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Exercise Sheet no. 1 of "Topology"

Exercise E1

(a) Show that all metrics d on a finite set define the discrete topology.
(b) Show that all finite Hausdorff spaces are discrete.

Exercise E2

Find an example of a countable metric space (X, d) for which the topology τ_d is not discrete.

Exercise E3

Show that a subset M of a topological space X is open if and only if it is a neighborhood of all points $x \in M$.

Exercise E4

Let Y be a subset of a topological space (X, τ). Show that $\tau|_Y = \{O \cap Y : O \in \tau\}$ defines a topology on Y.

Exercise E5

Let $a < b \leq c$ be real numbers. Show that

$$d(f, g) := \int_a^b |f(x) - g(x)| \, dx$$

defines a semimetric on the space $C([a, c], \mathbb{R})$ of continuous real-valued functions on $[a, c]$. Show also that $d(f, g) = 0$ is equivalent to $f = g$ on $[a, b]$ and that d is a metric if and only if $b = c$.

Exercise E6

Let (X, d) be a metric space and $Y \subseteq X$ be a subset. Show that the subspace topology $\tau_d|_Y$ on Y coincides with the topology defined by the restricted metric $d_Y := d|_{Y \times Y}$.

Exercise E7

Hausdorff's neighborhood axioms

Let (X, τ) be a topological space. Show that the collected $\mathfrak{U}(x)$ of neighborhoods of a point $x \in X$ satisfies:

(N1) $x \in U$ for all $U \in \mathfrak{U}(x)$ and $X \in \mathfrak{U}(x)$.

(N2) $U \in \mathfrak{U}(x)$ and $V \supseteq U$ implies $V \in \mathfrak{U}(x)$.

(N3) $U_1, U_2 \in \mathfrak{U}(x)$ implies $U_1 \cap U_2 \in \mathfrak{U}(x)$.

(N4) Each $U \in \mathfrak{U}(x)$ contains a $V \in \mathfrak{U}(x)$ with the property that $U \in \mathfrak{U}(y)$ for each $y \in V$.

Exercise E8

Let X be a set and suppose that we have for each $x \in X$ a subset $\mathcal{U}(x) \subseteq \mathbb{P}(X)$, such that the conditions (N1)-(N4) from the above exercise are satisfied. We then call a subset $O \subseteq X$ open if $O \in \mathcal{U}(x)$ holds for each $x \in O$. Show that the set τ of open subsets of X defines a topology on X for which $\mathcal{U}(x)$ is the set of all neighborhoods of x.

Exercise E9

For each norm $\|\cdot\|$ on $\mathbb{R}^n$, the metric $d(x,y) := \|x-y\|$ defines the same topology. Hint: Use that each norm is equivalent to $\|x\|_\infty := \max\{|x_i| \colon i = 1,\ldots,n\}$ (cf. Analysis II).

Exercise E10

Cofinite topology

Let X be a set and

$$\tau := \{\emptyset\} \cup \{A \subseteq X \colon |A^c| < \infty\}.$$

Show that τ defines a topology on X. When is this topology hausdorff?

Exercise E11

p-adic metric

Let p be a prime number. For $q \in \mathbb{Q}^\times$ we define $|q|_p := p^{-n}$ if we can write $q = p^n \frac{a}{b}$, where $a \in \mathbb{Z}, 0 \neq b \in \mathbb{Z}$ are not multiples of p. Note that this determines a unique $n \in \mathbb{Z}$. We also put $|0|_p := 0$. Show that

$$d(x,y) := |x-y|_p$$

defines a metric on $\mathbb{Q}$ for which the sequence $(p^n)_{n\in\mathbb{N}}$ converges to 0.

Exercise E12

Let d_1 and d_2 be two metrics on the set X and write $B^j_r(x)$ for the balls with respect to d_j, $j = 1, 2$. Show that d_1 and d_2 define the same topology on X if and only if for each $p \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ with

$$B^1_\delta(p) \subseteq B^2_\varepsilon(p)$$

and for each $p \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ with

$$B^2_\delta(p) \subseteq B^1_\varepsilon(p).$$

Exercise E13

Equivalent bounded metrics

Let (X,d) be a metric space. Show that:

(a) The function $f \colon \mathbb{R}_+ \to [0,1[, f(t) := \frac{t}{1+t}$ is continuous with continuous inverse $g(t) := \frac{t}{1-t}$. Moreover, f is subadditive, i.e., $f(x+y) \leq f(x) + f(y)$ for $x,y \in \mathbb{R}_+$.

(b) $d'(x,y) := \frac{d(x,y)}{1+d(x,y)}$ is a metric on X with $\sup_{x,y\in X} d'(x,y) \leq 1$.

(c) d' and d induce the same topology on X.