Fachbereich Mathematik
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9th exercise sheet Set Theory
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(E9.1) [Ordinals]

Call a set A $\in$-connected, if

$$\forall x, y \in A \,:\, x \in y \text{ or } x = y \text{ or } y \in x.$$

Prove that a set A is an ordinal iff it is transitive, grounded and $\in$-connected.

(E9.2) [Ordinals]

(i) Prove that for ordinals α and β,

$$\alpha \leqslant \beta \text{ iff } \alpha \subseteq \beta.$$

(ii) Prove that for ordinals α and β,

$$\alpha < \beta \text{ iff } \alpha \subsetneq \beta \text{ iff } \alpha \in \beta$$

(iii) Prove that $\alpha + 1 = \alpha \cup \{\alpha\}$ and $\sup\{\alpha_i \,:\, i \in I\} = \bigcup_{i \in I} \alpha_i$.

(E9.3) [Mostowski collapse]

(In this exercise, we assume the axiom **(DC)** of dependent choice.)

If one is given two grounded graphs $\mathcal{G} = (G, \rightarrow)$ and $\mathcal{H} = (H, \rightarrow)$, a node $g_0 \in G$ and a node $h_0 \in H$, one can play the "Bisimilarity Game". It is played by two players, called Skeptic and Believer, who move in turns. A position in the game is a pair of nodes (g, h), with $g \in G$ and $h \in H$; the starting position is (g_0, h_0).

Skeptic moves first: in the position (g, h), he first has to choose one of the two graphs. If he chooses $\mathcal{G}$, he has to select a predecessor g' of g (i.e., a node such that there is an edge $g' \rightarrow g$); if he chooses $\mathcal{H}$, he has to select a predecessor h' of h. The Believer has to reply by choosing a predecessor in the other graph: if the Skeptic chose a predecessor g' of g, the Believer has to reply by choosing a predecessor h' of h; if the Skeptic chose a predecessor h' of h, the Believer has to reply by choosing a predecessor g' of g. The new position is then (g', h') and it is again the Skeptic's turn, etcetera.

The player who cannot move (has no legal moves) looses.

(i) Show that every possible play ends after a finite number of steps in a loss for either of the two players.

   Hint: use groundedness of the two graphs.

In view of exercise E4.7 this means that one of the two players has a winning strategy. If Believer has a winning strategy, the nodes g_0 and h_0 are called *bisimilar*.

(ii) Prove that g_0 and h_0 are bisimilar iff they are decorated by the same set.

(E9.4) [Absoluteness]

(i) Show that the following formulas are absolute:

$$Z = X \times Y, Z = X - Y, Z = X \cap Y, Z = \bigcup X.$$

(ii) Show that the following formulas are absolute:

$$X \text{ is a relation}, f \text{ is a function}, Z = \text{dom } f, Z = \text{ran } f, y = f(x), g = f \restriction X,$$
$$f \text{ is an injective (surjective, bijective) function.}$$

(iii) Show that finiteness is absolute.

(E9.5) [Replacement]

Verify that $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ are all elements of $V_{\omega+\omega}$, and that at least 99% of mathematics takes places in $V_{\omega+\omega}$.