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Exercise Sheet no. 6 of "Topology"

Exercise E52

If X is a compact topological space and $\mathcal{A} \subseteq C(X, \mathbb{R})$ is a subalgebra, then its closure also is a subalgebra. Hint: If $f_n \to f$ and $g_n \to g$ uniformly, then also $f_n + g_n \to f + g, \lambda f_n \to \lambda f$ and $f_n g_n \to fg$ uniformly.

Exercise E53

Let $[a, b] \subseteq \mathbb{R}$ be a compact interval. Show that the space

$$\mathcal{A} := \Big\{ f|_{[a,b]} \colon (\exists a_0, \ldots, a_n \in \mathbb{R}, n \in \mathbb{N})\ f(x) = \sum_{i=0}^{n} a_i x^i \Big\}$$

of polynomial functions on $[a, b]$ is dense in $C([a, b], \mathbb{R})$ with respect to $\| \cdot \|_\infty$.

Exercise E54

Let (Y, d) be a complete metric space and X a topological space.

(a) Let $\mathcal{B}(X, Y) := \big\{ f : X \to Y | \operatorname{diam}_d(f(X)) = \sup_{x,x' \in X} d(f(x), f(x')) < \infty \big\}$ be the space of bounded maps from X to Y. Show that $(\mathcal{B}(X, Y), d_\infty)$ is a complete metric space, where $d_\infty(f, g) := \sup_{x \in X} d(f(x), g(x))$.

(b) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of Y-valued continuous maps on X and $f \colon X \to Y$ a map with

$$d_\infty(f_n, f) \to 0,$$

i.e., the sequence f_n converges uniformly to f. Show that f is continuous.

(c) Now let X be compact. Conclude that $(C(X, Y), d_\infty)$ is a complete metric space. In particular, the normed vector space $(C(X, \mathbb{K}), \| \cdot \|_\infty)$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a Banach space.

Exercise E55

Alexandroff compactification / one point compactification

Let X be a locally compact space. Show that:

(i) There exists a compact topology on the set $X_\omega := X \cup \{\omega\}$, where ω is a symbol of a point not contained in X. Hint: A subset $O \subseteq X_\omega$ is open if it either is an open subset of X or $\omega \in O$ and $X \setminus O$ is compact.

(ii) The inclusion map $\eta_X \colon X \to X_\omega$ is a homeomorphism onto an open subset of X_ω.

(iii) If Y is a compact space and $f \colon X \to Y$ a continuous map which is a homeomorphism onto the complement of a point in Y, then there is a homeomorphism $F \colon X_\omega \to Y$ with $F \circ \eta_X = f$.

X_ω is called the *Alexandroff compactification* or the *one point compactification* of X.

Exercise E56

Stereographic projection

We consider the n-dimensional sphere

$$\mathbb{S}^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \colon x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$

We call the unit vector $e_0 := (1, 0, \ldots, 0)$ the *north pole* of the sphere and $-e_0$ the *south pole*. We then have the corresponding *stereographic projection maps*

$$\varphi_+ \colon U_+ := \mathbb{S}^n \setminus \{e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 - y_0} y \quad \text{and}$$

$$\varphi_- \colon U_- := \mathbb{S}^n \setminus \{-e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 + y_0} y.$$

(a) Show that φ_+ and φ_- are homeomorphisms with inverse maps

$$\varphi_\pm^{-1}(x) = \Big(\pm \frac{\|x\|_2^2 - 1}{\|x\|_2^2 + 1}, \frac{2x}{1 + \|x\|_2^2} \Big).$$

(b) Show that $(\mathbb{R}^n)_\omega$, the one-point compactification of $\mathbb{R}^n$, is diffeomorphic to $\mathbb{S}^n$.

Exercise E57

Show that the one-point compactification of an open interval $]a, b[\subseteq \mathbb{R}$ is homeomorphic to $\mathbb{S}^1$.

Exercise E58

Let X be a compact space and $A \subseteq X$ be a compact subset. The space X/A is defined as the topological quotient space $X/\sim$, defined by the equivalence relation $x \sim y$ if either $x = y$ or $x, y \in A$. This means that we are collapsing A to a point. Show that:

(i) X/A is compact. (The main point is to see that X/A is T_2 by Lemma 4.4.4, Proposition 4.1.6.)

(ii) X/A is homeomorphic to the one-point compactification of the locally compact space $X \setminus A$.

Exercise E59

Let $K \subseteq \mathbb{R}^n$ be a compact subset. Show that the space $\mathcal{A}$ consisting of all restrictions of polynomial functions

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{R}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

to K is dense in $C(X, \mathbb{R})$ with respect to $\|\cdot\|_\infty$.

Exercise E60

Let $\mathbb{S}^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ and

$$\mathcal{A} := \Big\{ f|_{\mathbb{S}^1} \colon (\exists a_0, \ldots, a_n \in \mathbb{C}, n \in \mathbb{N})\ f(z) = \sum_{j=0}^{n} a_n z^n \Big\}.$$

Show that $\mathcal{A}$ is not dense in $C(\mathbb{S}^1, \mathbb{C})$. Hint: Consider the function $f(z) := z^{-1}$ on $\mathbb{S}^1$ and try to approximate it by elements f_n of $\mathcal{A}$; then consider the complex path integrals $\int_{|z|=1} f_n(z)\, dz$. Why does the Stone–Weierstraß Theorem not apply?