \in M_n(\mathbb{K})\colon (\forall t \in \mathbb{R})\, 1 = \det(e^{tx}) = e^{t\,\mathrm{Tr}\, x}\}\\ &= \{x \in M_n(\mathbb{K})\colon\ \mathrm{Tr}\, x = 0\}.\end{aligned}$$

Lemma 7.3.2. *Let V and W be Banach spaces and $\beta\colon V \times V \to W$ a continuous bilinear map. For $(x, y) \in \mathfrak{gl}(V) \times \mathfrak{gl}(W)$, the following are equivalent:*

(a) $e^{ty}\beta(v, v') = \beta(e^{tx}v, e^{tx}v')$ *for all* $t \in \mathbb{R}$ *and all* $v, v' \in V$.

(b) $y\beta(v, v') = \beta(xv, v') + \beta(v, xv')$ *for all* $v, v' \in V$.

Proof. (a) $\Rightarrow$ (b): Taking the derivative in $t = 0$, the relation (a) leads to

$$y.\beta(v, v') = \beta(xv, v') + \beta(v, xv'),$$

where we use the Product and the Chain Rule (Exercise 7.1.1).

(b) $\Rightarrow$ (a): If (b) holds, then we obtain inductively

$$y^n\beta(v, v') = \sum_{k=0}^{n} \binom{n}{k} \beta(x^k v, x^{n-k}v').$$

For the exponential series this leads with the general Cauchy Product Formula (Exercise 7.1.3) to

$$\begin{aligned} e^y\beta(v,v') &= \sum_{n=0}^{\infty} \frac{1}{n!} y^n \beta(v,v') = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{k=0}^{n} \binom{n}{k} \beta(x^k v, x^{n-k} v) \Big) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta\left(\frac{1}{k!} x^k v, \frac{1}{(n-k)!} x^{n-k} v' \right) \\ &= \beta\left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k v, \sum_{m=0}^{\infty} \frac{1}{m!} x^m v' \right) = \beta\left(e^x v, e^x v'\right). \end{aligned}$$

Since (b) also holds for the pair (tx, ty) for all $t \in \mathbb{R}$, this completes the proof. □

Proposition 7.3.3. *Let V and W be Banach space and $\beta\colon V \times V \to W$ a continuous bilinear map. For the group*

$$\mathrm{O}(V,\beta) = \{g \in \mathrm{GL}(V) \colon (\forall v, v' \in V)\ \beta(gv, gv') = \beta(v,v')\},$$

we then have

$$\mathfrak{o}(V,\beta) := \mathbf{L}(\mathrm{O}(V,\beta)) = \{x \in \mathfrak{gl}(V) \colon (\forall v, v' \in V)\ \beta(xv, v') + \beta(v, xv') = 0\}.$$

Proof. We only have to observe that $X \in \mathbf{L}(\mathrm{O}(V,\beta))$ is equivalent to the pair $(X, 0)$ satisfying condition (a) in Lemma 7.3.2. □

Example 7.3.4. (a) Let $B \in M_n(\mathbb{K})$, $\beta(v,w) = v^\top B w$, and

$$G := \{g \in \mathrm{GL}_n(\mathbb{K}) \colon g^\top B g = B\} \cong \mathrm{O}(\mathbb{K}^n, \beta).$$

Then Proposition 7.3.3 implies that

$$\begin{aligned} \mathbf{L}(G) &= \{x \in \mathfrak{gl}_n(\mathbb{K}) \colon (\forall v, v' \in V)\ \beta(xv, v') + \beta(v, xv') = 0\} \\ &= \{x \in \mathfrak{gl}_n(\mathbb{K}) \colon (\forall v, v' \in V)\ v^\top x^\top B v' + v^\top B x v' = 0\} \\ &= \{x \in \mathfrak{gl}_n(\mathbb{K}) \colon x^\top B + Bx = 0\}. \end{aligned}$$

In particular, we obtain for the *orthogonal group*

$$\mathrm{O}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K}) \colon g^\top = g^{-1}\}$$

the Lie algebra

$$\mathfrak{o}_n(\mathbb{K}) := \mathbf{L}(\mathrm{O}_n(\mathbb{K})) = \{x \in \mathfrak{gl}_n(\mathbb{K}) \colon x^\top = -x\} =: \mathrm{Skew}_n(\mathbb{K}).$$

Let $q := n - p$ and let $I_{p,q}$ denote the corresponding matrix

$$I_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \in M_{p+q}(\mathbb{R}).$$

Then we obtain for the *indefinite orthogonal group*

$$\mathrm{O}_{p,q}(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R})\colon g^\top I_{p,q} g = I_{p,q}\},$$

the Lie algebra

$$\mathfrak{o}_{p,q}(\mathbb{K}) := \mathbf{L}(\mathrm{O}_{p,q}(\mathbb{K})) = \{x \in \mathfrak{gl}_{p+q}(\mathbb{K})\colon x^\top I_{p,q} + I_{p,q} x = 0\},$$

and for the *symplectic group*

$$\mathrm{Sp}_{2n}(\mathbb{K}) := \{g \in \mathrm{GL}_{2n}(\mathbb{K})\colon g^\top B g = B\}, \quad B = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix},$$

we find

$$\mathfrak{sp}_{2n}(\mathbb{K}) := \mathbf{L}(\mathrm{Sp}_{2n}(\mathbb{K})) := \{x \in \mathfrak{gl}_{2n}(\mathbb{K})\colon x^\top B + Bx = 0\}.$$

(b) Applying Proposition 7.3.3 with $V = \mathbb{C}^n$ and $W = \mathbb{C}$, considered as real vector spaces, we also obtain for a hermitian form

$$\beta\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, \quad (z, w) \mapsto w^* I_{p,q} z:$$

$$\begin{aligned}\mathfrak{u}_{p,q}(\mathbb{C}) &:= \mathbf{L}(\mathrm{U}_{p,q}(\mathbb{C})) \\ &= \{x \in \mathfrak{gl}_n(\mathbb{C})\colon (\forall z, w \in \mathbb{C}^n)\, w^* I_{p,q} x z + w^* x^* I_{p,q} z = 0\} \\ &= \{x \in \mathfrak{gl}_n(\mathbb{C})\colon I_{p,q} x + x^* I_{p,q} = 0\}.\end{aligned}$$

In particular, we get

$$\mathfrak{u}_n(\mathbb{C}) := \mathbf{L}(\mathrm{U}_n(\mathbb{C})) = \{x \in \mathfrak{gl}_n(\mathbb{C})\colon x^* = -x\}.$$

(c) If $\mathcal{H}$ is a complex Hilbert space, then $\mathrm{U}(\mathcal{H})$ is a closed subgroup of $\mathrm{GL}(\mathcal{H})$, and we obtain for its Lie algebra

$$\mathfrak{u}(\mathcal{H}) := \mathbf{L}(\mathrm{U}(\mathcal{H})) = \{x \in \mathfrak{gl}(\mathcal{H})\colon x^* = -x\}.$$

Example 7.3.5. Let $\mathfrak{g}$ be a Banach Lie algebra and

$$\mathrm{Aut}(\mathfrak{g}) := \{g \in \mathrm{GL}(\mathfrak{g})\colon (\forall x, y \in \mathfrak{g})\ g[x, y] = [gx, gy]\}.$$

Then $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$, hence a linear group. To calculate the Lie algebra of G, we use Lemma 7.3.2 with $V = W = \mathfrak{g}$ and $\beta(x, y) = [x, y]$. Then we see that $D \in \mathfrak{aut}(\mathfrak{g}) := \mathbf{L}(\mathrm{Aut}(\mathfrak{g}))$ is equivalent to (D, D) satisfying the conditions in Lemma 7.3.2, and this leads to

$$\mathfrak{aut}(\mathfrak{g}) = \mathbf{L}(\mathrm{Aut}(\mathfrak{g})) = \{D \in \mathfrak{gl}(\mathfrak{g})\colon (\forall x, y \in \mathfrak{g})\, D[x, y] = [Dx, y] + [x, Dy]\}$$

The elements of this Lie algebra are called *derivations of* $\mathfrak{g}$, and $\mathfrak{aut}(\mathfrak{g})$ is also denoted $\mathrm{der}(\mathfrak{g})$. Note that the condition on an endomorphism of $\mathfrak{g}$ to be a derivation resembles the Leibniz Rule (Product Rule).

Remark 7.3.6. If $\mathcal{A}$ is a complex unital Banach algebra, we call a linear group $G \subseteq \mathcal{A}^\times$ a *complex linear group* if $\mathbf{L}(G) \subseteq \mathcal{A}$ is a complex subspace, i.e., $i\,\mathbf{L}(G) \subseteq \mathbf{L}(G)$. Since Proposition 7.2.4 only ensures that $\mathbf{L}(G)$ is a real subspace, this requirement is not automatically satisfied.

If $\mathcal{H}$ is a complex Hilbert space, then the linear group $\mathrm{U}(\mathcal{H}) \subseteq \mathrm{GL}(\mathcal{H})$ is not a complex linear group because

$$i\mathfrak{u}(\mathcal{H}) = \mathrm{Herm}(\mathcal{H}) \not\subseteq \mathfrak{u}(\mathcal{H}).$$

This is due to the fact that the scalar product on $\mathcal{H}$ whose automorphism group is $\mathrm{U}(\mathcal{H})$, is not complex bilinear. For any complex bilinear form $\beta\colon V \times V \to \mathbb{C}$, the corresponding group $\mathrm{O}(V,\beta)$ is a complex linear group because

$$\mathfrak{o}(V,\beta) = \{X \in \mathfrak{gl}(V)\colon (\forall v, w \in V)\, \beta(Xv, w) + \beta(v, Xw) = 0\}$$

is a complex subspace of $\mathfrak{gl}(V)$.

7.4 Smooth Functions Defined by Power Series

Lemma 7.4.1. *If A is a unital Banach algebra, then we endow the vector space $TA := A \oplus A$ with the norm $\|(a,b)\| := \|a\| + \|b\|$ and the multiplication*

$$(a,b)(a',b') := (aa', ab' + a'b).$$

Then TA is a unital Banach algebra with identity $(\mathbf{1}, 0)$.

Writing $\varepsilon := (0, \mathbf{1})$, then each element of TA can be written in a unique fashion as $(a,b) = a + b\varepsilon$ and the multiplication satisfies

$$(a + b\varepsilon)(a' + b'\varepsilon) = aa' + (ab' + a'b)\varepsilon.$$

In particular, $\varepsilon^2 = 0$.

Proof. That TA is a unital algebra is a trivial verification. That the norm is submultiplicative follows from

$$\begin{aligned}\|(a,b)(a',b')\| &= \|aa'\| + \|ab' + a'b\| \leq \|a\| \cdot \|a'\| + \|a\| \cdot \|b'\| + \|a'\| \cdot \|b\| \\ &\leq (\|a\| + \|b\|)(\|a'\| + \|b'\|) = \|(a,b)\| \cdot \|(a',b')\|.\end{aligned}$$

This proves that $(TA, \|\cdot\|)$ is a unital normed algebra, the unit being $\mathbf{1} = (\mathbf{1}, 0)$. The completeness of TA follows easily from the completeness of A (Exercise). □

Lemma 7.4.2. *Let $c_n \in \mathbb{K}$ and $r > 0$ with $\sum_{n=0}^\infty |c_n| r^n < \infty$. Further let A be a unital Banach algebra. Then*

$$f\colon B_r(0) := \{x \in A\colon \|x\| < r\} \to A, \quad x \mapsto \sum_{n=0}^\infty c_n x^n$$

defines a smooth function. Its derivative is given by

$$\mathtt{d}f(x) = \sum_{n=0}^{\infty} c_n \mathtt{d}p_n(x),$$

where $p_n(x) = x^n$ *is the* n^{th} *power map whose derivative is given by*

$$\mathtt{d}p_n(x)y = x^{n-1}y + x^{n-2}yx + \ldots + xyx^{n-2} + yx^{n-1}.$$

For $\|x\| < r$ *and* $y \in A$ *with* $xy = yx$ *we obtain in particular*

$$\mathtt{d}p_n(x)y = nx^{n-1}y \quad \text{and} \quad \mathtt{d}f(x)y = \sum_{n=1}^{\infty} c_n n x^{n-1} y.$$

Proof. First we observe that the series defining $f(x)$ converges for $\|x\| < r$ by the Comparison Test (for series in Banach spaces). We shall prove by induction over $k \in \mathbb{N}$ that all such functions f are C^k-functions.

Step 1: First we show that f is a C^1-function. We define $\alpha_n \colon A \to A$ by

$$\alpha_n(h) := x^{n-1}h + x^{n-2}hx + \ldots + xhx^{n-2} + hx^{n-1}.$$

Then α_n is a continuous linear map with $\|\alpha_n\| \leq n\|x\|^{n-1}$. Furthermore

$$p_n(x+h) = (x+h)^n = x^n + \alpha_n(h) + r_n(h),$$

where

$$\begin{aligned}\|r_n(h)\| &\leq \binom{n}{2}\|h\|^2\|x\|^{n-2} + \binom{n}{3}\|h\|^3\|x\|^{n-3} + \ldots + \|h\|^n \\ &= \sum_{k\geq 2}\binom{n}{k}\|h\|^k\|x\|^{n-k}.\end{aligned}$$

In particular $\lim_{h\to 0} \frac{\|r_n(h)\|}{\|h\|} = 0$, and therefore p_n is differentiable in x with $\mathtt{d}p_n(x) = \alpha_n$. The series

$$\beta(h) := \sum_{n=0}^{\infty} c_n \alpha_n(h)$$

converges absolutely in $\operatorname{End}(A)$ by the Ratio Test since $\|x\| < r$:

$$\sum_{n=0}^{\infty} |c_n|\|\alpha_n\| \leq \sum_{n=0}^{\infty} |c_n| \cdot n \cdot \|x\|^{n-1} < \infty.$$

We thus obtain a linear map $\beta(x) \in \operatorname{End}(A)$ for each x with $\|x\| < r$.

Now let h satisfy $\|x\| + \|h\| < r$, i.e., $\|h\| < r - \|x\|$. Then

$$f(x+h) = f(x) + \beta(x)(h) + r(h), \quad r(h) := \sum_{n=2}^{\infty} c_n r_n(h),$$

where

$$\begin{aligned}\|r(h)\| &\leq \sum_{n=2}^{\infty} |c_n| \|r_n(h)\| \leq \sum_{n=2}^{\infty} |c_n| \sum_{k=2}^{n} \binom{n}{k} \|h\|^k \|x\|^{n-k} \\ &\leq \sum_{k=2}^{\infty} \Big(\sum_{n=k}^{\infty} |c_n| \binom{n}{k} \|x\|^{n-k} \Big) \|h\|^k < \infty\end{aligned}$$

follows from $\|x\| + \|h\| < r$ because

$$\sum_k \sum_{n \geq k} |c_n| \binom{n}{k} \|x\|^{n-k} \|h\|^k = \sum_n |c_n| (\|x\| + \|h\|)^n \leq \sum_n |c_n| r^n < \infty.$$

Therefore the continuity of real-valued functions represented by a power series yields

$$\lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = \sum_{k=2}^{\infty} \Big(\sum_{n=k}^{\infty} |c_n| \binom{n}{k} \|x\|^{n-k} \Big) 0^{k-1} = 0.$$

This proves that f is a C^1-function with the required derivative.

Step 2: To complete our proof by induction, we now show that if all functions f as above are C^k, then they are also C^{k+1}. In view of Step 1, this implies that they are smooth.

To set up the induction, we consider the Banach algebra TA from Lemma 7.4.1 and apply Step 1 to this algebra to obtain a smooth function

$$\begin{aligned} F &: \{x + \varepsilon h \in TA : \|x\| + \|h\| = \|x + \varepsilon h\| < r\} \to TA \\ F(x + \varepsilon h) &= \sum_{n=0}^{\infty} c_n \cdot (x + \varepsilon h)^n, \end{aligned}$$

We further note that $(x + \varepsilon h)^n = x^n + \mathtt{d}p_n(x)h \cdot \varepsilon$. This implies the formula

$$F(x + \varepsilon h) = f(x) + \varepsilon \mathtt{d}f(x)h,$$

i.e., that the extension F of f to TA describes the first order Taylor expansion of f in each point $x \in A$. Our induction hypothesis implies that F is a C^k-function.

Let $x_0 \in A$ with $\|x_0\| < r$ and pick a basis $h_1, \ldots, h_d$ of A with $\|h_i\| < r - \|x_0\|$. Then all functions $x \mapsto \mathtt{d}f(x)h_i$ are defined and C^k on a neighborhood of x_0, and this implies that the function

$$B_r(0) \to \mathrm{Hom}(A, A), \quad x \mapsto \mathtt{d}f(x)$$

is C^k. This in turn implies that f is C^{k+1}. □

The following proposition shows in particular that inserting elements of a Banach algebra in power series is compatible with composition.

Proposition 7.4.3. (a) *On the set P_R of power series of the form*

$$f(z) := \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{K}$$

and converging on the open disc $B_R(0) := \{z \in \mathbb{K}\colon |z| < R\}$, we define for $r < R$:

$$\|f\|_r := \sum_{n=0}^{\infty} |a_n| r^n.$$

Then $\|\cdot\|_r$ is a norm with the following properties:

(1) *$\|\cdot\|_r$ is submultiplicative: $\|fg\|_r \leq \|f\|_r \|g\|_r$.*

(2) *The polynomials $f_N(z) := \sum_{n=0}^{N} a_n z^n$ satisfy $\|f - f_N\|_r \to 0$.*

(3) *If A is a finite dimensional Banach algebra and $x \in A$ satisfies $\|x\| < R$, then $f(x) := \sum_{n=0}^{\infty} a_n x^n$ converges. We further have*

$$\|f(x)\| \leq \|f\|_r \quad \text{for} \quad \|x\| \leq r < R$$

and for $f, g \in P_R$ we have

$$(f \cdot g)(x) = f(x)g(x).$$

(b) *If $g \in P_S$ with $\|g\|_s < R$ for all $s < S$ and $f \in P_R$, then $f \circ g \in P_S$ defines an analytic function on the open disc of radius S, and for $x \in A$ with $\|x\| < S$ we have $\|g(x)\| < R$ and the Composition Formula*

$$f(g(x)) = (f \circ g)(x). \tag{7.6}$$

Proof. (1) First we note that P_R is the set of all power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $\|f\|_r < \infty$ holds for all $r < R$. We leave the easy argument that $\|\cdot\|_r$ is a norm to the reader. If $\|f\|_r, \|g\|_r < \infty$ holds for $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then the Cauchy Product Formula (Exercise 7.1.3) implies that

$$\|fg\|_r = \sum_{n=0}^{\infty} \Big| \sum_{k=0}^{n} a_k b_{n-k} \Big| r^n \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k|\, |b_{n-k}| r^k r^{n-k} = \|f\|_r \|g\|_r.$$

(2) follows immediately from $\|f - f_N\|_r = \sum_{n>N} |a_n| r^n \to 0$.

(3) The relation $\|f(x)\| \leq \|f\|_r$ follows from $\|a_n x^n\| \leq |a_n| r^n$ and the Domination Test for absolutely converging series in a Banach space. The relation $(f \cdot g)(x) = f(x)g(x)$ follows directly from the Cauchy Product Formula because the series $f(x)$ and $g(x)$ converge absolutely (Exercise 7.1.3).

(b) We may w.l.o.g. assume that $\mathbb{K} = \mathbb{C}$ because everything on the case $\mathbb{K} = \mathbb{R}$ can be obtained by restriction. Our assumption implies that $g(B_S(0)) \subseteq$

$B_R(0)$, so that $f \circ g$ defines a holomorphic function on the open disc $B_S(0)$. For $s < S$ and $\|g\|_s < r < R$ we then derive

$$\sum_{n=0}^{\infty} \|a_n g^n\|_s \leq \sum_{n=0}^{\infty} |a_n| \|g\|_s^n \leq \|f\|_r.$$

Therefore the series $f \circ g = \sum_{n=0}^{\infty} a_n g^n$ converges absolutely in P_S with respect to $\|\cdot\|_s$, and we thus obtain the estimate

$$\|f \circ g\|_s = \lim_{N \to \infty} \|\sum_{n=0}^{N} a_n g^n\|_s \leq \sum_{n=0}^{\infty} |a_n| \|g\|_s^n \leq \|f\|_r.$$

For $s := \|x\|$ we obtain $\|g(x)\| \leq \|g\|_s < R$, so that $f(g(x))$ is defined. For $s < r < R$ we then have

$$\|f(g(x)) - f_N(g(x))\| \leq \|f - f_N\|_r \to 0.$$

Likewise

$$\|(f \circ g)(x) - (f_N \circ g)(x)\| \leq \|(f \circ g) - (f_N \circ g)\|_s \leq \|f - f_N\|_r \to 0,$$

and we get

$$(f \circ g)(x) = \lim_{N \to \infty} (f_N \circ g)(x) = \lim_{N \to \infty} f_N(g(x)) = f(g(x))$$

because the Composition Formula trivially holds if f is a polynomial. $\square$

7.5 The Logarithm Function

After the preparations of the preceding section, it is now easy to see that the matrix exponential function of a unital Banach algebra defines a smooth map on $\mathcal{A}$. In this section we describe some elementary properties of this function.

Proposition 7.5.1. *Let $\mathcal{A}$ be a unital Banach algebra. Then the exponential function* $\exp\colon \mathcal{A} \to \mathcal{A}$ *is smooth. For $xy = yx$ we have*

$$\mathtt{d}\exp(x)y = \exp(x)y = y\exp(x). \tag{7.7}$$

Proof. To verify the formula for the differential, we note that for $xy = yx$, Lemma 7.4.2 implies that

$$\mathtt{d}\exp(x)y = \sum_{k=1}^{\infty} \frac{1}{k!} k x^{k-1} y = \sum_{k=0}^{\infty} \frac{1}{k!} x^k y = \exp(x)y.$$

For $x = 0$, the relation $\exp(0) = \mathbf{1}$ now implies in particular that $\mathtt{d}\exp(0)y = y$. $\square$

Proposition 7.5.2. *For each sufficiently small open neighborhood U of 0 in $\mathcal{A}$, the map*

$$\exp|_U : U \to \mathcal{A}^\times$$

is a diffeomorphism onto an open neighborhood of $\mathbf{1}$ in $\mathcal{A}^\times$.

Proof. We have already seen that exp is a smooth map, and that $\mathtt{d}\exp(\mathbf{0}) = \mathrm{id}_{\mathcal{A}}$. Therefore the assertion follows from the Inverse Function Theorem. □

If U is as in Proposition 7.5.2 and $V = \exp(U)$, we define

$$\log_V\colon = (\exp|_U)^{-1}\colon V \to U \subseteq \mathcal{A}.$$

We shall see below why this function deserves to be called a *logarithm function.*

Theorem 7.5.3. (No Small Subgroup Theorem) *There exists an open neighborhood V of $\mathbf{1}$ in $\mathcal{A}^\times$ such that $\{\mathbf{1}\}$ is the only subgroup of $\mathcal{A}^\times$ contained in V.*

Proof. Let U be as in Proposition 7.5.2 and assume further that U is convex and bounded. We set $U_1 := \frac{1}{2}U$. Let $G \subseteq V := \exp U_1$ be a subgroup of $\mathcal{A}^\times$ and $g \in G$. Then we write $g = \exp x$ with $x \in U_1$ and assume that $x \neq 0$. Let $k \in \mathbb{N}$ be maximal with $kx \in U_1$ (the existence of k follows from the boundedness of U). Then

$$g^{k+1} = \exp(k+1)x \in G \subseteq V$$

implies the existence of $y \in U_1$ with $\exp(k+1)x = \exp y$. Since $(k+1)x \in 2U_1 = U$ follows from $\frac{k+1}{2}x \in [0,k]x \subseteq U_1$, and $\exp|_U$ is injective, we obtain $(k+1)x = y \in U_1$, contradicting the maximality of k. Therefore $g = \mathbf{1}$. □

Next we apply the tools from Section 7.4 to the logarithm series. Since this series has the radius of convergence 1, it defines a smooth function $\mathcal{A}^\times \supseteq B_1(\mathbf{1}) \to \mathcal{A}$, and we shall see that it provides a smooth inverse of the exponential function.

Lemma 7.5.4. *The series $\log(\mathbf{1}+x) := \sum_{k=1}^\infty (-1)^{k+1}\frac{x^k}{k}$ converges for $x \in \mathcal{A}$ with $\|x\| < 1$ and defines a smooth function*

$$\log\colon B_1(\mathbf{1}) \to \mathcal{A}.$$

For $\|x\| < 1$ and $y \in \mathcal{A}$ with $xy = yx$ we have

$$(\mathtt{d}\log)(\mathbf{1}+x)y = (\mathbf{1}+x)^{-1}y.$$

Proof. The convergence follows from

$$\sum_{k=1}^\infty (-1)^{k+1}\frac{r^k}{k} = \log(1+r) < \infty$$

for $|r| < 1$, so that the smoothness follows from Lemma 7.4.2.

If x and y commute, then the formula for the derivative in Lemma 7.4.2 leads to

$$(\mathtt{d}\log)(\mathbf{1}+x)y = \sum_{k=1}^{\infty}(-1)^{k+1}x^{k-1}y = (\mathbf{1}+x)^{-1}y$$

(see the proof of Proposition prop:1.1.7). $\square$

Proposition 7.5.5. (a) *For $x \in \mathcal{A}$ with $\|x\| < \log 2$ we have*

$$\log(\exp x) = x.$$

(b) *For $g \in \mathrm{GL}_d(\mathbb{K})$ with $\|g - \mathbf{1}\| < 1$ we have* $\exp(\log g) = g$.

Proof. (a) We apply Proposition 7.4.3 with $g = \exp \in P_S$, $S = \log 2$, $R = e^{\log 2} = 2$ and $\|\exp\|_s \leq e^s \leq e^S = 2$ for $s < S$. We thus obtain $\log(\exp x) = x$ for $\|x\| < \log 2$.

(b) Next we apply Proposition 7.4.3 with $f = \exp$, $S = 1$ and $g(z) = \log(1+z)$ to obtain $\exp(\log g) = g$. $\square$

Part II

Additional Material

Chapter 8

Complex Methods

8.1 Hilbert Spaces of Holomorphic Functions

The constructions in Chapter 3 do not refer to any differentiable structure on the set X, but positive definite kernels are also quite useful in complex analysis, where one studies holomorphic functions. One possibility to introduce holomorphic functions in an infinite dimensional context is the following:

Definition 8.1.1. (a) Let E be a complex locally convex space and $\mathcal{D} \subseteq \mathbb{C}$ be an open subset. A map $\gamma\colon \mathcal{D} \to E$ is called a *holomorphic curve* if it is continuous and

$$\gamma'(z) := \lim_{h \to 0} \frac{\gamma(z+h) - \gamma(z)}{h}$$

exists in every $z \in \mathcal{D}$. It is called *antiholomorphic* of the map $z \mapsto \gamma(\overline{z})$ is holomorphic.

(b) Let E_1 and E_2 be a locally convex spaces and $\mathcal{D} \subseteq E_1$ be an open subset. A map $f\colon \mathcal{D} \to E_2$ is said to be *(anti-)holomorphic* if it is continuous and for every holomorphic curve $\gamma\colon \mathcal{D}_0 \to \mathcal{D}$, the composition $f \circ \gamma\colon \mathcal{D}_0 \to \mathcal{D}$ is (anti-)holomorphic. [1]

We write $\mathcal{O}(\mathcal{D}, E)$ for the space of holomorphic E-valued functions on the domain $\mathcal{D}$. For $E = \mathbb{C}$, we simply write $\mathcal{O}(\mathcal{D}) := \mathcal{O}(\mathcal{D}, \mathbb{C})$ for the space of holomorphic functions on $\mathcal{D}$, which is a complex algebra with respect to the pointwise operations (Exercise 8.2.1).

Remark 8.1.2. Compositions of holomorphic functions are holomorphic: If E_1, E_2 and E_3 are complex locally convex spaces $\mathcal{D}_j \subseteq E_j$, $j = 1, 2$, open subsets and $\varphi\colon \mathcal{D}_1 \to \mathcal{D}_2$, $\psi\colon \mathcal{D}_2 \to E_3$ holomorphic, then their composition

$$\psi \circ \varphi\colon \mathcal{D}_1 \to E_3$$

[1] For more details on holomorphic function in infinite dimensional spaces, we refer to Hervé's book [He89]. The preceding concept applies in particular to $X = \mathbb{C}^n$. In this case the continuity requirement on f can be dropped. It is a deep result in the theory of several complex variables, called *Hartog's Theorem*, that continuity follows from the partial holomorphy of f.

is also holomorphic. In fact, for every holomorphic curve $\gamma\colon \mathcal{D}_0 \to \mathcal{D}_1$, the composition $\varphi \circ \gamma$ is a holomorphic curve in $\mathcal{D}_2$, so that the holomorphy of ψ implies that $\psi \circ (\varphi \circ \gamma)$ is holomorphic.

Similarly one argues that compositions of holomorphic maps with antiholomorphic maps are antiholomorphic and compositions of two antiholomorphic maps are holomorphic.

Lemma 8.1.3. *Let $X_0, X_1, \ldots, X_n$ be complex locally convex spaces and*

$$\beta\colon X_1 \times \cdots \times X_n \to X_0$$

be a continuous n-linear map. Then β is holomorphic.

Proof. Let $\mathcal{D} \subseteq \mathbb{C}$ be an open subset and $\gamma\colon \mathcal{D} \to X$ be a holomorphic curve, i.e., $\gamma = (\gamma_1, \ldots, \gamma_n)$ for holomorphic curves $\gamma_j : \mathcal{D} \to X_j$. We have to show that the curve

$$f(z) := \beta(\gamma_1(z), \ldots, \gamma_n(z))$$

is holomorphic. For $z + h \in \mathcal{D}$ we derive from the n-linearity of β that

$$\begin{aligned} f(z+h) - f(z) &= \beta(\gamma_1(z+h), \ldots, \gamma_n(z+h)) - \beta(\gamma_1(z), \ldots, \gamma_n(z)) \\ &= \beta(\gamma_1(z+h), \ldots, \gamma_n(z+h)) - \beta(\gamma_1(z), \gamma_2(z+h), \ldots, \gamma_n(z+h)) \\ &\quad + - \cdots - \beta(\gamma_1(z), \ldots, \gamma_n(z)) \\ &= \sum_{j=1}^n \beta(\gamma_1(z), \ldots, \gamma_j(z+h) - \gamma_j(z), \ldots, \gamma_n(z+h)). \end{aligned}$$

This implies that

$$\begin{aligned} \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \sum_{j=1}^n \beta(\gamma_1(z), \ldots, \frac{\gamma_j(z+h) - \gamma_j(h)}{h}, \ldots, \gamma_n(z)) \\ &= \sum_{j=1}^n \beta(\gamma_1(z), \ldots, \gamma_j'(z), \ldots, \gamma_n(z)). \end{aligned}$$

Therefore β is holomorphic. □

Theorem 8.1.4. (Principle of Analytic Continuation) *Let $\mathcal{D}$ be an open connected subset of the complex locally convex space X and $f, g\colon \mathcal{D} \to Y$ be two holomorphic functions into a locally convex space Y. Then $f = g$ if one of the following conditions is satisfied:*

(a) *There exists a non-empty open subset $U \subseteq \mathcal{D}$ with $f|_U = g|_U$.*

(b) *There exists an antilinear involution σ on X such that $f = g$ holds on $\mathcal{D} \cap (p + X^\sigma)$ for some $p \in \mathcal{D}$, where $X^\sigma := \{v \in X : \sigma v = v\}$ denotes the real subspace of σ-fixed vectors.*

Proof. Since the continuous linear functionals on Y separate the points, it suffices to show that $\alpha \circ f$ vanishes for every $\alpha \in Y'$. Therefore we may w.l.o.g. assume that $Y = \mathbb{C}$. Replacing f by $f - g$, we may also assume that $g = 0$.

(a) Let V denote the interior of the set $\{x \in \mathcal{D} \colon f(x) = 0\}$ and $F := \overline{V}$ denote its closure. Then the continuity of $f - g$ implies that f vanishes on the closed set F. This set is non-empty because it contains U. We claim that F is also open, so that the connectedness of $\mathcal{D}$ implies that $F = \mathcal{D}$, and hence that $f = 0$.

Pick $x \in F$ and let $W \subseteq \mathcal{D}$ be an open convex neighborhood of x. We want to show that f vanishes on W by reducing this assertion to the one-dimensional case. Let $w \in W$ and $v \in V \cap W$. Such a point exists because $x \in \overline{V}$. Then the map

$$\gamma_v \colon \mathbb{C} \to X, \quad z \mapsto v + z(w - v)$$

is affine, hence holomorphic, and $\Omega_v := \gamma_v^{-1}(W)$ is an open convex subset of $\mathbb{C}$. From $\gamma_v(0) = v$ we derive that the holomorphic function $f \circ \gamma_v$ on Ω_v vanishes on a neighborhood 0, hence on all of Ω_v by the one-dimensional version of (a). This implies that $f(w) = 0$, and because $w \in W$ was arbitrary, we derive that f vanishes on W.

(b) In view of (a), it suffices to show that f vanishes on some neighborhood of p. We may therefore assume that $\mathcal{D}$ is a convex neighborhood of p invariant under the antiholomorphic involution $\tau(v) := p + \sigma(v - p)$ fixing p. For $q \in \mathcal{D}$, the convexity of $\mathcal{D}$ implies that

$$q_0 := \frac{1}{2}(q + \tau(q)) = p + \frac{1}{2}((v - p) + \sigma(v - p)) \in \mathcal{D} \cap (p + X^\sigma)$$

is fixed by τ, so that

$$\sigma(q_0 - p) = \tau(q_0) - p = q_0 - p$$

implies that

$$q_0 \in p + X^\sigma.$$

Further,

$$2\sigma(q - q_0) = \sigma(q - \tau(q)) = \sigma(q - \sigma(q)) = \sigma(q) - q = \tau(q) - q = 2(q_0 - q)$$

yields $q - q_0 \in iX^\sigma = X^{-\sigma}$. Now the map

$$\gamma_q \colon \mathbb{C} \to X, \quad z \mapsto q_0 + iz(q - q_0)$$

is holomorphic and $\Omega_q := \gamma_q^{-1}(\mathcal{D})$ is an open convex subset of $\mathbb{C}$. For $z \in \mathbb{R} \cap \Omega_q$ we have

$$\gamma_q(z) \in \mathcal{D} \cap (q_0 + \mathbb{R}i(q - q_0)) \subseteq \mathcal{D} \cap (p + X^\sigma),$$

so that the holomorphic functions $f \circ \gamma_q$ vanishes on this open interval, and this implies that it vanishes on Ω_q because its set of zeros is not discrete. $\square$

Definition 8.1.5. Let X be a complex locally convex space. The *conjugate linear space* $\overline{X}$ is defined to be the same set X with the same addition, but the scalar multiplication defined by $(\lambda, v) \mapsto \overline{\lambda}v$. Then $\mathrm{id}_X : X \to \overline{X}$ is an antilinear isomorphism of complex vector spaces.

A hermitian kernel $K\colon \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ is said to be *holomorphic* if it is holomorphic as a function on the open subset $\mathcal{D} \times \mathcal{D}$ of the complex product space $X \times \overline{X}$.

Proposition 8.1.6. *Let $\mathcal{D}$ be an open subset of the locally convex space X and $K \in \mathcal{P}(\mathcal{D})$ be a holomorphic positive definite kernel. Then $\mathcal{H}_K \subseteq \mathcal{O}(\mathcal{D})$.*

Proof. First we use Proposition 3.3.5 to see that $\mathcal{H}_K \subseteq C(\mathcal{D}, \mathbb{C})$. Next we observe that, for each $w \in \mathcal{D}$, the function $K_w(z) := K(z, w)$ is holomorphic on $\mathcal{D}$. Let $f \in \mathcal{H}_K$ and $\gamma\colon \mathcal{D}_0 \to \mathcal{D}$ be a holomorphic curve. Then for each compact subset $S_0 \subseteq \mathcal{D}_0$, the image $S := \gamma(S_0)$ in $\mathcal{D}$ is compact, so that $C := \sup_{x \in S} K(x, x) < \infty$. Therefore convergence in $\mathcal{H}_K$ implies uniform convergence on S.

Let $f \in \mathcal{H}_K$ and $f_n \in \mathcal{H}_K^0$ with $f_n \to f$. Then, for each $n \in \mathbb{N}$, the function $f_n \circ \gamma\colon \mathcal{D} \to \mathbb{C}$ is holomorphic, and $f_n \circ \gamma \to f \circ \gamma$ uniformly on compact subsets of $\mathcal{D}_0$, so that $f \circ \gamma$ is also holomorphic. This proves that $f \in \mathcal{O}(\mathcal{D})$. $\square$

Examples 8.1.7. (a) The Kernel $K(z, w) := \langle z, w\rangle$ on a complex Hilbert space $\mathcal{H}$ is holomorphic. Indeed, as a function on the product space $\mathcal{H} \times \overline{\mathcal{H}}$, it is complex bilinear and continuous, hence holomorphic (Lemma 8.1.3).

(b) If $K \in \mathcal{P}(\mathcal{D})$ is a holomorphic kernel with $|K(z, w)| < r$ for $z, w \in \mathcal{D}$, where r is the radius of convergence of the complex power series $f(z) = \sum_{n=0}^\infty a_n z^n$ with $a_n \geq 0$ for each $n \in \mathbb{N}$, then the positive definite kernel $f(K(z, w)) = \sum_{n=0}^\infty a_n K(z, w)^n$ is also holomorphic because composition of holomorphic functions (on domains in $\mathbb{C}$) are holomorphic (Example 3.3.6 (e), (f)).

(c) Let $\mathcal{H}$ be a complex Hilbert space. Then the kernels $e^{\lambda\langle z,w\rangle}$, $\lambda > 0$, on $\mathcal{H}$ and $(1 - \langle z, w\rangle)^{-s}$, $s \geq 0$, on the open unit ball of $\mathcal{H}$ are holomorphic.

The following lemma is an important tool to verify holomorphy of vector-valued functions.

Lemma 8.1.8. *If $\mathcal{D} \subseteq E$ is an open subset of the complex locally convex space E and V a sequentially complete locally convex space. Then a map $\gamma\colon \mathcal{D} \to V$ is holomorphic if and only if it is weakly holomorphic, i.e., for each $\lambda \in V'$, the composition $\lambda \circ \gamma$ is holomorphic.*

Proof. Since holomorphy is tested by composition with holomorphic curves, we may w.l.o.g. assume that $E = \mathbb{C}$. Clearly, the weak holomorphy of γ follows from its holomorphy (Remark 8.1.2 and Lemma 8.1.3). So let us assume that γ is weakly holomorphic. Let $\alpha \in V'$, $z \in \mathcal{D}$ and assume that $z + h \in \mathcal{D}$ holds for $|h| \leq 2\varepsilon$. Then, for $|h| < \varepsilon$, the Cauchy Integral Formula

$$f(z + h) = \frac{1}{2\pi i}\oint_{|\zeta|=2\varepsilon} \frac{f(z + \zeta)}{\zeta - h}\, d\zeta$$

yields

$$\frac{1}{h}\langle\alpha, f(z+h)-f(z)\rangle = \frac{1}{2\pi i}\oint_{|\zeta|=2\varepsilon}\frac{(\alpha\circ f)(z+\zeta)}{(\zeta-h)\zeta}\,d\zeta$$

because

$$\frac{1}{h}\Big(\frac{1}{\zeta-h}-\frac{1}{\zeta}\Big) = \frac{\zeta-(\zeta-h)}{h\zeta(\zeta-h)} = \frac{1}{\zeta(\zeta-h)}.$$

Since V is sequentially complete, the V-valued integral

$$\oint_{|\zeta|=2\varepsilon}\frac{f(z+\zeta)}{(\zeta-h)\zeta}\,d\zeta$$

exists (cf. Lemma 8.2.4 below), and since V' separates the points of V, we derive the relation

$$\frac{1}{h}\big(f(z+h)-f(z)\big) = \frac{1}{2\pi i}\oint_{|\zeta|=2\varepsilon}\frac{f(z+\zeta)}{(\zeta-h)\zeta}\,d\zeta.$$

Finally, the fact that the function

$$F\colon \{h\in\mathbb{C}\colon |h|\leq\varepsilon\}\times\{\zeta\in\mathbb{C}\colon |\zeta|=2\varepsilon\}\to V, \quad (h,\zeta)\mapsto\frac{f(z+\zeta)}{(\zeta-h)\zeta}$$

is continuous entails that

$$\lim_{h\to 0}\frac{1}{h}\big(f(z+h)-f(z)\big) = \frac{1}{2\pi i}\oint_{|\zeta|=2\varepsilon}\frac{f(z+\zeta)}{\zeta^2}\,d\zeta$$

holds in V (Exercise!). This means that $f\colon\mathcal{D}\to V$ is holomorphic. □

The following result sharpens Proposition 8.1.6 considerably. It will be used below to derive irreducibility criteria for unitary representations on reproducing kernel spaces.

Proposition 8.1.9. *If $K\in\mathcal{P}(\mathcal{D})$ is a holomorphic positive definite kernel, then the map*

$$\gamma\colon\mathcal{D}\to\mathcal{H}, \quad z\mapsto K_z$$

is antiholomorphic.

Proof. In view of Lemma 8.1.8, it suffices to show that γ is weakly antiholomorphic, i.e., all compositions with continuous linear functionals are antiholomorphic. This is an immediate consequence of Proposition 8.1.6, which asserts that, for each $f\in\mathcal{H}_K$, the map

$$\mathcal{D}\to\mathbb{C}, \quad z\mapsto\langle\gamma(z), f\rangle = \overline{f(z)}$$

is antiholomorphic. □

Theorem 8.1.10. (S. Kobayashi's Irreducibility Criterion) *Let $\mathcal{D}$ be an open subset of a complex locally convex space, $\sigma\colon G\times X\to X$ be a group action, $J\colon G\times\mathcal{D}\to\mathbb{C}^\times$ a corresponding cocycle, and $K\in\mathcal{P}(\mathcal{D},\sigma,J)$ be a holomorphic positive definite kernel. If G acts transitively on $\mathcal{D}$, then the representation $(\pi_K,\mathcal{H}_K)$ is irreducible.*

Proof. Suppose that $\mathcal{K} \subseteq \mathcal{H}_K$ is a closed G-invariant subspace and let Q be its reproducing kernel, so that $\mathcal{K} = \mathcal{H}_Q$. Then Proposition 5.1.6(b) implies that $Q \in \mathcal{P}(X, \sigma, J)$, and from Proposition 5.1.11 we derive that $Q = K^B$ holds for some $B \in \pi(G)'$. This shows that

$$Q(z,w) = \langle BK_w, K_z\rangle \quad \text{for} \quad z, w \in \mathcal{D}.$$

To see that Q is holomorphic, we recall from Proposition 8.1.9 that the map $\mathcal{D} \to \mathcal{H}, z \mapsto K_z$ is antiholomorphic. Hence the map

$$\mathcal{H} \times \overline{\mathcal{H}} \supseteq \mathcal{D} \times \mathcal{D} \to \mathcal{H} \times \overline{\mathcal{H}}, \quad (z,w) \mapsto (K_w, K_z)$$

is holomorphic, and since the map $\mathcal{H} \times \overline{\mathcal{H}} \to \mathbb{C}, (z,w) \mapsto \langle Bz, w\rangle$ is continuous bilinear, hence holomorphic (Lemma 8.1.3), composition of these two maps implies that Q is holomorphic,

Pick $x_0 \in \mathcal{D}$. If $K_{x_0} = 0$, then also $Q(x_0, x_0) = 0$, so that there exists a $c \geq 0$ with $Q(x_0, x_0) = \lambda K(x_0, x_0)$. Then the invariance condition implies that

$$Q(g.x_0, g.x_0) = |J(g, x_0)|^2 Q(x_0, x_0) = c|J(g, x_0)|^2 K(x_0, x_0) = cK(g.x_0, g.x_0).$$

Our assumption that G acts transitively on $\mathcal{D}$ now implies that $Q(z,z) = cK(z,z)$ for all $z \in \mathcal{D}$.

To derive from this relation that the two holomorphic functions cK and Q on $\mathcal{D} \times \overline{\mathcal{D}}$ coincide, we note that the flip involution $\sigma(z,w) := (z,w)$ on $\mathcal{H} \times \overline{\mathcal{H}}$ is antilinear and $\Delta_{\mathcal{H}} = \{(z,z) \colon z \in \mathcal{H}\}$ is the set of its fixed points. Therefore $Q = cK$ follows from Theorem 8.1.4. If $c = 0$, then $Q = 0$ and $\mathcal{H}_Q = \{0\}$, and if $c > 0$, then $K = c^{-1}Q$ implies that $\mathcal{H}_K = \mathcal{H}_Q$. This proves the irreducibility of the representation $(\pi_K, \mathcal{H}_K)$. ☐

Example 8.1.11. Kobayashi's Theorem can be used in particular to prove the irreducibility of the representation of the Heisenberg group $\mathrm{Heis}(\mathcal{H})$ on the Fock space $\mathcal{F}(\mathcal{H})$. In fact, we apply it with $G = \mathrm{Heis}(\mathcal{H})$, $\mathcal{D} = \mathcal{H}$, $\sigma((t,v),z) = v + z$ and

$$J((t,v),z) = e^{it + \langle z,v\rangle - \frac{1}{2}\langle v,v\rangle}.$$

8.2 Appendix: Vector-valued Riemann Integrals

Definition 8.2.1. Let E be a locally convex space. A sequence $(x_n)_{n\in\mathbb{N}}$ in E is called a *Cauchy sequence* if for every 0-neighborhood U of E there exists an $N_U \in \mathbb{N}$ with

$$x_n - x_m \in U \quad \text{for} \quad n, m > N_U.$$

The space E is said to be *sequentially complete* if every Cauchy sequence in E has a limit.

Remark 8.2.2. Since the sets of the form

$$U(p, \varepsilon) := \{x \in E \colon p(x) < \varepsilon\},$$

where p is a continuous seminorm on E form a basis of zero-neighborhoods, a sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if for every continuous seminorm p and every $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ with

$$p(x_n - x_m) < \varepsilon \quad \text{for} \quad n, m > N_\varepsilon.$$

Definition 8.2.3. Let E be a locally convex space. Then a curve $\gamma\colon [a,b] \to E$ is said to be *differentiable in* t if the limit

$$\gamma'(t) := \lim_{h\to 0} \frac{1}{h}(\gamma(t+h) - \gamma(t))$$

exists in E. Then $\gamma'(t)$ is called the *derivative of* γ *in* t. The curve γ is said to be *differentiable* if it is differentiable in every point, and it is called a C^1*-curve*, if, in addition, the curve $\gamma'\colon [a,b] \to E$ is also continuous.

Lemma 8.2.4. *Let E be a sequentially complete locally convex space and $\gamma\colon [a,b] \to E$ be a continuous curve. Then there exists a unique C^1-curve $\Gamma\colon [a,b] \to E$ with $\Gamma' = \gamma$ and $\Gamma(a) = 0$. It is denoted*

$$\Gamma(t) := \int_a^t \gamma(\tau)\, d\tau.$$

Proof. For any partition $T = (t_0, t_1, \ldots, t_n)$ of the interval $[a,b]$, we write

$$S(T) := \int_{j=0}^{n-1} \gamma(t_j)(t_{j+1} - t_j)$$

for the corresponding Riemann sum. We call

$$\delta(T) := \max\{t_{j+1} - t_j \colon j = 0, \ldots, n-1\}$$

the *width of* T. We want to show that if (T_m) is a sequence of partitions with $\delta(T_m) \to 0$, then the sequence $(S(T_n))$ in E converges. Since E is sequentially complete, it suffices to show that it is a Cauchy sequence. So let p be a continuous seminorm on E. Since γ is uniformly continuous with respect to p (Exercise), there exists for each $\varepsilon > 0$ a $\delta > 0$ with

$$p(\gamma(t) - \gamma(s)) \leq \varepsilon \quad \text{for} \quad |t - s| \leq \delta.$$

Now let T and T' be two partitions with $\delta(T), \delta(T') < \delta$. Then their common refinement $\widetilde{T}$ also has this property and we have

$$p(S(T) - S(\widetilde{T})) \leq \sum_j \varepsilon(\widetilde{t}_{j+1} - \widetilde{t}_j) = \varepsilon(b-a),$$

and since $p(S(T') - S(\widetilde{T}))$ satisfies the same estimate, we obtain

$$p(S(T) - S(T')) \leq 2\varepsilon(b-a).$$

This proves that $(S(T_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in E, and our estimates show that the limit does not depend on the choice of the sequence (T_n). We write

$$\int_a^b \gamma(\tau)\,d\tau := \lim_{n\to\infty} S(T_n)$$

for this limit. We also observe that

$$p\Big(\int_a^b \gamma(\tau)\,d\tau\Big) = \lim_{n\to\infty} p(S(T_n)) \leq \lim_{n\to\infty} \sum_{j=0}^{n-1} p(\gamma(t_j))(t_{j+1}-t_j) = \int_a^b p(\gamma(t))\,dt$$

for every continuous seminorm p.

Finally, we put

$$\Gamma(t) := \int_a^t \gamma(\tau)\,d\tau.$$

For $a \leq t+h \leq b$ we then have

$$\begin{aligned} p\Big(\frac{\Gamma(t+h)-\Gamma(t)}{h} - \gamma(t)\Big) &= p\Big(\frac{1}{h}\int_t^{t+h} \gamma(\tau)-\gamma(t)\,d\tau\Big) \\ &\leq \frac{1}{h}\int_t^{t+h} p(\gamma(\tau)-\gamma(t))\,d\tau\Big) \to 0 \end{aligned}$$

because $p(\gamma(\tau)-\gamma(t)) \to 0$ for $\tau \to t$. We conclude that Γ is a C^1-curve with $\Gamma' = \gamma$ and $\Gamma(a) = 0$.

If $\widetilde{\Gamma}\colon [a,b] \to E$ is another C^1-curve with these properties, then $F := \widetilde{\Gamma} - \Gamma$ is a C^1-curve with $F' = 0$ and $F(a) = 0$. Then we obtain for every continuous linear functional $\lambda \in E'$ a C^1-curve $\lambda \circ F\colon [a,b] \to \mathbb{R}$ with

$$(\lambda \circ F)(a) = 0 \quad \text{and} \quad (\lambda \circ F)' = 0.$$

Now basic calculus implies that $\lambda \circ F = 0$, and hence that $\lambda(F(t)) = 0$ for each $\lambda \in E'$ and $t \in [a,b]$. Since the continuous linear functionals on E separate the points, we obtain $F = 0$. ☐

Exercises for Section 8.1

Exercise 8.2.1. Let $\mathcal{D}$ be an open subset of a complex locally convex space. Show that $\mathcal{O}(\mathcal{D})$ is a complex unital algebra with respect to pointwise multiplication.

Exercise 8.2.2. Let $\mathcal{H}$ be a complex Hilbert space and $\overline{\mathcal{H}}$ be its complex conjugate space with the scalar multiplication defined by $\lambda * v := \overline{\lambda}v$. Show that the map

$$\varphi\colon \overline{\mathcal{H}} \to \mathcal{H}^*, \quad \varphi(v)(w) := \langle w, v\rangle$$

is a linear isomorphism.

Exercise 8.2.3. Let V be a complex locally convex space and $\sigma\colon V \to V$ be a continuous antilinear involution. Further, let $\mathcal{D} \subseteq V$ be a convex σ-invariant domain and $\mathcal{H}_K \subseteq \mathcal{O}(\mathcal{D})$ be a Hilbert space of holomorphic functions with kernel K. Then

$$\mathcal{D}^\sigma := \{v \in V \colon \sigma(v) = v\}$$

is an open domain in the real locally convex space V^σ. Show that the restriction map

$$r\colon \mathcal{O}(\mathcal{D}) \to C(\mathcal{D}^\sigma), \qquad f \mapsto f|_{\mathcal{D}^\sigma}$$

restricts to a unitary map $r\colon \mathcal{H}_K \to \mathcal{H}_Q$, where $Q := K|_{\mathcal{D}^\sigma \times \mathcal{D}^\sigma}$.

Exercise 8.2.4. Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{H}_\mathbb{C}$ be its complexification. Show that the restriction map

$$R\colon \mathcal{F}(\mathcal{H}_\mathbb{C}) \to \mathcal{F}(\mathcal{H}), \quad f \mapsto f|_\mathcal{H}$$

is unitary, where $\mathcal{F}(\mathcal{H})$ is defined as the reproducing kernel space of the real-valued kernel $K(x,y) := e^{\langle x,y\rangle}$.

8.3 More Representations of $\mathrm{U}(\mathcal{H})$

We have already seen that each complex Hilbert space $\mathcal{H}$ can be realized as a reproducing kernel space $\mathcal{H}_Q$ on $X = \mathcal{H}$ with the kernel $Q(z,w) = \langle w, z\rangle$. In the same spirit, we have defined n-fold tensor products $\mathcal{H}_1 \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{H}_n$ as reproducing kernel spaces on $X = \prod_{j=1}^n X_j$ with the kernel $Q(z,w) := \prod_{j=1}^n \langle w_j, z_j\rangle$. Specializing this construction to $\mathcal{H}_j = \mathcal{H}$ for $j = 1,\ldots,n$, we obtain the space

$$\mathcal{H}^{\otimes n} := \mathcal{H}_Q \subseteq \mathbb{C}^{\mathcal{H}^n} \quad \text{with} \quad Q(z,w) := \prod_{j=1}^n \langle w_j, z_j\rangle.$$

Since the kernel Q is invariant under the natural action of the unitary group $\mathrm{U}(\mathcal{H})$ on $\mathcal{H}^n$ by $g.(z_1,\ldots,z_n) := (gz_1,\ldots,gz_n)$, we obtain a unitary representation

$$\pi(g) = g \otimes g \otimes \cdots \otimes g$$

of $\mathrm{U}(\mathcal{H})$ on $\mathcal{H}^{\otimes n}$. For $n > 1$ this representation is no longer irreducible, and it can be shown to be multiplicity free with finitely many irreducible summands whose precise description is part of the so-called *Schur–Weyl theory* of decompositions of tensor products of representations.

That the representation is not irreducible for $n > 1$ is easy to see by showing that its commutant is larger than $\mathbb{C}\mathbf{1}$. This is due to the fact that the symmetric group S_n also has a unitary representation on $\mathcal{H}^{\otimes n}$. To see this action, we first note that S_n acts on the product space $\mathcal{H}^n$ by permuting the factors:

$$\sigma.(z_1,\ldots,z_n) := (z_{\sigma^{-1}(1)},\ldots,z_{\sigma^{-1}(n)}),$$

and that this action preserves the kernel Q. Hence

$$(\rho(\sigma)f)(z_1,\ldots,z_n) := f(z_{\sigma(1)},\ldots,z_{\sigma(n)})$$

defines a unitary representation of S_n on $\mathcal{H}^{\otimes n}$. In terms of tensor products, this action is given by

$$(\rho(\sigma)(v_1\otimes\cdots\otimes v_n))(z_1,\ldots,z_n) = v_1(z_{\sigma(1)})\cdots v_n(z_{\sigma(n)}) = v_{\sigma^{-1}}(z_1)\cdots v_{\sigma^{-1}(n)}(z_n),$$

i.e.,

$$\rho(\sigma)(v_1\otimes\cdots\otimes v_n) = v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(n)}.$$

8.3.1 Symmetric and Exterior Powers

Clearly, $\rho(S_n) \subseteq \pi(\mathrm{U}(\mathcal{H}))'$ is non-trivial, which leads to a decomposition of the representation π. The two most important subspaces of $\mathcal{H}^{\otimes n}$ are the two eigenspaces of S_n for the two characters of this group. For the trivial character we obtain the subspace

$$S^n(\mathcal{H}) := (\mathcal{H}^{\otimes n})^{S_n}$$

of S_n-invariant vectors. It is called the *n-th symmetric power of $\mathcal{H}$*, and for the signature character $\mathrm{sgn}\colon S_n \to \{\pm 1\}$, we obtain the subspace

$$\Lambda^n(\mathcal{H}) := (\mathcal{H}^{\otimes n})^{\mathrm{sgn},S_n} = \{v\in\mathcal{H}^{\otimes n}\colon (\forall\sigma\in S_n)\ \rho(\sigma)v = \mathrm{sgn}(\sigma)v\}.$$

It is called the *n-th exterior power of $\mathcal{H}$*.

It is easy to write down projections onto these subspaces using the Haar measures on the finite group S_n:

$$P_+ := \frac{1}{n!}\sum_{\sigma\in S_n}\rho(\sigma)$$

is the projection onto $S^n(\mathcal{H})$ and

$$P_- := \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\rho(\sigma)$$

is the projection onto $\Lambda^n(\mathcal{H})$. To verify these claims, one first verifies that

$$\rho(\tau)P_+ = P_+ \quad \text{and} \quad \rho(\tau)P_- = \mathrm{sgn}(\tau)P_-$$

by direct calculation. This proves that $\mathrm{im}(P_+) \subseteq S^n(\mathcal{H})$, and that $P_+v = v$ holds for each $v \in S^n(\mathcal{H})$ is an immediate consequence of the definition of P_+. Therefore P_+ is a projection, and that it is hermitian follows from

$$P_+^* = \frac{1}{n!}\sum_{\sigma\in S_n}\rho(\sigma^{-1}) = \frac{1}{n!}\sum_{\sigma\in S_n}\rho(\sigma) = P_+.$$

Similarly one argues that P_- is an orthogonal projection onto $\Lambda^n(\mathcal{H})$.

For $v_1, \ldots, v_n \in \mathcal{H}$, we define the *symmetric product*

$$v_1 \vee \cdots \vee v_n := P_+(v_1 \otimes \cdots \otimes v_n)$$

and the *exterior (=alternating) product* by

$$v_1 \wedge \cdots \wedge v_n := P_-(v_1 \otimes \cdots \otimes v_n).$$

These products define continuous complex n-linear maps $\mathcal{H}^n \to S^n(\mathcal{H})$ and $\mathcal{H}^n \to \Lambda^n(\mathcal{H})$. It follows directly from the definition that the $\vee$-product is symmetric and the wedge product $\wedge$ is alternating, i.e.,

$$v_{\sigma(1)} \vee \cdots \vee v_{\sigma(n)} = v_1 \vee \cdots \vee v_n$$

and

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \operatorname{sgn}(\sigma) v_1 \vee \cdots \vee v_n.$$

The inner products of such elements are given by

$$\begin{aligned}\langle v_1 \vee \cdots \vee v_n, w_1 \vee \cdots \vee w_n\rangle &= \langle v_1 \vee \cdots \vee v_n, w_1 \otimes \cdots \otimes w_n\rangle \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \langle v_{\sigma(1)}, w_1\rangle \cdots \langle v_{\sigma(n)}, w_m\rangle\end{aligned}$$

and likewise

$$\begin{aligned}&\langle v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_n\rangle = \langle v_1 \wedge \cdots \wedge v_n, w_1 \otimes \cdots \otimes w_n\rangle \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \langle v_{\sigma(1)}, w_1\rangle \cdots \langle v_{\sigma(n)}, w_m\rangle = \frac{1}{n!} \det(\langle v_i, w_j\rangle)_{1 \le i,j \le n}.\end{aligned}$$

8.3.2 The Representation of U($\mathcal{H}$) on $S^n(\mathcal{H})$

We first take a closer look at the subspace $S^n(\mathcal{H})$. As a closed subspace of the reproducing kernel space $\mathcal{H}_Q = \mathcal{H}^{\otimes n}$, it also is a reproducing kernel space on $\mathcal{H}^n$, and the evaluation functionals on $S^n(\mathcal{H})$ are obtained as the orthogonal projections of Q_w, $w \in \mathcal{H}^n$, to $S^n(\mathcal{H})$:

$$Q^+(w,z) := P_+(Q_z)(w) = \frac{1}{n!} \sum_{\sigma \in S_n} Q_z(\sigma^{-1}.w) = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{j=1}^n \langle z_j, w_{\sigma(j)}\rangle.$$

To evaluate this expression, we note that the elements of $S^n(\mathcal{H})$ are n-linear functions on $\mathcal{H}$ which are invariant under the symmetric group, hence completely determined by their values on the diagonal

$$\Delta_{\mathcal{H}} = \{(v, \cdots, v) \colon v \in \mathcal{H}\}$$

(Exercise). Therefore the restriction map

$$R \colon S^n(\mathcal{H}) \to \mathbb{C}^{\mathcal{H}}, \quad (Rf)(z) := f(z, \cdots, z)$$

is injective, and it maps $S^n(\mathcal{H})$ in a $\mathrm{U}(\mathcal{H})$-equivariant fashion onto the Hilbert space with reproducing kernel

$$\widetilde{Q}_+(z,w) := Q^+((z,\cdots,z),(w,\cdots,w)) = \langle z,w\rangle^n.$$

This implies that the unitary representation $(S^n, S^n(\mathcal{H}))$ of $\mathrm{U}(\mathcal{H})$, given by

$$S^n(g)(v_1 \vee \cdots \vee v_n) = gv_1 \vee \cdots \vee gv_n$$

is equivalent to the representation on the dual space $\mathcal{F}_n(\mathcal{H})' \cong \overline{\mathcal{F}_n(\mathcal{H})}$ of the set n-homogeneous functions in the Fock space. In particular, we derive from Example 5.3.17 that this representation is irreducible.

8.3.3 The Representation of $\mathrm{U}(\mathcal{H})$ on $\Lambda^n(\mathcal{H})$

Now we turn to the representation Λ^n of $\mathrm{U}(\mathcal{H})$ on the exterior power, given by

$$\Lambda^n(g)(v_1 \wedge \cdots \wedge v_n) = gv_1 \wedge \cdots \wedge gv_n.$$

The subspace $\Lambda^n(\mathcal{H})$ is generated by elements of the form $v_1 \wedge \cdots \wedge v_n$. Since the wedge product is alternating, this product vanishes if $v_1,\ldots,v_n$ are linearly independent. We also see that, if $w_1,\ldots,w_n$ is an orthonormal basis of $\mathrm{span}\{v_1,\ldots,v_n\}$, then

$$v_1 \wedge \cdots \wedge v_n \in \mathbb{C}w_1 \wedge \cdots \wedge w_n.$$

We now assume that $\dim \mathcal{H} \geq n$ because otherwise $\Lambda^n(\mathcal{H})$ vanishes. Let $e_1,\ldots,e_n$ be orthonormal. Then each other orthonormal tuple $(v_1,\cdots,v_n)$ in $\mathcal{H}$ is of the form $(ge_1,\ldots,ge_n)$ for some $g \in \mathrm{U}(\mathcal{H})$. This implies that

$$\mathrm{U}(\mathcal{H}).(e_1 \wedge \cdots \wedge e_n) \cup \{0\} \supseteq \{v_1 \wedge \cdots \wedge v_n \colon v_1,\ldots,v_n \in \mathcal{H}\},$$

so that

$$v_0 := e_1 \wedge \cdots \wedge e_k \in \Lambda^k(\mathcal{H})$$

is a cyclic vector for $\mathrm{U}(\mathcal{H})$.

Let $(e_j)_{j\in J}$ be an orthonormal basis containing $e_1,\ldots,e_n$ and, accordingly, $\{1,\ldots,n\} \subseteq J$. We also assume that J carries a linear order $\leq$ satisfying

$$1 \leq 2 \leq \ldots \leq n \leq j \quad \text{for} \quad j \in J \setminus \{1,\ldots,n\}.$$

For each n-element subset $F = \{j_1,\ldots,j_n\} \subseteq J$ with $j_1 < j_2 < \ldots < j_n$, we then obtain an element

$$e_F := e_{j_1} \wedge \cdots \wedge e_{j_n} \in \Lambda^n(\mathcal{H}),$$

and, expanding wedge products $v_1 \wedge \cdots \wedge v_n$ with respect to the orthonormal basis $(e_j)_{j\in J}$, we see that the finite wedge products

$$e_{j_1} \wedge \cdots \wedge e_{j_n}, \quad j_1,\ldots,j_n \in J,$$

form a total subset of $\Lambda^n(\mathcal{H})$. If $|\{j_1, \ldots, j_n\}| < n$, then $e_{j_1}, \ldots, e_{j_n}$ are linearly dependent, so that $e_{j_1} \wedge \cdots \wedge e_{j_n} = 0$. If $|\{j_1, \ldots, j_n\}| = n$, we put $F := \{j_1, \ldots, j_n\}$ and note that if $\sigma \in S_n$ satisfies $j_{\sigma(1)} > j_{\sigma(2)} > \cdots > j_{\sigma(n)}$, then

$$e_{j_1} \wedge \cdots \wedge e_{j_n} = \mathrm{sgn}(\sigma) e_F.$$

This proves that the e_F, $F \subseteq J$ an n-element subset, form a total subset of $\Lambda^n(\mathcal{H})$. Since

$$\langle e_F, e_{F'} \rangle = \det(\delta_{j_k, j'_m})_{1 \leq k,m \leq n} = \begin{cases} 0 & \text{for } \ F \neq F' \\ 1 & \text{for } \ F = F', \end{cases}$$

we see that the e_F even form an orthonormal basis of $\Lambda^n(\mathcal{H})$.

To show that the representation of $\mathrm{U}(\mathcal{H})$ on $\Lambda^n(\mathcal{H})$ is irreducible, we consider the restriction to the subgroup $T \cong T^J$ of diagonal operators with respect to our orthonormal basis. For $F = \{j_1, \ldots, j_n\}$ we have

$$t.e_F = t_{j_1} e_{j_1} \wedge \cdots \wedge t_{j_n} e_{j_n} = (t_{j_1} \cdots t_{j_n}) e_F.$$

Defining

$$\chi_F \in \widehat{T} \cong \mathbb{T}^J \quad \text{by} \quad \chi_F(t) := \prod_{f \in F} t_f,$$

we conclude that

$$\Lambda^n(\mathcal{H})^{T, \chi_F} = \mathbb{C} e_F$$

because different n-element subsets $F, F' \subseteq J$ lead to different characters of T. As we have seen above, each e_F is a cyclic vector for $\mathrm{U}(\mathcal{H})$, so that Proposition 5.3.11 now implies that the representation of $\mathrm{U}(\mathcal{H})$ on $\Lambda^n(\mathcal{H})$ is irreducible.

For the cyclic vector v_0 we have

$$\varphi(g) = \langle g v_0, v_0 \rangle = \langle g e_1 \wedge \cdots \wedge g e_n, e_1 \wedge \cdots \wedge e_n \rangle = \frac{1}{n!} \det(\langle g e_j, e_k \rangle)_{1 \leq j,k \leq n}.$$

Writing g as a block matrix with respect to the decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}, \quad \mathcal{H}_0 := \mathrm{span}\{e_1, \ldots, e_n\},$$

this means that

$$\varphi(g) = \det(a) \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Chapter 9

Projective Unitary Representations

In this chapter we discuss an important issue arising in many situations where unitary operators are only determined up to a "phase factor" in the circle group $\mathbb{T}$. To understand the underlying structures, we take a brief look at central group extensions for which the unitary group $\mathrm{U}(\mathcal{H})$, as a $\mathbb{T}$-extension of the projective unitary group $\mathrm{PU}(\mathcal{H})$, is a paradigmatic example.

We have already seen in Section 5.2 that for a complex Hilbert space $\mathcal{H}$, we do not get a unitary representation of the translation group $(\mathcal{H}, +)$ on the Fock space $\mathcal{F}(\mathcal{H})$. Instead we have to extend this group to the Heisenberg group $\mathrm{Heis}(\mathcal{H})$ which has a natural representation on $\mathcal{F}(\mathcal{H})$. In this section we take a closer look at the underlying structures. In particular, we shall define projective unitary representations and explore how they are related to central group extensions.

9.1 Central Group Extensions

Definition 9.1.1. An *extension of groups* is a short exact sequence

$$\mathbf{1} \to N \xrightarrow{\iota} \widehat{G} \xrightarrow{q} G \to \mathbf{1}$$

of group homomorphisms. We then speak of an extension of G by N. Since $\iota\colon N \to \widehat{G}$ is injective, we may identify N with its image, so that the surjectivity of q leads to $G \cong \widehat{G}/N$.

An extension is called *central* if $\iota(N)$ is central in $\widehat{G}$. We call two extensions $N \hookrightarrow \widehat{G}_1 \twoheadrightarrow G$ and $N \hookrightarrow \widehat{G}_2 \twoheadrightarrow G$ of G by the group N *equivalent* if there exists a homomorphism $\varphi\colon \widehat{G}_1 \to \widehat{G}_2$ such that the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "\iota_1"] \arrow[d, "\mathrm{id}_N"] & \widehat{G}_1 \arrow[r, "q_1"] \arrow[d, "\varphi"] & G \arrow[d, "\mathrm{id}_G"] \\ N \arrow[r, "\iota_2"] & \widehat{G}_2 \arrow[r, "q_2"] & G \end{tikzcd}$$

It is easy to see that any such φ is in particular an isomorphism of groups (Exercise). We write $\mathrm{Ext}(G,N)$ for the set of equivalence classes of group extensions of G by N.

We call an extension $q\colon \widehat{G} \to G$ *split* or *trivial* if there exists a group homomorphism $\sigma\colon G \to \widehat{G}$ with $q \circ \sigma = \mathrm{id}_G$. In this case the map

$$N \rtimes_S G \to \widehat{G}, \quad (n,g) \mapsto n\sigma(g)$$

is an isomorphism, where the semidirect product structure is defined by the homomorphism

$$S\colon G \to \mathrm{Aut}(N), \quad S(g)(n) := \sigma(g)n\sigma(g)^{-1}$$

(Exercise).

When dealing with topological groups, one requires all homomorphisms to be continuous, ι to be an embedding and q to be a quotient map.

In this section we shall only deal with central extensions. We start by introducing product coordinates on such extensions:

Remark 9.1.2. Let $q\colon \widehat{G} \to G$ be a central group extension with kernel Z and $\sigma\colon G \to \widehat{G}$ be a section of q which is *normalized* in the sense that $\sigma(\mathbf{1}) = \mathbf{1}$. Then the map

$$\Phi\colon Z \times G \to \widehat{G}, \quad (z,g) \mapsto z\sigma(g)$$

is a bijection and it becomes an isomorphism of groups if we endow $Z \times G$ with the multiplication

$$(z,g)(z',g') = (zz'\omega(g,g'), gg'), \tag{9.1}$$

where

$$\omega\colon G \times G \to Z, \quad (g,g') \mapsto \sigma(g)\sigma(g')\sigma(gg')^{-1}. \tag{9.2}$$

The map ω satisfies

$$\sigma(g)\sigma(g') = \omega(g,g')\sigma(gg'), \tag{9.3}$$

and the cocycle condition

$$\omega(g,g')\omega(gg',g'') = \omega(g',g'')\omega(g,g'g'') \tag{9.4}$$

If $\sigma'\colon G \to \widehat{G}$ is another normalized section, then there exists a function $\theta\colon G \to Z$ with $\theta(\mathbf{1}) = \mathbf{1}$ and $\sigma' = \sigma\theta$. Then we find the relation

$$\omega'(g,g') = \omega(g,g')\frac{\theta(g)\theta(g')}{\theta(gg')}.$$

Definition 9.1.3. If Z is an abelian group, G a group and $\omega\colon G \times G \to Z$ a map satisfying (9.4), then we call ω a 2-*cocycle*. It is said to be normalized if $\omega(\mathbf{1},g) = \omega(g,\mathbf{1}) = \mathbf{1}$ for any $g \in G$. We write $Z^2(G,Z)$ for the set of normalized 2-cocycles. It is easy to see that this is an abelian group under pointwise multiplication.

Functions of the form

$$b(g,g') := \frac{\theta(g)\theta(g')}{\theta(gg')}$$

for some normalized function $\theta\colon G \to Z$ are always normalized cocycles (Exercise). They are called 2-*coboundaries.* The set $B^2(G,Z)$ of 2-coboundaries is a subgroup of $Z^2(G,Z)$, and the quotient group

$$H^2(G,Z) := Z^2(G,Z)/B^2(G,Z)$$

is called the *second cohomology group of G with values in Z*. We write $[f]$ for the image of $f \in Z^2(G,Z)$ in $H^2(G,Z)$. It is called the *cohomology class of f*.

Theorem 9.1.4. *For any $\omega \in Z^2(G,Z)$, we obtain on $Z \times G$ a group structure by*

$$(z,g)(z',g') := (zz'\omega(g,g'), gg').$$

We write $Z \times_\omega G$ for this group. Then $\iota(z) := (z,\mathbf{1})$ and $q(z,g) := g$ define a central extension

$$Z \xrightarrow{\iota} Z \times_\omega G \xrightarrow{q} G.$$

Every central extension of G by Z is equivalent to one of these, and $Z \times_\omega G$ is equivalent to $Z \times_{\omega'} G$ if and only if $[\omega] = [\omega']$.

In this sense the group $H^2(G,Z)$ parameterizes the equivalence classes of central extensions of G by Z.

Proof. The associativity of the multiplication in $Z \times_\omega G$ follows immediately from the cocycle property of f. To see that we actually obtain a group, we observe that the conditions $\omega(g,\mathbf{1}) = \omega(\mathbf{1},g) = \mathbf{1}$ imply that $\mathbf{1} := (\mathbf{1},\mathbf{1})$ is an identity element of $Z \times_f G$ and

$$(z,g)^{-1} = (z^{-1}\omega(g,g^{-1})^{-1}, g^{-1}) = (z^{-1}\omega(g^{-1},g)^{-1}, g^{-1})$$

is the inverse of (z,g). Note that $\omega(g,g^{-1}) = \omega(g^{-1},g)$ follows from the cocycle relation, applied to the triple (g,g^{-1},g). It is obvious that ι and q are group homomorphisms and that $\iota(Z) = Z \times \{\mathbf{1}\}$ is central. Therefore $Z \times_\omega G$ defines a central extension of G by Z.

The preceding remark implies that any central extension is equivalent to some $Z \times_\omega G$ because for any normalized section $\sigma\colon G \to \widehat{G}$ and

$$\omega(g,g') := \sigma(g)\sigma(g')\sigma(gg')^{-1}$$

the map

$$\Phi\colon Z \times_\omega G \to \widehat{G}, \quad (z,g) \mapsto z\sigma(g)$$

is an equivalence of central extensions.

Finally we note that an equivalence

$$\Psi\colon Z \times_{\omega'} G \to Z \times_\omega G$$

has the form $\Psi(z,g) = (z\theta(g), g)$ for some normalized function $\theta\colon G \to Z$. Now the requirement that Ψ is a group homomorphism implies that

$$zz'\theta(g)\theta(g')\omega(g,g') = zz'\theta(gg')\omega'(g,g') \quad \text{for} \quad g,g' \in G,$$

which is equivalent to

$$\frac{\theta(g)\theta(g')}{\theta(gg')} = \omega(g,g')^{-1}\omega'(g,g').$$

We conclude that such an equivalence of central extension exists if and only if $[\omega] = [\omega']$. □

Example 9.1.5. (a) If G and Z are abelian groups, then every biadditive map

$$\omega\colon G \times G \to Z$$

is a group cocycle because

$$\omega(g+g',g'')+\omega(g,g') = \omega(g,g'')+\omega(g',g'')+\omega(g,g') = \omega(g,g'+g'')+\omega(g',g'').$$

(b) The Heisenberg group $\mathrm{Heis}(\mathcal{H})$ of a Hilbert space is a central extension of the additive group $(\mathcal{H},+)$ by $\mathbb{R}$, defined by the cocycle

$$\omega(v,w) = -\frac{1}{2}\,\mathrm{Im}\langle v,w\rangle.$$

Remark 9.1.6. If a central extension $q\colon \widehat{G} \to G$ of G by Z is trivial, then any homomorphic section $\sigma\colon G \to \widehat{G}$ leads to an isomorphism of groups

$$\Phi\colon Z \times G \to \widehat{G}, \quad (z,g) \mapsto z\sigma(g).$$

In particular, there exists a homomorphism $p\colon \widehat{G} \to Z$ with $p|_Z = \mathrm{id}_Z$. As Z is abelian, it follows that

$$(\widehat{G},\widehat{G}) \cap Z = \{\mathbf{1}\} \tag{9.5}$$

holds for the commutator group of $\widehat{G}$. This is a necessary condition for the triviality of a central extension.

If, conversely, this condition is satisfied, then q restricts to a group isomorphism

$$q|_{(\widehat{G},\widehat{G})}\colon (\widehat{G},\widehat{G}) \to (G,G),$$

so that its inverse provides a homomorphism

$$\sigma\colon (G,G) \to \widehat{G} \quad \text{with} \quad q \circ \sigma = \mathrm{id}_{(G,G)}\,.$$

However, in general σ does not extend to a homomorphic section on all of G.

An instructive example is the surjective homomorphism

$$q\colon C_4 := \{z \in \mathbb{C}\colon z^4 = 1\} \to C_2 = \{\pm 1\}, \quad z \mapsto z^2.$$

It defines a central extension of C_2 by C_2 which is non-trivial because $C_2 \times C_2$ contains no element of order 4. As C_4 is abelian, its commutator group is trivial. Therefore (9.5) is not sufficient for a central extension to split.

9.2 Projective Unitary Representations

Definition 9.2.1. Let $\mathcal{H}$ be a complex Hilbert space, $\mathrm{PU}(\mathcal{H}) := \mathrm{U}(\mathcal{H})/\mathbb{T}\mathbf{1}$ be the *projective unitary group* and $q\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H}), g \mapsto [g] := \mathbb{T}g$, be the quotient map. We endow $\mathrm{PU}(\mathcal{H})$ with the quotient topology with respect to the strong operator topology on $\mathrm{U}(\mathcal{H})$ which turns it into a topological group (Exercise 9.3.1).

Proposition 9.2.2. (a) *The topology on* $\mathrm{PU}(\mathcal{H})$ *is the coarsest topology for which all functions*

$$h_{v,w}\colon \mathrm{PU}(\mathcal{H}) \to \mathbb{R}, \quad [g] \mapsto |\langle gv, w\rangle|$$

are continuous.

(b) *The quotient map* $q\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$ *has continuous local sections, i.e., each* $[g] \in \mathrm{PU}(\mathcal{H})$ *has an open neighborhood* U *on which there exists a continuous section* $\sigma\colon U \to \mathrm{U}(\mathcal{H})$ *of* q.

Proof. (a) Let $q\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$ denote the quotient map. Then all functions $h_{v,w} \circ q = f_{v,w}$ are continuous on $\mathrm{U}(\mathcal{H})$, which implies that the functions $h_{v,w}$ are continuous on $\mathrm{PU}(\mathcal{H})$.

Let τ denote the coarsest topology on $\mathrm{PU}(\mathcal{H})$ for which all functions $h_{v,w}$ are continuous. We know already that this topology is coarser than the quotient topology. Next we observe that the relations

$$h_{v,w}([g][g']) = h_{g'v,w}([g]) = h_{v,g^{-1}w}([g'])$$

imply that left and right multiplications are continuous in τ. To see that τ coincides with the quotient topology, it therefore remains to see that $[g_i] \to \mathbf{1}$ in τ implies that $[g_i] \to \mathbf{1}$ in the quotient topology.

For a net $([g_i])_{i\in I}$ in $\mathrm{PU}(\mathcal{H})$ we consider a lift $(g_i)_{i\in I}$ in $\mathrm{U}(\mathcal{H})$. Since the closed operator ball $\mathcal{B} := \{A \in B(\mathcal{H})\colon \|A\| \leq 1\}$ is compact in the weak operator topology (Exercise 9.3.2), there exists a convergent subnet $g_{\alpha(j)} \to g_0 \in \mathcal{B}$. For $v, w \in \mathcal{H}$ we then have

$$h_{v,w}(g_{\alpha(j)}) \to h_{v,w}(\mathbf{1}) = |\langle v, w\rangle|$$

and also

$$h_{v,w}(g_{\alpha(j)}) = |\langle g_{\alpha(j)}v, w\rangle| \to |\langle g_0 v, w\rangle|,$$

hence $|\langle g_0 v, w\rangle| = |\langle v, w\rangle|$. This implies in particular that for each non-zero vector v, we have

$$g_0 v \in (v^\perp)^\perp = \mathbb{C}v,$$

so that each vector is an eigenvector, and this implies that $g_0 = t\mathbf{1}$ for some $t \in \mathbb{C}$ (Exercise). If $v = w$ is a unit vector, we obtain $|t| = |\langle g_0 v, v\rangle| = 1$. Therefore we have $g_{\alpha(j)} \to t\mathbf{1}$ in $\mathrm{U}(\mathcal{H})$, and this implies that $[g_{\alpha(j)}] \to [\mathbf{1}]$ in $\mathrm{PU}(\mathcal{H})$.

If the net $(g_i)_{i\in I}$ does not converge to $\mathbf{1}$ in $\mathrm{PU}(\mathcal{H})$, then there exists an open $\mathbf{1}$-neighborhood U for which the set $I_U := \{i \in I : g_i \notin U\}$ is cofinal, which leads to a subnet $(g_i)_{i\in I_U}$ converging to $\mathbf{1}$ in τ and contained in the closed subset U^c. Applying the preceding argument to this subnet now leads to a contradiction since it cannot have any subnet converging to $\mathbf{1}$ because U^c is closed.

(b) Since we can move sections with left multiplication maps, it suffices to assume that $g = \mathbf{1}$. Pick $0 \neq v_0 \in \mathcal{H}$. Then

$$\Omega := \{g \in \mathrm{U}(\mathcal{H})\colon \langle gv_0, v_0\rangle \neq 0\}$$

is an open $\mathbf{1}$-neighborhood in $\mathrm{U}(\mathcal{H})_s$ with $\Omega\mathbb{T} = \Omega$. Therefore $\overline{\Omega} := \{[g]\colon g \in \Omega\}$ is an open $\mathbf{1}$-neighborhood of $\mathrm{PU}(\mathcal{H})$ and for each $g \in \Omega$ there exists a unique $t \in \mathbb{T}$ with

$$tg \in \Omega_+ := \{g \in \mathrm{U}(\mathcal{H})\colon \langle gv_0, v_0\rangle > 0\}.$$

We now define a map

$$\sigma\colon \overline{\Omega} \to \Omega, \quad [g] \mapsto g \quad \text{for} \quad g \in \Omega_+.$$

To see that σ is continuous, it suffices to observe that the map

$$\Omega \to \Omega_+, \quad g \mapsto \frac{|\langle gv_0, v_0\rangle|}{\langle gv_0, v_0\rangle} g$$

is continuous and constant on the cosets of $\mathbb{T}$. Hence it factors through a continuous map $\overline{\Omega} \to \Omega_+$ which is σ. This proves that the quotient map

$$q\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H}), \quad g \mapsto [g]$$

has a continuous section in some $\mathbf{1}$-neighborhood of $\mathrm{PU}(\mathcal{H})$. □

Remark 9.2.3. (a) For each complex Hilbert space $\mathcal{H}$, we have

$$Z(\mathrm{U}(\mathcal{H})) = \mathbb{T}\mathbf{1}.$$

In fact, for each unit vector $v \in \mathcal{H}$, the operator

$$\sigma_v(x) := x - \langle v, x\rangle v$$

is unitary with the eigenvector v for the eigenvalue -1 and $v^\perp$ is pointwise fixed. Therefore any linear map $A \in B(\mathcal{H})$ commuting with σ_v preserves the line $\mathbb{C}v$. This implies that for any element $z \in Z(\mathrm{U}(\mathcal{H}))$ each unit vector v is an eigenvector, and therefore $z \in \mathbb{C}\mathbf{1}$, so that $z \in \mathbb{T}\mathbf{1}$ by unitarity.

The projection

$$q\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H}), \quad g \mapsto [g] = \mathbb{T}g$$

defines a central extension of $\mathrm{PU}(\mathcal{H})$ by $\mathbb{T}$. We have seen in Remark 9.1.6 that the subgroup

$$Z(\mathrm{U}(\mathcal{H})) \cap (\mathrm{U}(\mathcal{H}), \mathrm{U}(\mathcal{H})) = \mathbb{T}\mathbf{1} \cap (\mathrm{U}(\mathcal{H}), \mathrm{U}(\mathcal{H}))$$

is a measure of non-triviality for this central extension.

(b) To determine this group, we fist assume that $\dim \mathcal{H} = n$ is finite. Then $\mathrm{U}(\mathcal{H}) \cong \mathrm{U}_n(\mathbb{C})$ and we have a homomorphism

$$\det\colon \mathrm{U}_n(\mathbb{C}) \to \mathbb{T}$$

vanishing on all commutators. We write

$$\mathrm{SU}_n(\mathbb{C}) := \{g \in \mathrm{U}_n(\mathbb{C})\colon \det(g) = \mathbf{1}\}$$

for the *special unitary group*. Then

$$(\mathrm{U}_n(\mathbb{C}), \mathrm{U}_n(\mathbb{C})) \subseteq \mathrm{SU}_n(\mathbb{C})$$

implies that

$$(\mathrm{U}_n(\mathbb{C}), \mathrm{U}_n(\mathbb{C})) \cap \mathbb{T}\mathbf{1} \subseteq \{z\mathbf{1} \in \mathbb{T}\mathbf{1}\colon z^n = 1\} =: C_n.$$

We claim that we actually have equality. Let $\zeta \in C_n$. Then we define two unitary operators $p, q \in \mathrm{U}_n(\mathbb{C})$ by

$$q(e_j) = \zeta^j e_j, \quad j = 1, \ldots, n \quad \text{and} \quad p(e_j) := \begin{cases} e_{j+1} & \text{for } j < n \\ e_1 & \text{for } j = n. \end{cases}$$

For $1 < j$ we then have

$$(q, p)e_j = qpq^{-1}p^{-1}e_j = \zeta^{-(j-1)}qpe_{j-1} = \zeta^{-(j-1)}qe_j = \zeta e_j$$

and

$$(q, p)e_1 = \zeta^{-n}qpe_n = qe_1 = \zeta e_1.$$

This proves that $(p, q) = \zeta\mathbf{1}$, and hence that every element of C_n is a commutator of two unitary operators. We conclude that

$$(\mathrm{U}_n(\mathbb{C}), \mathrm{U}_n(\mathbb{C})) \cap \mathbb{T}\mathbf{1} = C_n\mathbf{1}. \tag{9.6}$$

(c) If $\dim \mathcal{H} = \infty$, we have $\mathcal{H} \cong \ell^2(\mathbb{Z}, \mathbb{C})\widehat{\otimes}\mathcal{K}$ for some other Hilbert space $\mathbb{K}$. In fact, if J is an index set of an ONB, then there exists a bijection $\mathbb{Z} \times J \to J$ ([La93, Thm. 3.3 in App. 2]). Alternatively, one can use Zorn's Lemma directly to show that each infinite set J can be written as a disjoint union $J = \bigcup_{i \in I} J_i$ of infinite countable sets, which leads to a bijection $I \times \mathbb{Z} \to J$ and hence to a unitary map $\ell^2(J, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathbb{C})\widehat{\otimes}\ell^2(I, \mathbb{C})$.

Let $\zeta \in \mathbb{T}$. Then we define two unitary operators $p, q \in \mathrm{U}(\mathcal{H})$ by

$$q(e_j \otimes v) = \zeta^j e_j \otimes v \quad \text{and} \quad p(e_j \otimes v) := e_{j+1} \otimes v \quad \text{for} \quad j \in \mathbb{Z}, v \in \mathcal{K}.$$

We then have

$$(q, p)(e_j \otimes v) = qpq^{-1}(e_{j-1} \otimes v) = \zeta^{-(j-1)}qp(e_{j-1} \otimes v) = \zeta^{-(j-1)}q(e_j \otimes v) = \zeta e_j \otimes v,$$

so that $(q, p) = \zeta\mathbf{1}$. This proves that every element of $\mathbb{T}\mathbf{1}$ is a commutator group of two unitary operators. We conclude that

$$\mathbb{T}\mathbf{1} \subseteq (\mathrm{U}(\mathcal{H}), \mathrm{U}(\mathcal{H})). \tag{9.7}$$

Definition 9.2.4. Let G be a (topological) group. A *projective unitary representation* is a (continuous) homomorphism $\pi\colon G \to \mathrm{PU}(\mathcal{H})$.

Remark 9.2.5. Clearly, every continuous unitary representation

$$\pi\colon G \to \mathrm{U}(\mathcal{H})$$

also defines a continuous projective representation,

$$\overline{\pi}\colon G \to \mathrm{PU}(\mathcal{H}), \quad g \mapsto [\pi(g)].$$

One can even say more, namely that, if $N := \{g \in G\colon \pi(g) \in \mathbb{T}\mathbf{1}\}$, then we obtain a continuous unitary representation of the factor group G/N:

$$\overline{\pi}\colon G/N \to \mathrm{PU}(\mathcal{H}), \quad gN \mapsto [\pi(g)].$$

Example 9.2.6. (a) We have seen in Proposition 5.2.1 that the Heisenberg group $\mathrm{Heis}(\mathcal{H})$ has a continuous unitary representation on the Fock space $\mathcal{F}(\mathcal{H})$ defined by

$$(\pi(t,v)f)(z) = e^{it+\langle z,v\rangle - \frac{1}{2}\langle v,v\rangle} f(z-v).$$

Since $\pi(t,0) = e^{it}\mathbf{1} \in \mathbb{T}\mathbf{1}$, we obtain a continuous projective representation

$$\overline{\pi}\colon \mathcal{H} \to \mathrm{PU}(\mathcal{F}(\mathcal{H})), \quad v \mapsto [\pi(t,v)].$$

(b) Let G be a locally compact abelian group and $\mathcal{H} := L^2(G,\mu_G)$. Then we have the right regular representation $(\pi_r, L^2(G))$ of G, given by $(\pi_r(g)f)(x) = f(xg)$, but we also have a unitary representation

$$\rho\colon \widehat{G} \to \mathrm{U}(L^2(G)), \quad (\rho(\chi)f)(g) := \chi(g)f(g).$$

We consider the corresponding map

$$\widehat{\pi}\colon G \times \widehat{G} \to \mathrm{U}(L^2(G)), \quad (g,\chi) \mapsto \pi_r(g)\rho(\chi).$$

This map is not a group homomorphism because

$$\pi_r(g)\rho(\chi) = \chi(g)\rho(\chi)\pi_r(g) \quad \text{for} \quad g \in G, \chi \in \widehat{G}.$$

From this relation we immediately derive that

$$\overline{\pi}\colon G \times \widehat{G} \to \mathrm{PU}(\mathcal{H}), \quad (g,\chi) \mapsto [\pi_r(g)\rho(\chi)]$$

defines a projective representation of the abelian group $G \times \widehat{G}$ on $L^2(G)$.

Proposition 9.2.7. *Let $\alpha\colon G \to \mathrm{PU}(\mathcal{H})$ be a continuous projective unitary representation of G on $\mathcal{H}$. Then*

$$\widehat{G} := \alpha^* \mathrm{U}(\mathcal{H}) := \{(g,u) \in G \times \mathrm{U}(\mathcal{H})\colon \alpha(g) = [u]\},$$

endowed with the product topology, is a topological group and

$$q\colon \widehat{G} \to G, \quad (g,u) \mapsto g$$

is a quotient map with central kernel $\mathbb{T}$*, hence a central* $\mathbb{T}$*-extension of* G*. By*

$$\pi\colon \widehat{G} \to \mathrm{U}(\mathcal{H}), \quad (g,u) \mapsto u$$

we obtain a continuous unitary representation of G *on* $\mathcal{H}$ *with* $\overline{\pi} = \alpha$.

Proof. Clearly, $\widehat{G}$ is a closed subgroup of $G \times \mathrm{U}(\mathcal{H})$, hence in particular a topological group. Next we recall from Proposition 9.2.2(b) that there exists a $\mathbf{1}$-neighborhood $V \subseteq \mathrm{PU}(\mathcal{H})$ and a continuous section $\sigma\colon V \to \mathrm{U}(\mathcal{H})$ of the quotient map $\mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H}), g \mapsto [g]$. Therefore we obtain on $\alpha^{-1}(V) \subseteq G$ a continuous section

$$\alpha^{-1}(V) \to \widehat{G}, \quad g \mapsto (g, \sigma(\alpha(g))),$$

showing that q has continuous local sections, hence in particular that q is a quotient map (Exercise). The remaining assertions are trivial. $\square$

Examples 9.2.8. (a) For the projective unitary representation

$$\overline{\pi}\colon \mathcal{H} \to \mathrm{PU}(\mathcal{F}(\mathcal{H}))$$

on the Fock space of $\mathcal{H}$, we know already that we have a corresponding unitary representation $\pi\colon \mathrm{Heis}(\mathcal{H}) \to \mathrm{U}(\mathcal{H})$ of the Heisenberg group. This implies that $\overline{\pi}^* \mathrm{U}(\mathcal{H}) \cong \mathrm{Heis}(\mathcal{H})$.

(b) If G is a locally compact abelian group and $\overline{\pi}\colon G \times \widehat{G} \to \mathrm{PU}(L^2(G))$ the projective unitary representation from Example 9.2.6, then

$$\overline{\pi}^* \mathrm{U}(\mathcal{H}) \cong \mathbb{T} \times_\omega (G \times \widehat{G}),$$

where the cocycle $\omega \in Z^2(G \times \widehat{G}, \mathbb{T})$ is given by

$$\omega((g,\chi),(g',\chi')) := \chi(g')^{-1}.$$

9.3 Projective Invariance of Kernels

Definition 9.3.1. For a kernel K on the set X, we define its *projective invariance group* by

$$\mathrm{PAut}(X,K) := \{\varphi \in S_X : (\exists \theta\colon X \to \mathbb{C}^\times)\, (\theta,\varphi).K = K\},$$

where

$$((\theta,\varphi).K)(x,y) := \theta(x) K(\varphi^{-1}(x), \varphi^{-1}(y)) \overline{\theta(y)}$$

is the natural action of $(\mathbb{C}^\times)^X \rtimes S_X$ on the space of kernels (cf. Definition 5.1.2). By definition, we then have a surjective homomorphism

$$q\colon \mathrm{Aut}(X,K) \to \mathrm{PAut}(X,K), \quad (f,\varphi) \mapsto \varphi.$$

Its kernel is the set of all functions $\theta\colon X \to \mathbb{C}^\times$ leaving K invariant in the sense that

$$\theta(x)K(x,y)\overline{\theta(y)} = K(x,y) \quad \text{for all} \quad x,y \in X. \tag{9.8}$$

We now take a closer look at the kernel of q.

Definition 9.3.2. We call a kernel $K\colon X \times X \to \mathbb{C}$ *of transitive type* if $K(x,x) \neq 0$ for $x \in X$, and the equivalence relation on X generated by

$$\{(x,y) \in X \times X \colon K(x,y) \neq 0\}$$

coincides with $X \times X$. This means that, for $x, y \in X$, there exist $x_0, x_1, \ldots, x_n \in X$ with $x_0 = x$, $x_n = y$ and

$$K(x_i, x_{i+1}) \neq 0 \quad \text{for} \quad i = 0, \ldots, n-1.$$

Proposition 9.3.3. *Suppose that the kernel K is of transitive type and positive definite. Then the following assertions hold:*

(a) $\ker q \cong \mathbb{T}$ *is the set of constant functions with values in* $\mathbb{T}$. *In particular,*

$$q\colon \operatorname{Aut}(X,K) \to \operatorname{PAut}(X,K)$$

defines a central $\mathbb{T}$*-extension of* $\operatorname{PAut}(X,K)$.

(b) $\pi\colon \operatorname{PAut}(X,K) \to \operatorname{PU}(\mathcal{H}_K)$, *defined by*

$$\pi(\varphi) := [\pi(\theta,\varphi)] \quad \textit{for} \quad (\theta,\varphi) \in \operatorname{Aut}(X,K), \quad (\pi(\theta,\varphi)f) = \theta \cdot (\varphi_* f)$$

defines a projective unitary representation of $\operatorname{PAut}(X,K)$ *on* $\mathcal{H}_K$.

Proof. (a) Suppose that $f \in \ker q$, i.e., that (9.8) holds. If the kernel K is of transitive type, then $K(x,x) > 0$ implies $|f(x)| = 1$, and for $K(x,y) \neq 0$ we obtain $f(x) = f(y)$. Therefore the kernel relation of f is an equivalence relation on X containing all pairs (x,y) with $K(x,y) \neq 0$, hence equals $X \times X$, and this means that f is constant.

(b) follows from (a) and the discussion in Definition 5.1.2. □

Remark 9.3.4. For the diagonal kernel $K(x,y) = \delta_{x,y}$ on a set X, the condition

$$\theta(x)K(x,y)\overline{\theta(y)}$$

is satisfied for ever function $\theta\colon X \to \mathbb{T}$ and the kernel K is invariant under the whole group S_X. Therefore $\operatorname{PAut}(X,K) = S_X$ and

$$\operatorname{Aut}(X,K) \cong \mathbb{T}^X \rtimes S_X.$$

In particular, the kernel of the quotient homomorphism

$$q\colon \operatorname{Aut}(X,K) \to \operatorname{PAut}(X,K)$$

is not central because S_X acts non-trivially on $\ker q \cong \mathbb{T}^X$. As the preceding lemma shows, the kernel of q is minimal if K is of transitive type.

Definition 9.3.5. Let $\sigma\colon G \times X \to X$ be an action of G on X. The kernel K on X is called *projectively G-invariant* if $\sigma_g \in \mathrm{PAut}(X, K)$ holds for every $g \in G$.

Proposition 9.3.6. *If K is a projectively invariant kernel of transitive type, then the group*

$$\widehat{G} := \{(f, g) \in (\mathbb{C}^\times)^X \rtimes G \colon (f, g).K = K\}$$

is a central extension of G by the circle group $\mathbb{T}$, i.e., the projection

$$p\colon \widehat{G} \to G, \quad (f, g) \mapsto g$$

is a surjective homomorphism with central kernel $\ker p \cong \mathbb{T}$.

Proof. The projective invariance of K implies that the action σ defines a homomorphism $\sigma\colon G \to \mathrm{PAut}(X, K)$. This implies that $p(\widehat{G}) = G$. The kernel of p is the set of all functions f with $(f, \mathbf{1}).K = K$, which is equivalent to f being constant with values in $\mathbb{T}$. Since G acts trivially on constant functions $\ker p$ is central in the semidirect product group $(\mathbb{C}^\times)^X \rtimes G$ and therefore also in $\widehat{G}$. □

Exercises for Chapter 9

Exercise 9.3.1. Let G be a topological group and $N \trianglelefteq G$ be a closed normal subgroup, so that we can form the quotient group G/N with the quotient map $q\colon G \to G/N$. We endow G/N with the quotient topology, i.e., $O \subseteq G/N$ is open if and only if $q^{-1}(O) \subseteq G$ is open. Show that G/N is a topological group with respect to the quotient topology. Here are some steps to follow:

(a) Show that q is open. Hint: Exercise 5.3.7.

(b) To see that G/N is Hausdorff, argue that for $y \notin xN$ there exists an open $\mathbf{1}$-neighborhood U in G with $U^{-1}Uy \cap xN = \emptyset$ and derive that $\pi(Uy) \cap \pi(Ux) = \emptyset$.

(c) Use Exercise 1.1.4 to complete the proof. To verify (iii), pick for any open $\mathbf{1}$-neighborhood O in G/N an open $\mathbf{1}$-neighborhood U in G with $UU \subseteq q^{-1}(O)$ and argue that $\pi(U) = \pi(UN)$ is open.

Exercise 9.3.2. For a Hilbert space $\mathcal{H}$, show that the closed operator ball

$$\mathcal{B} := \{A \in B(\mathcal{H}) \colon \|A\| \leq 1\}$$

is compact with respect to the weak operator topology. Hint: Consider the topological embedding

$$\eta\colon \mathcal{B} \to \mathbb{C}^{\mathcal{H} \times \mathcal{H}}, \quad \eta(A)(v, w) := \langle Av, w \rangle$$

and show that its image is compact by applying Tychonov's Theorem.

Exercise 9.3.3. Let $K\colon X \times X \to \mathbb{C}$ be a positive definite kernel. Let $X_0 := \{x \in X\colon K(x,x) = 0\}$ and $X_+ := X \setminus X_0$. On X_+ we consider the equivalence relation generated by

$$x \sim x' \quad \text{if} \quad K(x,x') \neq 0.$$

Show that:

(a) On each equivalence class $[x]$, the kernel $K^{[x]} := K|_{[x]\times[x]}$ is of transitive type.

(b) $\mathcal{H}^{[x]} := \mathcal{H}_{K^{[x]}}$ is a closed subspace of $\mathcal{H}_K$ and

$$\mathcal{H}^{[x]} \perp \mathcal{H}^{[y]} = \{0\} \quad \text{for} \quad [x] \neq [y],$$

so that

$$\mathcal{H}_K = \widehat{\bigoplus}_{[x], x \in X_+} \mathcal{H}^{[x]}.$$

(c) The equivalence relation on X_+ and the subset X_0 are invariant under the group $\mathrm{Aut}(X,K)$.

Exercise 9.3.4. Let X be a compact space and $x\colon (I,\leq) \to X$ be a net, i.e., $(I,\leq)$ is a directed set. Show that if all convergent subnets $x \circ \alpha\colon (J,\leq) \to X$ of x converge to the same point $p_0 \in X$, then $x_i \to p_0$. Hint: If $x_i \not\to p_0$, then there exists an open neighborhood U of p_0 for which $I_U := \{i \in I\colon x_i \in U^c\}$ is cofinal. Use the compactness of U^c to see that $x|_{I_U}$ has a convergent subnet, and verify that this is also a subnet of x.

Exercise 9.3.5. Let G be a *perfect group*, i.e., $G = (G,G)$. Show that a central extension $q\colon \widehat{G} \to G$ with kernel Z splits if and only if

$$Z \cap (\widehat{G}, \widehat{G}) = \{\mathbf{1}\}$$

(cf. Remark 9.1.6).

Chapter 10

Negative Definite Kernels and Affine Actions

10.1 Negative Definite Kernels

Definition 10.1.1. Let X be a set. A hermitian kernel $Q\colon X \times X \to \mathbb{K}$ is called *negative definite* if for $(x_1, c_1), \ldots, (x_n, c_n)$ in $X \times \mathbb{C}$ with $\sum_j c_j = 0$, we have

$$\sum_{j,k=1}^{n} c_j \overline{c_k} Q(x_j, x_k) \leq 0.$$

We write $\mathcal{N}(X) = \mathcal{N}(X, \mathbb{K})$ for the set of negative definite kernels on X. If Q is negative definite, then the kernel $-Q$ is said to be *conditionally positive definite*.

Remark 10.1.2. (a) If $f\colon X \to \mathbb{K}$ is a function, then the kernel

$$Q_f(x, y) := f(x) + \overline{f(y)}$$

is negative definite because $\sum_{j=1}^{n} c_j = 0$ implies that

$$\sum_{j,k=1}^{n} c_j \overline{c_k} (f(x_j) + \overline{f(x_k)}) = \Big(\sum_{j=1}^{n} c_j f(x_j)\Big) \sum_{k=1}^{n} \overline{c_k} + \Big(\sum_{k=1}^{n} \overline{c_k}\overline{f(x_k)}\Big) \sum_{j=1}^{n} c_j = 0.$$

We even conclude that $\pm Q_f \in \mathcal{N}(X)$. Note that for $c \in \mathbb{R}$, we have $Q_{f+ic} = Q_f$, and that $Q_f = 0$ implies that f is constant with values in $i\mathbb{R}$.

(b) If K is a positive definite kernel, then $-K$ is negative definite, i.e.,

$$-\mathcal{P}(X) \subseteq \mathcal{N}(X).$$

(c) Clearly, positive definiteness implies conditional positive definiteness, but if $K\colon X \times X \to \mathbb{K}$ is conditionally positive definite and K_{x_0} vanishes for some $x_0 \in X$, then $K(x_0, x) = 0 = K(x, x_0)$ for any $x \in X$ because K is hermitian. This implies that for $x_1, \ldots, x_n$ and $c_1, \ldots, c_n \in \mathbb{K}$, we may put $c_0 := -(c_1 + \ldots + c_n)$ to derive

$$0 \leq \sum_{j,k=0}^{n} c_j \overline{c_k} K(x_j, x_k) = \sum_{j,k=1}^{n} c_j \overline{c_k} K(x_j, x_k),$$

so that K is positive definite.

(d) The preceding argument can be used to write a negative definite kernel Q as $Q_f - K$, where K is positive definite. Pick $x_0 \in X$ and put $f(x) := Q(x, x_0) - \frac{1}{2}Q(x_0, x_0)$. Then (a) implies that

$$K := Q_f - Q, \quad K(x, y) = Q(x, x_0) + Q(x_0, y) - Q(x_0, x_0) - Q(x, y).$$

is conditionally positive definite, but our construction also implies that $K_{x_0} = 0$, so that K is positive definite by (c).

If $h: X \to \mathbb{C}$ is another function for which $Q - Q_h$ vanishes in x_0, then Q_{f-h} vanishes in x_0, which implies that $f - h$ is constant purely imaginary, so that $Q_f = Q_h$.

Lemma 10.1.3. *For each map $\gamma: X \to \mathcal{H}$ of a set into a Hilbert space $\mathcal{H}$,*

$$Q_\gamma(x, y) := \|\gamma(x) - \gamma(y)\|^2 = \|\gamma(x)\|^2 + \|\gamma(y)\|^2 - 2\,\mathrm{Re}\langle\gamma(x), \gamma(y)\rangle$$

is a negative definite real-valued kernel on X vanishing on the diagonal.

Proof. We write $Q = Q_1 + Q_2$, where

$$Q_1(x, y) := \|\gamma(x)\|^2 + \|\gamma(y)\|^2 \quad \text{and} \quad Q_2(x, y) := -2\,\mathrm{Re}\langle\gamma(x), \gamma(y)\rangle.$$

The first kernel is negative definite by Remark 10.1.2(a), and the second one is negative definite because $-Q_2$ is positive definite, which follows from Proposition 3.2.1(e) and Remark 3.3.1(a). ☐

Definition 10.1.4. Let $\mathcal{H}$ be a real Hilbert space. A pair $(X, \gamma, \mathcal{H})$ consisting of a set X and a map $\gamma: X \to \mathcal{H}$ is called an *affine realization triple* if $\gamma(X)$ is not contained in a proper closed affine subspace of $\mathcal{H}$. Then

$$Q(x, y) := \|\gamma(x) - \gamma(y)\|^2$$

is called the corresponding negative definite kernel.

Before we turn to the uniqueness of the affine realizations of a negative definite kernel, we explain the connection between negative and positive definite real-valued kernels.

Lemma 10.1.5. *If $Q \in \mathcal{N}(X, \mathbb{R})$ vanishes on the diagonal and $x_0 \in X$, then there exists a unique positive definite kernel K on X with $K_{x_0} = 0$ and*

$$Q(x, y) = K(x, x) + K(y, y) - 2K(x, y).$$

It is given by

$$K(x,y) := \frac{1}{2}(Q(x,x_0) + Q(x_0,y) - Q(x,y)). \tag{10.1}$$

Proof. That the kernel K defined by (10.1) is positive definite follows from Remark 10.1.2(d). It satisfies $K_{x_0} = 0$ because Q vanishes on the diagonal, and we have

$$\begin{aligned}
&K(x,x) + K(y,y) - 2K(x,y) \\
&= Q(x,x_0) + Q(y,x_0) - Q(x,x_0) - Q(y,x_0) + Q(x,y) = Q(x,y).
\end{aligned}$$

That K is uniquely determined by these properties follows from Remark 10.1.2(d). □

Theorem 10.1.6. (Realization Theorem for Negative Definite Kernels) *Let X be a non-empty set. For each negative definite real-valued kernel Q on X vanishing on the diagonal, there exists an affine realization triple $(X, \gamma, \mathcal{H})$. For any other affine realization triple $(X, \gamma', \mathcal{H}')$ for Q, there exists a unique affine isometry $\varphi\colon \mathcal{H} \to \mathcal{H}'$ with $\varphi \circ \gamma = \gamma'$.*

Proof. **Existence:** Let $x_0 \in X$ and consider the real-valued kernel K from Lemma 10.1.5 satisfying $K_{x_0} = 0$,

$$Q(x,y) = K(x,x) + K(y,y) - 2K(x,y)$$

and

$$K(x,y) = \frac{1}{2}(Q(x,x_0) + Q(y,x_0) - Q(x,y)).$$

We consider the canonical realization triple $(X, \gamma_K, \mathcal{H}_K)$ of the positive definite kernel K with $\gamma_K(x) = K_x$. Then

$$\begin{aligned}
\|\gamma_K(x) - \gamma_K(y)\|^2 &= \|\gamma_K(x)\|^2 + \|\gamma_K(y)\|^2 - 2\,\mathrm{Re}\langle \gamma_K(x), \gamma_K(y)\rangle \\
&= K(x,x) + K(y,y) - 2K(x,y) = Q(x,y).
\end{aligned}$$

Since $\gamma_K(X)$ contains $0 = \gamma_K(x_0)$ and spans a dense subspace of $\mathcal{H}_K$, it is not contained in any proper closed affine subspace of $\mathcal{H}_K$. Therefore $(X, \gamma_K, \mathcal{H}_K)$ is an affine realization triple for Q.

Uniqueness: Let $(X, \gamma, \mathcal{H})$ be an affine realization triple for Q. Pick $x_0 \in X$ and consider the triple $(X, \eta, \mathcal{H})$, where $\eta(x) := \gamma(x) - \gamma(x_0)$. Then $\eta(x_0) = 0$ and there exists no proper closed affine subspace containing $\eta(X)$, which implies that $\eta(X)$ spans a dense subspace of $\mathcal{H}$. Therefore $(X, \eta, \mathcal{H})$ is an affine realization triple for Q and also a realization triple for the positive definite kernel

$$\begin{aligned}
&\langle \eta(y), \eta(x)\rangle = \frac{1}{2}(\|\eta(x)\|^2 + \|\eta(y)\|^2 - \|\eta(x) - \eta(y)\|^2) \\
&= \frac{1}{2}(\|\eta(x) - \eta(x_0)\|^2 + \|\eta(y) - \eta(x_0)\|^2 - \|\eta(x) - \eta(y)\|^2) \\
&= \frac{1}{2}(Q(x,x_0) + Q(y,x_0) - Q(x,y)) = K(x,y).
\end{aligned}$$

Now we use Theorem 10.1.6 to find a surjective isometry

$$\varphi\colon \mathcal{H} \to \mathcal{H}_K \quad \text{with} \quad \varphi \circ \eta = \gamma_K.$$

Then $\varphi \circ \gamma = \gamma_K + \varphi(\gamma(x_0))$ shows that the affine map $\psi(v) := \varphi(v) - \varphi(\gamma(x_0))$ satisfies $\psi \circ \gamma = \gamma_K$. □

Combining Theorem 10.1.6 with Lemma 10.1.3, we obtain:

Corollary 10.1.7. (Schoenberg) *A real-valued kernel Q on X vanishing on the diagonal is negative definite if and only if there exists a map $\gamma\colon X \to \mathcal{H}$ to some real Hilbert space $\mathcal{H}$ satisfying*

$$Q(x,y) = \|\gamma(x) - \gamma(y)\|^2, \quad x, y \in X.$$

Corollary 10.1.8. *A metric space (X,d) can be embedded isometry into a real Hilbert space if and only if the kernel $Q(x,y) := d(x,y)^2$ on X is negative definite.*

Theorem 10.1.9. (Schoenberg) *A kernel Q on $X \times X$ is negative definite if and only if, for each $t > 0$, the kernel e^{-tQ} is positive definite.*

Proof. Suppose first that Q is negative definite. In Remark 10.1.2(d), we have seen that $Q = Q_f - K$ holds for a function $f\colon X \to \mathbb{C}$ and a positive definite kernel K on X. Then the positive definiteness of

$$e^{-tQ}(x,y) = e^{-tQ_f(x,y)}e^{tK(x,y)} = e^{-tf(x)}\overline{e^{-tf(y)}}e^{tK(x,y)}$$

follows from the positive definiteness of e^{tK} (Corollary 3.2.2) and Remark 3.3.1(b).

If, conversely, e^{-tQ} is positive definite for any $t > 0$, and $E(x,y) := 1 = Q_{1/2}(x,y)$ is the constant kernel, then E is negative definite, and therefore

$$\frac{1}{t}(E - e^{-tQ})$$

is negative definite. This implies that the pointwise limit

$$Q = \lim_{t\to 0} \frac{1}{t}(E - e^{-tQ})$$

is also negative definite. □

Exercises for Section 10.1

Exercise 10.1.1. Show that for each $a \in \mathbb{R}$ the kernel $Q_a(x,y) := (a + x - y)^2$ on $\mathbb{R}$ satisfies the inequalities

$$\sum_{j,k=1}^{n} c_j c_k Q_a(x_j, x_k) \leq 0$$

for $x_1, \ldots, x_n \in \mathbb{R}$ and $c_1, \ldots, c_n \in \mathbb{R}$ with $\sum_j c_j = 0$. Nevertheless, Q_a is negative definite only for $a = 0$.

Chapter 11

Hilbert–Schmidt and Trace Class Operators

In this chapter, we collect some facts on Hilbert–Schmidt and trace class operators.

11.1 Hilbert–Schmidt Operators

For the following lemma, we note that for a finite rank operator on any vector space V, its trace is well defined by identifying the finite rank operators with $V \otimes V^*$ and the trace with the functional

$$\mathrm{tr}\colon V \otimes V^* \to \mathbb{C}, \quad x \otimes \alpha \mapsto \alpha(x).$$

Lemma 11.1.1. *For $x, y \in \mathcal{H}$, we write $P_{x,y}$ for the operator given by $P_{x,y}(v) = \langle v, y\rangle x$ and put $P_x := P_{x,x}$. Then the following assertions hold:*

(i) $\mathrm{tr}\, P_{x,y} = \langle x, y\rangle$.

(ii) $P_{x,y}^* = P_{y,x}$.

(iii) $P_{x,y}P_{z,w} = \langle z, y\rangle P_{x,w}$.

(iv) $AP_{x,y}B^* = P_{Ax,By}$ *for* $A, B \in B(\mathcal{H})$.

(v) $P_{Ax} = AP_xA^*$.

(vi) *If* $A = \sum_{j=1}^n \lambda_j P_{v_j,w_j}$, *where the finite sequences* $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ *are orthonormal, then* $\|A\| = \max\{|\lambda_j| \colon j = 1, \ldots, n\}$.

Proof. The simple proof of (i)–(v) is left to the reader.

(vi) From $Aw_j = \lambda_j v_j$ we obtain $\|A\| \geq |\lambda_j|$ for all $j = 1, \ldots, n$. Conversely, we obtain for $L := \max\{|\lambda_j| \colon j = 1, \ldots, n\}$ with

$$A^*A = \sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} P^*_{v_k,w_k} P_{v_j,w_j} = \sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} P_{w_k,v_k} P_{v_j,w_j}$$
$$= \sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} \langle v_j, v_k\rangle P_{w_k,w_j} = \sum_{j=1}^{n} |\lambda_j|^2 P_{w_j}$$

that $\|Av\|^2 = \langle A^*Av, v\rangle \leq \sum_{j=1}^n |\lambda_j|^2 |\langle v, w_j\rangle|^2 \leq L^2 \sum_{j=1}^n |\langle v, w_j\rangle|^2 \leq L^2\|v\|^2$, and hence $\|A\| \leq L$. Thus $\|A\| = L$. ☐

Lemma 11.1.2. *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, $(e_j)_{j\in J}$ an orthonormal basis in $\mathcal{H}$, and $(f_k)_{k\in K}$ an orthonormal basis in $\mathcal{K}$. For $A \in B(\mathcal{H}, \mathcal{K})$, we then have*

$$\sum_{j\in J} \|Ae_j\|^2 = \sum_{k\in K} \|A^*.f_k\|^2.$$

Proof. $\sum_j \|Ae_j\|^2 = \sum_{j,k} |\langle Ae_j, f_k\rangle|^2 = \sum_{j,k} |\langle e_j, A^*.f_k\rangle|^2 = \sum_k \|A^*.f_k\|^2$. ☐

Definition 11.1.3. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $(e_j)_{j\in J}$ an orthonormal basis. An operator $A \in B(\mathcal{H}, \mathcal{K})$ is called a *Hilbert–Schmidt operator* if

$$\|A\|_2 := \Big(\sum_{j\in J} \|Ae_j\|^2\Big)^{\frac{1}{2}} < \infty.$$

In view of Lemma 11.1.2, the preceding expression does not depend on the choice of the orthonormal basis in $\mathcal{H}$. We write $B_2(\mathcal{H}, \mathcal{K})$ for the space of Hilbert–Schmidt operators in $B(\mathcal{H}, \mathcal{K})$, $B_2(\mathcal{H})$ for the Hilbert–Schmidt operators in $B(\mathcal{H})$, and $B_{\mathrm{fin}}(\mathcal{H}) = \mathrm{span}\{P_{x,y} \colon x, y \in \mathcal{H}\}$ for the space of continuous finite rank operators on $\mathcal{H}$.

Proposition 11.1.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces.*

(i) *For $A \in B_2(\mathcal{H}, \mathcal{K})$, we have $\|A\| \leq \|A\|_2 = \|A^*\|_2$.*

(ii) *If $A, B \in B_2(\mathcal{H}, \mathcal{K})$ and $(e_j)_{j\in J}$ is an orthonormal basis of $\mathcal{H}$, then*

$$\langle A, B\rangle := \sum_j \langle B^*Ae_j, e_j\rangle$$

converges and defines the structure of a complex Hilbert space on $B_2(\mathcal{H}, \mathcal{K})$ such that $\langle A, A\rangle = \|A\|_2^2$.

(iii) *$\langle A, B\rangle$ as in* (ii) *does not depend on the chosen basis.*

If $\mathcal{H} = \mathcal{K}$, then

(iv) *$\langle A, B\rangle = \langle B^*, A^*\rangle$ for $A, B \in B_2(\mathcal{H})$.*

(v) *If* $A \in B(\mathcal{H})$ *and* $B, C \in B_2(\mathcal{H})$, *then* $AB \in B_2(\mathcal{H})$ *with*

$$\|AB\|_2 \leq \|A\| \cdot \|B\|_2 \quad \text{and} \quad \langle AB, C\rangle = \langle B, A^*C\rangle.$$

(vi) *Hilbert–Schmidt operators are compact, i.e.,* $B_2(\mathcal{H}) \subseteq K(\mathcal{H})$.

Proof. (i) The relation $\|A\|_2 = \|A^*\|_2$ is immediate from the proof of Lemma 11.1.2. To prove that $\|A\| \leq \|A\|_2$, let $\varepsilon > 0$ and $(e_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}$ such that $\|Ae_{j_0}\| \geq \|A\| - \varepsilon$ for an element $j_0 \in J$. Then

$$\|A\|_2^2 = \sum_{j\in J} \|Ae_j\|^2 \geq (\|A\| - \varepsilon)^2.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $\|A\|_2 \geq \|A\|$.

(ii) One easily checks that the subspace

$$E := \Big\{(v_j)_{j\in J} \in \mathcal{K}^J \colon \sum_j \|v_j\|^2 < \infty\Big\}$$

is a Hilbert space with scalar product given by $\langle v, w\rangle := \sum_{j\in J}\langle v_j, w_j\rangle$. Further it is clear from the definition that $\Phi\colon B_2(\mathcal{H},\mathcal{K}) \to E, A \mapsto (Ae_j)_{j\in J}$ is an isometric embedding. That Φ is surjective follows from the observation that for $v \in E$ and $x \in \mathcal{H}$ the prescription $Ax := \sum_{j\in J}\langle x, e_j\rangle v_j$ defines a bounded operator $\mathcal{H} \to \mathcal{K}$ with

$$\|Ax\|^2 \leq \sum_{j\in J} |\langle x, e_j\rangle|^2 \sum_j \|v_j\|^2 = \|x\|^2\|v\|^2$$

(Cauchy–Schwarz inequality). Hence $A \in B_2(\mathcal{H},\mathcal{K})$ with $\Phi(A) = v$. This shows that Φ is an isometric bijection, and therefore that $B_2(\mathcal{H},\mathcal{K})$ is a Hilbert space with scalar product given by $\langle A, B\rangle = \sum_{j\in J}\langle Ae_j, Be_j\rangle = \sum_{j\in J}\langle B^*Ae_j, e_j\rangle$.

(iii) This follows from the fact that the scalar product on the Hilbert space $B_2(\mathcal{H},\mathcal{K})$ is uniquely determined by the norm via the polarization identity.

(iv) We know already that $A \mapsto A^*$ is an isometry of $B_2(\mathcal{H})$. Hence both sides in (iv) are hermitian forms on $B_2(\mathcal{H})$ which define the same norm. Since the scalar product is uniquely determined by the norm, the assertion follows.

(v) The first part follows from

$$\|AB\|_2^2 = \sum_{j\in J} \|ABe_j\|^2 \leq \sum_{j\in J} \|A\|^2\|Be_j\|^2 = \|A\|^2\|B\|_2^2.$$

For the second part, we calculate

$$\langle AB, C\rangle = \sum_{j\in J}\langle C^*ABe_j, e_j\rangle = \sum_{j\in J}\langle (A^*C)^*Be_j, e_j\rangle = \langle B, A^*C\rangle.$$

(vi) If $A \in B(\mathcal{H},\mathcal{K})$ is a Hilbert–Schmidt operator and $(e_j)_{j\in J}$ is an orthonormal basis of $\mathcal{H}$, then for each finite subset $F \subseteq J$ we consider the operator A_F with $A_F.e_j = Ae_j$ for $j \in F$ and $A_F.e_j = 0$ otherwise. Then the net $(A_F)_{F\subseteq J}$ converges to A in $B_2(\mathcal{H},\mathcal{K})$ and hence, in particular, with respect to the operator norm. Therefore A is a limit of finite rank operators, hence compact. $\square$

11.2 Trace Class Operators

Definition 11.2.1. We say that an operator $A \in B(\mathcal{H})$ is of *trace class* if $A \in B_2(\mathcal{H})$ and

$$\|A\|_1 := \sup\{|\langle A, B\rangle| \colon B \in B_2(\mathcal{H}), \|B\| \leq 1\}$$

is finite. We write $B_1(\mathcal{H}) \subseteq B_2(\mathcal{H})$ for the subspace of trace class operators. It follows easily from the definition that $\|\cdot\|_1$ defines a norm on $B_1(\mathcal{H})$.

Proposition 11.2.2. *The following assertions hold:*

(i) *If $A \in B(\mathcal{H})$ and $B \in B_1(\mathcal{H})$, then $AB \in B_1(\mathcal{H})$ with $\|AB\|_1 \leq \|A\| \cdot \|B\|_1$.*

(ii) *$\|A\|_2 \leq \|A\|_1$ for $A \in B_1(\mathcal{H})$.*

(iii) *$B_1(\mathcal{H})$ is invariant under taking adjoints, and $\|A^*\|_1 = \|A\|_1$.*

(iv) *$B_2(\mathcal{H})B_2(\mathcal{H}) \subseteq B_1(\mathcal{H})$.*

(v) *$\|P_{x,y}\| = \|P_{x,y}\|_1 = \|x\| \cdot \|y\|$ for $x, y \in \mathcal{H}$.*

Proof. (i) This follows from $|\langle AB, X\rangle| = |\langle B, A^*X\rangle| \leq \|B\|_1\|A\| \cdot \|X\|$ for $X \in B_2(\mathcal{H})$ with $\|X\| \leq 1$.

(ii) From $\|X\| \leq \|X\|_2$ for $X \in B_2(\mathcal{H})$ (Proposition 11.1.4(i)), it follows that

$$\{X \in B_2(\mathcal{H}) \colon \|X\| \leq 1\} \supseteq \{X \in B_2(\mathcal{H}) \colon \|X\|_2 \leq 1\}.$$

Hence the assertion follows from $\|A\|_2 = \sup\{|\langle A, X\rangle| \colon \|X\|_2 \leq 1\}$.

(iii) From $|\langle A^*, X\rangle| = |\langle X^*, A\rangle| = |\langle A, X^*\rangle|$ and the fact that $X \mapsto X^*$ is an isometry of $B(\mathcal{H})$ and $B_2(\mathcal{H})$, we see that $A^* \in B_1(\mathcal{H})$ with $\|A^*\|_1 = \|A\|_1$.

(iv) If $A = BC$ with $B, C \in B_2(\mathcal{H})$, then we have for $X \in B_2(\mathcal{H})$ the estimate

$$|\langle A, X\rangle| = |\langle BC, X\rangle| = |\langle C, B^*X\rangle| \leq \|C\|_2\|B^*X\|_2 \leq \|C\|_2\|B\|_2\|X\|.$$

Hence $A \in B_1(\mathcal{H})$ with $\|A\|_1 \leq \|B\|_2\|C\|_2$.

(v) For $A \in B_2(\mathcal{H})$, we have

$$\langle A, P_{x,y}\rangle = \operatorname{tr}(AP_{x,y}^*) = \operatorname{tr}(AP_{y,x}) = \operatorname{tr} P_{Ay,x} = \langle Ay, x\rangle,$$

hence $\|P_{x,y}\| \leq \|P_{x,y}\|_1 \leq \|x\| \cdot \|y\|$. Moreover, $\|P_{x,y}(y)\| = \|y\|^2\|x\|$ shows that $\|P_{x,y}\| \geq \|x\| \cdot \|y\|$. This proves the desired equality. □

Proposition 11.2.3. *Let $(e_j)_{j\in J}$ be an orthonormal basis and $A \in B_1(\mathcal{H})$. Then the sum*

$$\operatorname{tr} A := \sum_{j\in J} \langle Ae_j, e_j\rangle$$

converges absolutely and has the following properties:

(i) $|\operatorname{tr} A| \leq \|A\|_1$, *i.e.,* tr *is a continuous linear functional on* $B_1(\mathcal{H})$ *and it is independent of the chosen orthonormal basis.*

(ii) $\langle A, B\rangle = \operatorname{tr}(AB^*)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ *for* $A, B \in B_2(\mathcal{H})$.

(iii) *For* $A \in B_1(\mathcal{H})$, *the function* $X \mapsto \operatorname{tr}(XA)$ *on* $B(\mathcal{H})$ *is continuous and extends the linear functional* $X \mapsto \langle X, A^*\rangle$ *on* $B_2(\mathcal{H})$. *Moreover,* $\operatorname{tr}(AX) = \operatorname{tr}(XA)$.

(iv) *Each* $A \in B_1(\mathcal{H})$ *can be written as* $A = \sum_{n=1}^\infty P_{v_n,w_n}$, *where* $(v_n)_{n\in\mathbb{N}}$ *and* $(w_n)_{n\in\mathbb{N}}$ *are orthogonal sequences. Then* $\|A\|_1 = \sum_{n=1}^\infty \|v_n\| \cdot \|w_n\|$ *and* $\operatorname{tr} A = \sum_{n=1}^\infty \langle v_n, w_n\rangle$.

(v) $B_{\mathrm{fin}}(\mathcal{H})$ *is dense in* $B_1(\mathcal{H})$.

(vi) $B_1(\mathcal{H}) \cong K(\mathcal{H})'$ *and* $B_1(\mathcal{H})' \cong B(\mathcal{H})$, *where the pairings are given by the bilinear form* $(A, B) \mapsto \operatorname{tr}(AB)$.

(vii) $\big(B_1(\mathcal{H}), \|\cdot\|_1\big)$ *is a Banach space.*

(viii) *The ultraweak operator topology on* $B(\mathcal{H})$ *coincides with the weak-$*$-topology with respect to the identification* $B(\mathcal{H}) \cong B_1(\mathcal{H})'$.

(ix) *If* $(e_j)_{j\in J}$ *is an orthonormal basis, then* $\|A\|_1 \leq \sum_{i,j\in J} |\langle Ae_i, e_j\rangle|$ *holds for all* $A \in B(\mathcal{H})$.

Proof. Let $J_0 \subseteq J$ be a finite subset and $\lambda_j, j \in J_0$, be complex numbers with $|\lambda_j| = 1$ and $\lambda_j\langle Ae_j, e_j\rangle = |\langle Ae_j, e_j\rangle|$. Then, in view Lemma 11.1.1(vi),

$$\sum_{j\in J_0} |\langle Ae_j, e_j\rangle| = \sum_{j\in J_0} \lambda_j\langle Ae_j, e_j\rangle = \langle A, \sum_{j\in J_0} \lambda_j P_{e_j}\rangle \leq \|A\|_1 \| \sum_{j\in J_0} \lambda_j P_{e_j}\| \leq \|A\|_1.$$

This proves the estimate under (i) and the absolute convergence of the series. To see that $\operatorname{tr} A$ does not depend on the chosen basis, let $(f_k)_{k\in K}$ be another basis and calculate

$$\begin{aligned}\sum_k \langle Af_k, f_k\rangle &= \sum_{k,j} \langle Af_k, e_j\rangle\langle e_j, f_k\rangle = \sum_{k,j} \langle f_k, A^*e_j\rangle\langle e_j, f_k\rangle \\ &= \sum_j \langle e_j, A^*e_j\rangle = \sum_j \langle Ae_j, e_j\rangle.\end{aligned}$$

(ii) The first part is precisely Proposition 11.1.4(ii). The second follows from Proposition 11.1.4(iv):

$$\operatorname{tr}(AB) = \langle A, B^*\rangle = \langle B, A^*\rangle = \operatorname{tr}(BA).$$

(iii) For $A \in B_1(\mathcal{H})$ and $B \in B(\mathcal{H})$, we use Proposition 11.2.2(i) to see that $AB \in B_1(\mathcal{H})$ with $|\operatorname{tr}(XA)| \leq \|XA\|_1 \leq \|X\| \cdot \|A\|_1$. This proves the first part of (iii).

To show that $\mathrm{tr}(AX) = \mathrm{tr}(XA)$ holds for $A \in B_1(\mathcal{H})$ and $X \in B(\mathcal{H})$, we note that, since both sides define complex bilinear forms, we may assume that A is hermitian. Then the fact that A is compact (Proposition 11.1.4(vi)) shows that there exists an orthogonal basis consisting of eigenvectors for A. Thus we may assume that $Ae_j = \lambda_j e_j$. Then

$$\begin{aligned}\mathrm{tr}(AX) &= \sum_j \langle AX.e_j, e_j\rangle = \sum_j \langle X.e_j, Ae_j\rangle = \sum_j \lambda_j \langle X.e_j, e_j\rangle \\ &= \sum_j \langle XAe_j, e_j\rangle = \mathrm{tr}(XA).\end{aligned}$$

(iv) Since A is compact, it can be written as $A = \sum_{n=1}^\infty \lambda_n P_{v_n,w_n}$, as required (cf. [We76, Satz 7.6]). Now Proposition 11.2.2 yields $\|A\|_1 \leq \sum_{n=1}^\infty \|v_n\| \cdot \|w_n\|$.

To obtain the converse estimate, we consider the operator $X_n = \sum_{j=1}^n c_j P_{v_j,w_j}$, where $c_j = \frac{1}{\|v_j\|\cdot\|w_j\|}$ if $P_{v_j,w_j} \neq 0$. Then $\|X\| \leq 1$ follows from Lemma 11.1.1(vi). Moreover, we have

$$\|A\|_1 \geq \langle A, X_n\rangle = \sum_{j=1}^n \langle v_j, X_n.w_j\rangle = \sum_{j=1}^n c_j \|v_j\|^2 \|w_j\|^2 = \sum_{j=1}^n \|v_j\| \cdot \|w_j\|.$$

Since n was arbitrary, we obtain $\|A\|_1 \geq \sum_{j=1}^\infty \|v_j\|\cdot\|w_j\|$ and therefore equality. It follows, in particular, that $A = \lim_{m\to\infty} A_m$ with $A_m = \sum_{n=1}^m P_{v_n,w_n}$ because $\|A - A_m\| = \sum_{j>m} \|v_j\| \cdot \|w_j\|$. Therefore

$$\mathrm{tr}\, A = \lim_{m\to\infty} \mathrm{tr}\, A_m = \lim_{m\to\infty} \sum_{j=1}^m \langle v_j, w_j\rangle = \sum_{j=1}^\infty \langle v_j, w_j\rangle.$$

(v) Since $B_1(\mathcal{H})$ is invariant under taking adjoints, it suffices to show that each symmetric element A in $B_1(\mathcal{H})$ can be approximated by finite rank operators with respect to $\|\cdot\|_1$. We write $A = \sum_{n=1}^\infty \lambda_n P_{v_n}$, where $(v_n)_{n\in\mathbb{N}}$ is an orthonormal system and conclude as in (iv) that $A_n \to A$.

(vi) The continuity of the pairing $B_1(\mathcal{H}) \times B(\mathcal{H}) \to \mathbb{C}, (A, B) \mapsto \mathrm{tr}(AB)$ follows from $|\mathrm{tr}(AB)| \leq \|AB\|_1 \leq \|A\|_1\|B\|$. First we show that this pairing yields an isomorphism of $B_1(\mathcal{H})$ with $K(\mathcal{H})'$. So let $f \in K(\mathcal{H})'$. Then $f|_{B_2(\mathcal{H})}$ is a linear functional with $|f(X)| \leq \|f\|\cdot\|X\| \leq \|f\|\cdot\|X\|_2$ (Proposition 11.1.4(i)), hence can be represented by an element $Y \in B_2(\mathcal{H})$. Then $f(X) = \langle X, Y\rangle = \mathrm{tr}(XY^*)$ holds for all $X \in B_2(\mathcal{H})$, and with $|f(X)| \leq \|f\| \cdot \|X\|$ we obtain $Y \in B_1(\mathcal{H})$ with $\|Y\|_1 = \|Y^*\|_1 \leq \|f\|$. The converse follows from the density of $B_{\mathrm{fin}}(\mathcal{H}) \subseteq B_2(\mathcal{H})$ in $K(\mathcal{H})$.

Next we show that $B_1(\mathcal{H})' \cong B(\mathcal{H})$. So we have to represent each continuous linear functional f on $B_1(\mathcal{H})$ by a bounded linear operator on $\mathcal{H}$. From Proposition 11.2.2 we recall that $\|P_{v,w}\|_1 = \|v\| \cdot \|w\|$. Therefore, for each $w \in \mathcal{H}$, the mapping $v \mapsto f(P_{v,w})$ is continuous and linear, hence can be represented by a vector a_w in the sense that $f(P_{v,w}) = \langle v, a_w\rangle$ holds for all $v \in \mathcal{H}$. Moreover, the above calculation shows that $\|a_w\| \leq \|f\| \cdot \|w\|$. Since the assignment $w \mapsto a_w$

is linear, we find a bounded operator A on $\mathcal{H}$ with $Aw = a_w$ for all $w \in \mathcal{H}$ and $\|A\| \leq \|f\|$. Now $f(P_{v,w}) = \langle v, Aw\rangle = \langle P_{v,w}, A\rangle$ holds for $v, w \in \mathcal{H}$. From that we obtain $f(X) = \operatorname{tr}(XA^*)$ for $X \in B_{\text{fin}}(\mathcal{H})$ and since, in view of (v), $B_{\text{fin}}(\mathcal{H})$ is dense in $B_1(\mathcal{H})$, we obtain $f(X) = \operatorname{tr}(XA^*)$ for all $X \in B_1(\mathcal{H})$. This proves (vi).

(vii) Since $B_1(\mathcal{H}) \cong K(\mathcal{H})'$ follows from (vi), the completeness of $B_1(\mathcal{H})$ follows from the fact that dual spaces of normed spaces are Banach spaces.

(viii) Let $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ satisfying $\sum_{n=1}^\infty \|v_n\|^2 < \infty$ and $\sum_{n=1}^\infty \|w_n\|^2 < \infty$. Then $\|P_{v_n,w_n}\|_1 = \|v_n\|\cdot\|w_n\|$ (Proposition 11.2.2(v)) implies that the series $A := \sum_{n=1}^\infty P_{v_n,w_n}$ converges absolutely in $B_1(\mathcal{H})$. Therefore

$$\sum_{n=1}^\infty \langle X.v_n, w_n\rangle = \sum_{n=1}^\infty \operatorname{tr}(XP_{v_n,w_n}) = \operatorname{tr}(XA).$$

This proves that each ultraweakly continuous linear functional on $B(\mathcal{H})$ is also weak-$*$-continuous.

If, conversely, $f(X) = \operatorname{tr}(AX)$ with $A \in B_1(\mathcal{H})$, then we want to show that f is ultraweakly continuous. Writing $A = B_+ - B_- + i(C_+ - C_-)$, where B_+, B_-, C_+ and C_- are positive trace class operators, we may assume that A is positive. Then $A = \sum_{n=1}^\infty P_{u_n}$, where $(u_n)_{n\in\mathbb{N}}$ is an orthogonal sequence consisting of eigenvectors of eigenvalue $\|u_n\|^2$, and, in view of (iv), $\|A\|_1 = \sum_{n=1}^\infty \|u_n\|^2 < \infty$. We conclude that $f(X) = \operatorname{tr}(XA) = \sum_{n=1}^\infty \langle X.u_n, u_n\rangle$ with $\sum_{n=1}^\infty \|u_n\|^2 = \|A\|_1 < \infty$. Hence f is ultraweakly continuous. Now the assertion follows from the fact that the weak-$*$-topology and the ultraweak topology are the coarsest topology for which the same set of linear functionals is continuous.

(ix) We may assume that the sum $\sum_{i,j} |\langle Ae_i, e_j\rangle|$ exists. In view of $|\langle Ae_i, e_j\rangle| \leq \|A\|$, this implies that $\sum_{i,j} |\langle Ae_i, e_j\rangle|^2 < \infty$, i.e., $A \in B_2(\mathcal{H})$.

Let $B = \sum_{i,j\in J} b_{i,j} P_{e_j,e_i}$ be a finite sum of the operators P_{e_j,e_i}. Then $B \in B_2(\mathcal{H})$, $|b_{i,j}| \leq \|B\|$ for all i, j, and

$$|\operatorname{tr}(AB)| \leq \sum_{i,j} |b_{i,j} \operatorname{tr}(AP_{e_i,e_j})| \leq \sum_{i,j} |b_{i,j}| \cdot |\langle Ae_i, e_j\rangle| \leq \|B\| \sum_{i,j} |\langle Ae_i, e_j\rangle|.$$

This prove that whenever the sum $\sum_{i,j} |\langle Ae_i, e_j\rangle|$ exists, then $A \in B_1(\mathcal{H})$ with $\|A\|_1 \leq \sum_{i,j} |\langle Ae_i, e_j\rangle|$. □

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