Fachbereich Mathematik
Prof. Dr. Karl-Hermann Neeb
Dipl.-Math. Hasan Gündoğan

TECHNISCHE UNIVERSITÄT DARMSTADT

Summer Semester 2009
June 16, 17 & 23, 2009

Exercise Sheet no. 5 of "Topology"

Exercise E42

Let (X, d) be a metric space. A subset $S \subseteq X$ is called *bounded*, if

$$\operatorname{diam}_d(S) := \sup\{d(x, y) \colon x, y \in S\} < \infty.$$

(a) Show that every compact subset $C \subseteq X$ is bounded and closed.

(b) Give an example of a metric space with a bounded and closed subset which is not compact.

Exercise E43

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the topological space X. Show that, if $\lim_{n\to\infty} x_n = x$, then the set $\{x_n \colon n \in \mathbb{N}\} \cup \{x\}$ is compact.

Exercise E44

The Cantor Set as a product space

We consider the compact product space $\{0, 1\}^{\mathbb{N}}$, where $\{0, 1\}$ carries the discrete topology. The image C of the function

$$f \colon \{0, 1\}^{\mathbb{N}} \to \mathbb{R}, \quad f(x) := 2\sum_{n=1}^{\infty} \frac{x_n}{3^n}$$

is called the *Cantor set*. Show that:

(a) f is continuous and injective.

(b) $f \colon \{0, 1\}^{\mathbb{N}} \to C$ is a homeomorphism and C is compact.

(c) $C = \bigcap_{n\in\mathbb{N}} C_n$, where

$$C_1 = [0, 1] \setminus \left]\frac{1}{3}, \frac{2}{3}\right[= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right],$$

each C_n is a union of 2^n closed intervals of length $\frac{1}{3^n}$, and C_{n+1} arises from C_n by deleting in each interval of C_n the open middle third.

Exercise E45

Let (X, d) be a compact metric space. Show that:

(1) X is separable, i.e., X contains a countable dense subset.

(2) If Y is a metric space and $f \colon X \to Y$ is continuous, then f is *uniformly continuous*, i.e., for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x \in X$ we have: $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Exercise E46

Let X be a set and

$$\tau := \{\emptyset\} \cup \{A \subseteq X \colon |A^c| < \infty\}$$

be the cofinite topology introduced in Exercise E10. Show that (X, τ) is quasicompact.

Exercise E47

On the compact space $X := [0;1] \subseteq \mathbb{R}$, we consider the equivalence relation defined by $x \sim y$ if either $x = y$ or ($x = 0$ and $y = 1$) or ($x = 1$ and $y = 0$). Show that:

(a) The quotient space $[X] := X/\sim$ is Hausdorff and compact.

(b) $[X]$ is homeomorphic to $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$. Hint: Consider $f\colon [X] \to \mathbb{S}^1, f([t]) := e^{2\pi i t}$ and Proposition 4.1.6.

Exercise E48

For $n \in \mathbb{N}$ let the group $\mathbb{Z}^n$ act on $\mathbb{R}^n$ by addition: $\sigma(z, r) := z + r$ for $z \in \mathbb{Z}^n$, $r \in \mathbb{R}^n$.

(a) Show that the quotient space $\mathbb{R}^n/\mathbb{Z}^n$ corresponding to σ is Hausdorff.

(b) Why is the quotient map $q : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $r \mapsto r + \mathbb{Z}^n$ open?

(c) Show that $[0;1]^n$ intersects every equivalence class in $\mathbb{R}^n/\mathbb{Z}^n$ non-trivially. Conclude with Exercise E29(b) that $\mathbb{R}^n/\mathbb{Z}^n$ is compact.

(d) Let $\mathbb{T}^n := \left(\mathbb{S}^1\right)^n$ be the n-fold topological product of the circle, the *n-dimensional torus*. Show that $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to $\mathbb{T}^n$.

Exercise E49

Let $\mathcal{H} := \prod_{n\in\mathbb{N}} \left[-\frac{1}{n}; \frac{1}{n}\right] \subseteq \ell^2(\mathbb{N}, \mathbb{R}) = \left\{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_2 := \sqrt{\sum_{n=1}^{\infty} x_n^2} < \infty\right\}$ be the *Hilbert cube*. Show that the inclusion $j : \mathcal{H} \to \ell^2(\mathbb{N}, \mathbb{R})$, $x \mapsto x$ is a topological embedding, i.e. the corestriction of j onto its image is a homeomorphism. Here, $\mathcal{H}$ is given the the product topology and $\ell^2(\mathbb{N}, \mathbb{R})$ the topology associated to the metric $d(x, y) := \|x - y\|_2$.

Exercise E50

Let X be a locally compact space and $Y \subseteq X$ be a subset. Show that Y is locally compact with respect to the subspace topology if and only if there exists an open subset $O \subseteq X$ and a closed subset A with $Y = O \cap A$. Hint: If Y is locally compact, write it as a union of compact subsets of the form $O_i \cap Y$, O_i open in X, where $O_i \cap Y$ has compact closure, contained in Y. Then put $O := \bigcup_{i\in I} O_i$ and $A := \overline{Y \cap O}$.

Exercise E51

Show that a locally compact space is regular, i.e., a T_3-space. Hint:Urysohn's Theorem.