Fachbereich Mathematik
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5th exercise sheet Set Theory
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(E5.1)

Show that the following instances of the axiom of choice are provable in $\mathbf{BST} + \mathbf{Infinity}$.

(i) $\forall x \in a \, \exists! y \in b \, \varphi(x, y) \to \exists f : a \to b \, \forall x \in a \, \varphi(x, fx)$
(Axiom of Unique Choice or Axiom of No Choice)

(ii) $\forall x \in a \, \exists y \in b \, \varphi(x, y) \to \exists f : a \to b \, \forall x \in a \, \varphi(x, fx)$, where a is a finite set
(Finite Axiom of Choice)

(E5.2)

Show that the statement

> "A graph $\mathcal{G} = (G, \to)$ is grounded iff there is no descending chain $w_0 \leftarrow w_1 \leftarrow w_2 \leftarrow \ldots$."

is equivalent to **(DC)**.

(E5.3)

Prove König's Lemma

> "Every infinite, finitely branching tree has at least one infinite branch."

(i) using **(DC)**.

(ii) using $(\mathbf{AC}_\omega)$.

(E5.4)

Let G be a game in which two players, Black and White, in turn make a move with White to move first. Assume that every possible play ends after a finite number of moves in a win for either of the two players.

Show that one of the two players has a winning strategy in G. (The proof uses **(DC)**.)

(E5.5)

(i) Show that ordinal addition is associative, but not commutative and that 0 acts as a unit.

(ii) Exactly one of the following two statements is correct. Which? Give a counterexample to the other statement.

$$\begin{aligned} \alpha + \beta = \alpha + \gamma &\Rightarrow \beta = \gamma \\ \beta + \alpha = \gamma + \alpha &= \beta = \gamma \end{aligned}$$

(E5.6)

(i) Show that ordinal multiplication is associative, but not commutative and that 1 acts as a unit.

(ii) Exactly one of the following two statements is correct. Which? Give a counterexample to the other statement.

$$\begin{aligned} \alpha(\beta + \gamma) &= \alpha\beta + \alpha\gamma \\ (\beta + \gamma)\alpha &= \beta\alpha + \gamma\alpha \end{aligned}$$

(E5.7)

(i) Check that $\mathbb{P}^{\mathbb{Q}}$ is a woset, if both $\mathbb{P}$ and $\mathbb{Q}$ are.

(ii) Show the following identities:

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^1 &= \alpha \\ \alpha^\gamma \beta^\gamma &= (\alpha\beta)^\gamma \\ (\alpha^\beta)^\gamma &= \alpha^{\beta\gamma} \\ \alpha^\beta \alpha^\gamma &= \alpha^{\beta+\gamma} \end{aligned}$$

(E5.8)

Show that an ordinal α is a cardinal iff it is a minimal well-order iff for all ordinals $\beta < \alpha$ we have $\beta <_c \alpha$. And show that the cardinality $\| X \|$ of X is the least ordinal α such that $X =_c \alpha$.

(E5.9)

Show that $2^\omega = \omega$ in ordinal arithmetic, but $2^\omega > \omega$ in cardinal arithmetic.