Fachbereich Mathematik
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10th exercise sheet Set Theory
Winter Term 2008/2009

(E10.1) [Rank]

Use the formula

$$\text{rank}(x) = \sup\{\text{rank}(y) + 1 \,:\, y \in x\}$$

to prove that the rank-function can be regarded as being defined by transfinite induction. Deduce that this function is absolute.

(E10.2) [Hereditarily finite sets]

Recall that

$$H_\kappa = \{x \,:\, |\text{tc}(x)| < \kappa\}.$$

The elements of H_ω are called the *hereditarily finite sets*.

(i) Prove that $V_\omega = H_\omega$.

(ii) Consider the model $(\mathbb{N}, \in)$ whose underlying set is the set of natural numbers, and where $\in$ is defined by:

$$x \in y \quad \Leftrightarrow \quad \text{the } x\text{th digit in the binary representation of } y \text{ is } 1.$$

Prove $(\mathbb{N}, \in) \cong (H_\omega, \in)$.

(E10.3) [Hereditarily countable sets]

In this exercise we assume the axiom of choice.

(i) If $\kappa > \omega$ is a regular cardinal, prove that

$$H_\kappa \models \textbf{ZFC} - \text{Powerset axiom}.$$

(ii) The elements of H_{ω_1} are called the *hereditarily countable sets*. Prove

$$H_{\omega_1} \models \textbf{ZFC} - \text{Powerset axiom} + \neg\text{Powerset axiom}.$$

(E10.4) [Strongly inaccessible cardinals]

In this exercise we assume the axiom of choice.

Let $\kappa > \omega$ be a regular cardinal. Prove that the following statements are equivalent:

(a) H_κ is a model of **ZFC**.

(b) $H_\kappa = V_\kappa$.

(c) κ is strongly inaccessible.