Fachbereich Mathematik
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December 17, 2008

8th exercise sheet Set Theory
Winter Term 2008/2009

(E8.1) [Normal functions]

A function $F : Ord \to Ord$ is called *normal*, if it is increasing

$$\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$$

and continuous at limit ordinals, i.e.

$$F(\lambda) = \sup\{F(\beta) \,:\, \beta < \lambda\},$$

when λ is a limit ordinal.

(i) Show that, if $F : Ord \to Ord$ is a normal function, $F(\alpha) \geqslant \alpha$ for all ordinals α, and $F(\lambda)$ is a limit ordinal for all limit ordinals λ.

(ii) Show that for $\alpha > 1$, the functions $\beta \mapsto \alpha + \beta$, $\beta \mapsto \alpha \cdot \beta$ and $\beta \mapsto \alpha^\beta$ in ordinal arithmetic are normal.

(iii) Show that normal functions are continuous: if F is normal and $\alpha = \sup\{\alpha_i \,:\, i \in I\}$, then
$$F(\alpha) = \sup\{F(\alpha_i) \,:\, i \in I\}.$$

(iv) Show that the composition of two normal functions is again normal.

(v) Show that a normal function has arbitrarily large fixed points.

(vi) Conclude that there are arbitrarily large ordinals α such that $\aleph_\alpha = \alpha$.

(E8.2) [Cantor normal form]

(i) Show that for every ordinal $\alpha > 0$ there are ordinals β and γ such that $\alpha = \omega^\beta + \gamma$ and $\gamma < \alpha$.

(ii) Deduce that every ordinal $\alpha > 0$ can be written in the form of a finite sum of powers of ω,

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \ldots + \omega^{\beta_s},$$

with $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_s$.

(iii) Prove that $\omega^\alpha + \omega^\beta = \omega^\beta$ if $\alpha < \beta$.

(iv) Show that every ordinal $\alpha > 0$ can be written *uniquely* as $\alpha = \omega^\beta + \gamma$ with $\gamma < \alpha$.

(v) Prove by induction on s that if $\beta \geqslant \beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_s$ and $\gamma = \omega^{\beta_1} + \omega^{\beta_2} + \ldots + \omega^{\beta_s}$, then $\gamma < \omega^\beta + \gamma$.

(vi) Deduce that the representation of an ordinal $\alpha > 0$ as a finite sum of non-increasing powers of ω is unique.

(E8.3) [Grounded graphs]

Let $\mathcal{G} = (G, \rightarrow)$ be a narrow graph and write

$$x \rightarrow^+ y,$$

if there is a path $x = g_0 \rightarrow g_1 \rightarrow \ldots \rightarrow g_n = y$ $(n > 0)$ in $\mathcal{G}$.

(i) Prove that $(G, \rightarrow)$ is grounded iff all graphs $(\mathrm{tc}(x), \rightarrow)$ are.

(ii) Prove that if $(G, \rightarrow)$ is grounded, then so is $(G, \rightarrow^+)$.

(E8.4) [Extensional collapse]

Call a narrow graph $\mathcal{G} = (G, \rightarrow)$ *extensional*, if

$$G_{\rightarrow a} = G_{\rightarrow b} \Longrightarrow a = b.$$

(i) Verify that the graph associated to a (possibly large) well-order is extensional, and that $(\mathrm{tc}(x), \in)$ is extensional for every set x.

(ii) Show that the decoration of an extensional grounded graph is injective, so that in that case we obtain a graph isomorphism $\mathcal{G} \cong d(\mathcal{G})$.

(iii) Verify that the only decoration of the graph $(\mathrm{tc}(x), \in)$, where x is a grounded set, is the identity.