Fachbereich Mathematik
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November 26, 2008

6th exercise sheet Set Theory
Winter Term 2008/2009

(E6.1)

Show that cardinal numbers κ, λ, μ satisfy the following high school equalities:

(a) $\kappa + 0 = \kappa$, $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$, $\kappa + \lambda = \lambda + \kappa$.

(b) $\kappa \cdot 0 = 0$, $\kappa \cdot 1 = \kappa$, $\kappa \cdot 2 = \kappa + \kappa$.

(c) $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$, $\kappa \cdot \lambda = \lambda \cdot \kappa$, $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$.

(d) $\kappa^0 = 1$, $\kappa^1 = \kappa$, $\kappa^2 = \kappa \cdot \kappa$.

(e) $(\kappa \cdot \lambda)^\mu =_c \kappa^\mu \cdot \lambda^\mu$, $\kappa^{(\lambda+\mu)} = \kappa^\lambda \cdot \kappa^\mu$, $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

Why does the cancellation law

$$\kappa + \mu = \lambda + \mu \Rightarrow \kappa = \lambda$$

fail?

(E6.2)

Show the following implications for all cardinal numbers κ, λ, μ.

$$\begin{array}{lcll} \kappa \leqslant \mu & \Rightarrow & \kappa + \lambda \leqslant \mu + \lambda & \\ \kappa \leqslant \mu & \Rightarrow & \kappa \cdot \lambda \leqslant \mu \cdot \lambda & \\ \lambda \leqslant \mu & \Rightarrow & \kappa^\lambda \leqslant \kappa^\mu & \quad (\kappa \neq 0) \\ \kappa \leqslant \lambda & \Rightarrow & \kappa^\mu \leqslant \lambda^\mu & \end{array}$$

For what values of λ, μ does the third implication fail when $\kappa = 0$?

(E6.3)

Show that $2^\omega = \omega$ in ordinal arithmetic, but $2^\omega > \omega$ in cardinal arithmetic.

(E6.4)

Show that every infinite cardinal is a limit ordinal.

(E6.5)

Show that the cardinals are closed under suprema of ordinals.

(E6.6)

(AC) Show that $A \leqslant_c B$ iff there is a surjection $p : B \to A$ or $A = \emptyset$.

(E6.7)

(AC) If κ is an infinite cardinal and we have an indexed family of sets $\{X_i : i \in I\}$, each of which has cardinality $\leqslant \kappa$, and the cardinality of the index set I is $\leqslant \kappa$, then the cardinality of both $\sum_{i \in I} X_i$ and $\bigcup_{i \in I} X_i$ is $\leqslant \kappa$.