Fachbereich Mathematik
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Exercise Sheet no. 2 of "Topology"

Exercise E14

Show that the topological spaces

$$I = [0,1] \quad \text{and} \quad \mathbb{S}^1 := \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$$

are not homeomorphic. Hint: Consider the connectedness properties if one point is removed.

Exercise E15

Show that if Y is a connected subset of the topological space X, then its closure $\overline{Y}$ is also connected.

Exercise E16

For a continuous function $f \colon]0,1] \to \mathbb{R}$, we consider its graph

$$\Gamma(f) := \{(x, f(x)) \colon 0 < x \leq 1\}.$$

Show that:

(a) $\Gamma(f)$ is an arcwise connected subset of $\mathbb{R}^2$.

(b) $\overline{\Gamma(f)} = \Gamma(f) \cup (\{0\} \times I_f)$, where $I_f \subseteq \mathbb{R}$ is the set of all those points y for which there exists a sequence $x_n \to 0$ in $]0,1]$ with $f(x_n) \to y$.

(c) $\overline{\Gamma(f)}$ is connected.

(d) For $f(x) := \sin(1/x)$, the set $\overline{\Gamma(f)}$ is not arcwise connected.

(e) $\overline{\Gamma(f)}$ is arcwise connected if and only if $|I_f| \leq 1$.

Exercise E17

Show that the connected components of a topological space are closed.

Exercise E18

Find an example of an arc-component of a topological space which is not closed.

Exercise E19

A topological space X is called *locally (arcwise) connected*, if each neighborhood U of a point x contains a connected (an arcwise connected) neighborhood V of x.
Show that in a locally connected space the connected components are open and in a locally arcwise connected space the arc-components are open and coincide with the connected components.

Exercise E20

In $\mathbb{R}^2$ we consider the set

$$X = ([0,1] \times \{1\}) \cup \Big(\Big\{\frac{1}{n} \colon n \in \mathbb{N}\Big\} \times [0,1]\Big) \cup (\{0\} \times [0,1]).$$

Sketch X and show that it is arcwise connected but not locally arcwise connected.

Exercise E21

For a non-empty subset A of the metric space (X, d), we consider the function

$$d_A(x) := \inf\{d(x, a) \colon a \in A\}.$$

Show that:

(i) $|d_A(x) - d_A(y)| \leq d(x, y)$ for $x, y \in X$. In particular, d_A is continuous.

(ii) $d_A(x) = 0$ if and only if $x \in \overline{A}$.

(iii) Every metric space (X, d) is normal. Hint: For two disjoint closed subsets $A, B \subseteq X$, consider the function $f := d_A - d_B$.

Exercise E22

(a) Which separation axioms does $\mathbb{N}$, equipped with the cofinite topology (cf. E10), fulfill?

(b) Is there a topology τ on $X = \{x, y\}$ such that (X, τ) is T_0, but not T_1?

(c) Is $\mathbb{R}$, equipped with the usual topology, a T_3-space?