Fachbereich Mathematik
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3nd exercise sheet Set Theory
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(E3.1)

(i) Show that there is a unique binary operation $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the following recursion equations hold:

$$\begin{aligned} n + 0 &= n \\ n + Sm &= S(n + m) \end{aligned}$$

(ii) Show that $+$ is associative, commutative and satisfies the cancellation law:

$$m + x = n + x \Rightarrow m = n.$$

(iii) Write down recursion equations for multiplication and exponentiation and prove the obvious things.

(E3.2)

A set A is called finite, if $A =_c \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. Otherwise A is called infinite.

Show that $\mathbb{N}$ is infinite (and make sure that your argument can be formalised in **BST** + **Infinity**!).

(E3.3)

Let $\mathcal{G} = (G, \to)$ be a directed graph. We call a set $T \subseteq G$ *transitive*, if

$$x \to y \in T \Rightarrow x \in T.$$

(i) Show that

$$x_0 \to x_1 \to \ldots \to x_n \in T \Rightarrow x_0 \in T$$

for any transitive T.

(ii) Use the Knaster-Tarski Fixed Point Theorem to show that any set $X \subseteq G$ there is a least (in the sense of $\subseteq$) transitive subset $T \subseteq G$ such that $X \subseteq T$. This set T is of course unique and called the *transitive closure* of X. We will write $\mathrm{tc}(X)$ for the transitive closure of X and $\mathrm{tc}(x)$ for the transitive closure of $\{x\}$.

(E3.4)

Let $\mathcal{G} = (G, \to)$ be a directed graph. For all $a \in G$ we write

$$G_{\to a} = \{x \in G \,:\, x \to a\}.$$

A directed graph $\mathcal{G} = (G, \to)$ is called *grounded* if every non-empty subset $A \subseteq X$ has a $\to$-least element: i.e., if every non-empty subset $A \subseteq X$ contains an element $a \in A$ such that $G_{\to a} \cap A = \emptyset$.

(i) A subset $A \subseteq G$ is called *inductive*, if

$$G_{\to a} \subseteq A \Rightarrow a \in A$$

for all $a \in G$. Prove that the following two statements are equivalent:

(a) $\mathcal{G}$ is grounded.

(b) The only inductive subset of G is G itself.

(ii) Show that if $\mathcal{G} = (G, \to)$ is grounded, then so is $\mathcal{G}^+ = (G, \to^+)$, where

$$\begin{aligned} x \to^+ y \quad &\Leftrightarrow \quad x \in \mathrm{tc}(G_{\to y}) \\ &\Leftrightarrow \quad \text{there is a path } x = a_0 \to a_1 \to a_2 \ldots \to a_n = y \text{ with } n > 0. \end{aligned}$$

(iii) Let $\mathcal{G}$ be a grounded graph, X be a set and $\varphi : \sum_{a \in G} X^{G_{\to a}} \to X$ be an operation. Prove that there is a unique map $f : G \to X$ such that

$$f(a) = \varphi(a, f \restriction G_{\to a})$$

for all $a \in G$.