Fachbereich Mathematik
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5th exercise sheet Set Theory
Winter Term 2008/2009

(E5.1)

Fill in the details of the proof of the Transfinite Recursion Theorem for large well-orders:

> Let $(W, \leqslant)$ be a large well-order and X a set equipped with an operation
>
> $$\varphi : \sum_{w \in W} X^{W_{<w}} \to X.$$
>
> There is a unique function $f : W \to X$ such that, for all $w \in W$,
>
> $$f(w) = \varphi(w, f \upharpoonright W_{<w}).$$

(E5.2)

Use the Axiom of Replacement to show that, if $(W, \leqslant)$ is a large well-order and $(X, \leqslant)$ a small one, then $(X, \leqslant) <_o (W, \leqslant)$.

(E5.3)

(i) Show that ordinal addition is associative, but not commutative and that 0 acts as a unit.

(ii) Exactly one of the following two statements is correct. Prove that one and give a counterexample to the other statement.

$$\alpha + \beta = \alpha + \gamma \quad \Rightarrow \quad \beta = \gamma$$
$$\beta + \alpha = \gamma + \alpha \quad = \quad \beta = \gamma$$

(iii) Exactly one of the following two statements is correct. Prove that one and give a counterexample to the other statement.

$$\sup\{\alpha_i : i \in I\} + \beta \quad = \quad \sup\{\alpha_i + \beta : i \in I\}$$
$$\beta + \sup\{\alpha_i : i \in I\} \quad = \quad \sup\{\beta + \alpha_i : i \in I\}$$

(E5.4)

(i) Show that ordinal multiplication is associative, but not commutative and that 1 acts as a unit.

(ii) Exactly one of the following two statements is correct. Prove that one and give a counterexample to the other statement.

$$\begin{aligned} \alpha(\beta+\gamma) &= \alpha\beta+\alpha\gamma \\ (\beta+\gamma)\alpha &= \beta\alpha+\gamma\alpha \end{aligned}$$

(iii) Exactly one of the following two statements is correct. Prove that one and give a counterexample to the other statement.

$$\begin{aligned} \sup\{\alpha_i : i \in I\}\,\beta &= \sup\{\alpha_i\beta : i \in I\} \\ \beta\sup\{\alpha_i : i \in I\} &= \sup\{\beta\alpha_i : i \in I\} \end{aligned}$$

(E5.5)

(i) Check that $\mathbb{P} \to_o \mathbb{Q}$ is a woset, if both $\mathbb{P}$ and $\mathbb{Q}$ are.

(ii) Show the following identities:

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^1 &= \alpha \\ \alpha^\gamma\beta^\gamma &= (\alpha\beta)^\gamma \\ (\alpha^\beta)^\gamma &= \alpha^{\beta\gamma} \\ \alpha^\beta\alpha^\gamma &= \alpha^{\beta+\gamma} \end{aligned}$$