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Exercise Sheet no. 3 of "Topology"

Exercise E23

A subset $\mathcal{A} \subseteq \mathbb{P}(X)$ is a basis for a topology on X if and only if

(1) $\bigcup \mathcal{A} = X$ and

(2) for each $x \in A \cap B$, $A, B \in \mathcal{A}$, there exists a $C \in \mathcal{A}$ with $x \in C \subseteq A \cap B$.

Exercise E24

Let $X_1, \ldots, X_n$ be topological spaces. Show that the sets of the form

$$U_1 \times \ldots \times U_n, \quad U_i \subseteq X_i \text{ open},$$

form a basis for the product topology on $X_1 \times \ldots \times X_n$ and for $A_i \subseteq X_i$, $1 \leq i \leq n$, we have

$$\overline{\prod_{i=1}^n A_i} = \prod_{i=1}^n \overline{A_i} \quad \text{and} \quad \left(\prod_{i=1}^n A_i\right)^0 = \prod_{i=1}^n A_i^0.$$

Exercise E25

Let X and Y be topological spaces and $x \in X$. Show that the maps

$$j_x \colon Y \to X \times Y, \quad y \mapsto (x, y)$$

are continuous, and the corestriction

$$j_x^{|Y \times \{x\}} \colon Y \to Y \times \{x\}$$

is a homeomorphism.

Exercise E26

Let $(X_i)_{i \in I}$ be a family of topological spaces and $X := \prod_{i \in I} X_i$ the topological product space. Show that:

(a) X is T_2 if and only if each X_i is T_2.

(b) X is T_3 if and only if each X_i is T_3. Hint: Use (a) and Proposition 1.4.3 for one direction and the fact that subspaces of T_3-spaces are T_3 (Why?) for the other.

Exercise E27

Let (X_i, d_i), $i = 1, \ldots, n$, be metric spaces. Show that the metrics

$$d(x, y) := \sum_{i=1}^n d_i(x_i, y_i) \quad \text{and} \quad d_\infty(x, y) := \max\{d_i(x_i, y_i) \colon i = 1, \ldots, n\}$$

both induce the product topology on $X := \prod_{i=1}^n X_i$.

Exercise E28

Let $(X_i, d_i)_{i\in\mathbb{N}}$ be a sequence of metric spaces and $X := \prod_{i\in\mathbb{N}} X_i$ their topological product. Show that the product topology coincides with the topology on X induced by the metric

$$d(x,y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$$

(cf. Exercise E13).

Show further that a sequence $(x^{(n)})_{n\in\mathbb{N}}$ in $X = \prod_{i\in\mathbb{N}} X_i$ converges if and only if all component sequences $(x_i^{(n)})_{n\in\mathbb{N}}$ converge.

Exercise E29

Let (X, τ) be a topological space, $\sim$ be an equivalence relation on X, $q\colon X \to [X] := X/\sim = \{[x]_\sim = \{y \in X \mid y \sim x\} \mid x \in X\}$ be the quotient map, and endow $[X]$ with the quotient topology.

(a) Show that, if $f : X \to Y$ is a continuous map satisfying

$$x \sim y \quad \Rightarrow \quad f(x) = f(y) \qquad \forall x, y \in X,$$

then there exists a unique continuous map $\overline{f}\colon [X] \to Y$ with $f = \overline{f} \circ q$.

(b) Assume there is a subset $Z \subseteq X$ such that for all $x \in X$ we have $[x]_\sim \cap Z \neq \emptyset$. Write $[Z]' := \{[x]'_\sim := \{y \in Z \mid y \sim x\} \mid x \in Z\}$ and endow $[Z]'$ with the quotient topology. Show that the map $\varphi : [Z]' \to [X]$, $[x]'_\sim \mapsto [x]_\sim$ is a well-defined continuous bijection. Also show that, if $q : X \to [X]$ is an open map, then $\varphi : [Z]' \to [X]$ is a homeomorphism.

Exercise E30

Let $(G, \cdot, 1)$ be a group, (X, τ) be a topological space and $\sigma : G \times X \to X$, $(g, x) \mapsto \sigma_g(x) =: g.x$ a *group action*, i.e. $G \to S(X)$, $g \mapsto \sigma_g$ is a morphism to the symmetric group of X.

For $x \in X$ the set $\mathcal{O}_x := \{g.x \mid g \in G\}$ is called the *orbit* of x with respect to σ and we write $X/G := \{\mathcal{O}_x \mid x \in X\}$ for the set of all orbits. A subset $A \subseteq X$ is a *system of representatives* if for all $x \in X$ the intersection $\mathcal{O}_x \cap A$ contains exactly one element.

(a) Show that the relation $\sim$ on X/G, defined by $x \sim y \;:\Longleftrightarrow\; y \in \mathcal{O}_x$, is an equivalence relation.

We endow X/G with the quotient topology with respect to $q : X \to [X] := X/G$ (cf. Exercise E29). Quotient topologies can be very bizarre as the following example shows:

(d) Consider the multiplicative group $G := \left(\mathbb{R}_+^\times, \cdot, 1\right)$ and the set $\mathbb{R}$. Show that $\sigma : G \times \mathbb{R} \to \mathbb{R}$, $(p, r) \mapsto p \cdot r$ is a group action.

(e) Show that $\mathbb{R}/G$ contains exactly three elements. Give an easy system of representatives.

(f) Show that $\mathbb{R}/G$ is not T_2.

Exercise E31

Let $(X_i, d_i)_{i\in I}$ be an *uncountable* family of non-trivial[1] metric spaces and $X := \prod_{i\in I} X_i$ their topological product. Show that the product topology does *not* coincide with the topology induced by any metric d on X.

Hint: Assume the converse, consider the metric subspace $\left(S := \prod_{i\in I} \{x_i, y_i\}, d_{|S}\right)$, where $x_i \neq y_i \in X_i$, and find a contradiction.

[1]Each X_i contains more than one element.