Fachbereich Mathematik
Prof. Dr. Karl-Hermann Neeb
Dipl.-Math. Hasan Gündoğan

TECHNISCHE UNIVERSITÄT DARMSTADT

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Exercise Sheet no. 7 of "Topology"

Exercise E64

If $f\colon X \to Y$ is a map between topological spaces and $X = X_1 \cup \ldots \cup X_n$ holds with closed subsets $X_1, \ldots, X_n$, then f is continuous if and only if all restrictions $f|_{X_i}$ are continuous.

Exercise E65

Show that the homotopy relation on $P(X, x_0, x_1)$ is an equivalence relation. Hint: Exercise E64 helps to glue homotopies.

Exercise E66

Show that for $n \geq 2$ the sphere $\mathbb{S}^n$ is simply connected. For the proof, proceed along the following steps:
(a) Let $\gamma : [0, 1] \to \mathbb{S}^n$ be continuous. Then there exists an $m \in \mathbb{N}$ such that $\|\gamma(t) - \gamma(t')\| < \frac{1}{2}$ for $|t - t'| < \frac{1}{m}$.
(b) Define $\widetilde{\alpha} : [0, 1] \to \mathbb{R}^{n+1}$ as the piecewise affine curve with $\widetilde{\alpha}(\frac{k}{m}) = \gamma(\frac{k}{m})$ for $k = 0, \ldots, m$. Then $\alpha(t) := \frac{1}{\|\widetilde{\alpha}(t)\|}\widetilde{\alpha}(t)$ defines a continuous curve $\alpha : [0, 1] \to \mathbb{S}^n$.
(c) $\alpha \sim \gamma$. Hint: Consider $H(t, s) := \frac{(1-s)\gamma(t)+s\alpha(t)}{\|(1-s)\gamma(t)+s\alpha(t)\|}$.
(d) α is not surjective. The image of α is the central projection of a polygonal arc on the sphere.
(e) If $\beta \in \Omega(\mathbb{S}^n, y_0)$ is not surjective, then $\beta \sim y_0$ (it is homotopic to a constant map). Hint: Let $p \in \mathbb{S}^n \setminus \operatorname{im} \beta$. Using stereographic projection, where p corresponds to the point at infinity, show that $\mathbb{S}^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$, hence contractible.
(f) $\pi_1(\mathbb{S}^n, y_0) = \{[y_0]\}$ for $n \geq 2$ and $y_0 \in \mathbb{S}^n$.

Exercise E67

Let X be a topological space, $x_0, x_1 \in X$ and $\alpha \in P(X, x_0, x_1)$ a path from x_0 to x_1. Show that the map

$$C\colon \pi_1(X, x_1) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]$$

is an isomorphism of groups. In this sense the fundamental group does not depend on the base point if X is arcwise connected.

Exercise E68

Let $\sigma\colon G \times X \to X$ be a continuous action of the topological group G on the topological space X and $x_0 \in X$. Then the orbit map $\sigma^{x_0}\colon G \to X, g \mapsto \sigma(g, x_0)$ defines a group homomorphism

$$\pi_1(\sigma^{x_0})\colon \pi_1(G) \to \pi_1(X, x_0).$$

Show that the image of this homomorphism is central, i.e., lies in the center of $\pi_1(X, x_0)$. Hint: Mimic the argument in the proof of Lemma 6.1.8.

Exercise E69

Let $F\colon I^2 \to X$ be a continuous map with $F(0, s) = x_0$ for $s \in I$ and define

$$\gamma(t) := F(t, 0), \quad \eta(t) := F(t, 1), \quad \alpha(t) := F(1, t), \quad t \in I.$$

Show that $\gamma * \alpha \sim \eta$. Hint: Consider the map

$$G\colon I^2 \to I^2, \quad G(t,s) := \begin{cases} (2t, s) & \text{for } 0 \leq t \leq \frac{1}{2}, s \leq 1-2t, \\ (1, 2t-1) & \text{for } \frac{1}{2} \leq t \leq 1, s \leq 2t-1, \\ (t + \frac{1-s}{2}, s) & \text{else} \end{cases}$$

and show that it is continuous. Take a look at the boundary values of $F \circ G$.

Exercise E70

Let $q\colon G \to H$ be an morphism of topological groups with discrete kernel Γ. Show that:

(1) If $V \subseteq G$ is an open $\mathbf{1}$-neighborhood with $(V^{-1}V) \cap \Gamma = \{\mathbf{1}\}$ and q is open, then $q|_V\colon V \to q(V)$ is a homeomorphism.

(2) If q is open and surjective, then q is a covering.

(3) If q is open and H is connected, then q is surjective, hence a covering.

Exercise E71

A map $f\colon X \to Y$ between topological spaces is called a *local homeomorphism* if each point $x \in X$ has an open neighborhood U such that $f|_U\colon U \to f(U)$ is a homeomorphism onto an open subset of Y.

(1) Show that each covering map is a local homeomorphism.

(2) Find a surjective local homeomorphism which is not a covering. Can you also find an example where X is connected?

Exercise E72

Let X be a topological space. The *cone over* X is the space

$$C(X) := (X \times [0,1])/(X \times \{1\}).$$

Show that $C(X)$ is always contractible.

Exercise E73

(Hawaiian earring)

In the euclidean plane $\mathbb{R}^2$, we write

$$C_r(m) := \{x \in \mathbb{R}^2 \colon \|x - m\|_2 = r\}$$

for the circle of radius r and center m. Consider the union

$$X := \bigcup_{n \in \mathbb{N}} C_{\frac{1}{n}}\Big(\frac{1}{n}, 0\Big).$$

Show that X is arcwise connected but not semilocally simply connected. Hint: Consider the point $(0,0) \in X$.