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Exercise Sheet no. 4 of "Topology"

Note: Exercise E37, E38, E41 are essentially identical to E29, E30, E31, respectively.

Exercise E32

Consider the two element set $X = \{x, y\}$, endowed with the indiscrete topology. Show that $\mathcal{F} = \{\{x\}, \{x, y\}\}$ is a filter on X converging to x and y. This shows that limits of filters need not be unique.

Exercise E33

Show that a topological space X is separated if and only if each filter $\mathcal{F}$ on X converges at most to one point.

Exercise E34

Let X be a finite set. Show that for each ultrafilter $\mathcal{U}$ on X there exists a point $x \in X$ with $\mathcal{U} = \{A \subseteq X : x \in A\}$.

Exercise E35

Let X be a topological space and $p \in X$.

(a) Let $(x_i)_{i\in I} \subseteq X$ be a net. For each $i \in I$, let $F_i := \{x_j : j \geq i\}$.
- Show that the F_i form a filter basis $\mathcal{F}$ on X.
- Show that $x_i \to p$ if and only if $\mathcal{F} \to p$.

(b) Let $\mathcal{F} \subseteq \mathbb{P}(X)$ be a filter basis. Define the relation $\preceq$ on $\mathcal{F}$ by $U \succeq V :\Leftrightarrow U \subseteq V$.
- Show that $(\mathcal{F}, \preceq)$ is a directed partially ordered set. So any choice $(x_U)_{U\in\mathcal{F}} \in \prod_{U\in\mathcal{F}} U$ determines a net $(x_U)_{U\in\mathcal{F}} \subseteq X$.
- Show that $\mathcal{F} \to p$ if and only if $x_U \to p$ for all choices $(x_U)_{U\in\mathcal{F}} \in \prod_{U\in\mathcal{F}} U$.

Exercise E36

Let X be a topological space and $A \subseteq X$ a subset.

(a) Show that
$$\overline{A} = \{\, x \in X \mid \text{There is a net } (x_i)_{i\in I} \subseteq A \text{ with } x_i \to x \in X.\}$$

(b) What happens with the statement in (a) when you replace the word "net" by "sequence"?

(c) Show that
$$\overline{A} = \{ x \in X \mid \text{There is a filter basis } \mathcal{F} \subseteq \mathbb{P}(A) \text{ with } \mathcal{F} \to x \in X.\}$$

Exercise E37

Let (X, τ) be a topological space, $\sim$ be an equivalence relation on X, $q \colon X \to [X] := X/\sim = \{\, [x]_\sim = \{\, y \in X \mid y \sim x\} \mid x \in X\}$ be the quotient map, and endow $[X]$ with the quotient topology.

(a) Show that, if $f : X \to Y$ is a continuous map satisfying

$$x \sim y \quad \Rightarrow \quad f(x) = f(y) \qquad \forall x, y \in X,$$

then there exists a unique continuous map $\overline{f} \colon [X] \to Y$ with $f = \overline{f} \circ q$.

(b) Assume there is a subset $Z \subseteq X$ such that for all $x \in X$ we have $[x]_\sim \cap Z \neq \emptyset$. Write $[Z]' := \{ [x]'_\sim := \{ y \in Z | \, y \sim x \} | \, x \in Z \}$ and endow $[Z]'$ with the quotient topology. Show that the map $\varphi : [Z]' \to [X]$, $[x]'_\sim \mapsto [x]_\sim$ is a well-defined continuous bijection. Also show that, if $q : X \to [X]$ is an open map, then $\varphi : [Z]' \to [X]$ is a homeomorphism.

Exercise E38

Let $(G, \cdot, 1)$ be a group, (X, τ) be a topological space and $\sigma : G \times X \to X$, $(g, x) \mapsto \sigma_g(x) =: g.x$ a *group action*, i.e. $G \to S(X)$, $g \mapsto \sigma_g$ is a morphism to the symmetric group of X.

For $x \in X$ the set $\mathcal{O}_x := \{ g.x | \, g \in G \}$ is called the *orbit* of x with respect to σ and we write $X/G := \{ \mathcal{O}_x | \, x \in X \}$ for the set of all orbits. A subset $A \subseteq X$ is a *system of representatives* if for all $x \in X$ the intersection $\mathcal{O}_x \cap A$ contains exactly one element.

(a) Show that the relation $\sim$ on X/G, defined by $x \sim y \; :\Longleftrightarrow \; y \in \mathcal{O}_x$, is an equivalence relation.

We endow X/G with the quotient topology with respect to $q : X \to [X] := X/G$.

(b) Show that, if G is a topological group and σ is continuous, then $q : X \to [X]$ is an open map.

Quotient topologies can be very bizarre as the following example shows:

(c) Let $G := \left(\mathbb{R}^\times_+, \cdot, 1 \right)$. Show that $\sigma : G \times \mathbb{R} \to \mathbb{R}$, $(p, r) \mapsto p \cdot r$ is a continuous group action.

(d) Show that $\mathbb{R}/G$ contains exactly three elements. Give an easy system of representatives.

(e) Show that $\mathbb{R}/G$ is not T_2.

Exercise E39

Let X be a topological space and define the *diagonal* of X to be $\Delta_X := \{ (x, x) \in X \times X | \, x \in X \}$.

(a) Show that X is separated if and only if Δ_X is closed in $X \times X$.

(b) Let $\sim$ be an equivalence relation on X and $q \colon X \to [X]$ the quotient map and endow $[X]$ with the quotient topology. We define the set $R := \{ (x, x') \in X \times X | \, x \sim x' \}$. Show that, if $q : X \to [X]$ is an open map, then $[X]$ is T_2 if and only if R is closed in $X \times X$.

Exercise E40

Let $(d_i)_{i \in I}$ be a family of semimetrics on the set X and $\tau := \bigcap_{i \in I} \tau_{d_i}$ be the topology defined by this family. Show that:

(a) The diagonal mapping $\eta \colon X \to \prod_{i \in I} (X, \tau_{d_i}), x \mapsto (x)_{i \in I}$ is a homeomorphism onto its image.

(b) A net $(x_j)_{j \in J}$ converges in (X, τ) to $p \in X$ if and only if $d_i(x_j, p) \to 0$ holds for each $i \in I$.

(c) (X, τ) is Hausdorff if and only if for $x \neq y$ there exists an i with $d_i(x, y) \neq 0$.

Exercise E41

Let $(X_i, d_i)_{i \in I}$ be an *uncountable* family of non-trivial[1] metric spaces and $X := \prod_{i \in I} X_i$ their topological product. Show that the product topology does *not* coincide with the topology induced by any metric d on X.

Hint: Assume the converse, consider the subspace $\left(S := \prod_{i \in I} \{ x_i, y_i \}, d_{|S} \right)$, where $x_i \neq y_i \in X_i$, and find a contradiction. Can you now give an example of a T_2-space which is not T_3?

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[1] Each X_i contains more than one element.