Fachbereich Mathematik
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2nd exercise sheet Set Theory
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(E2.1)

Show the following strengthened version of the Knaster-Tarski Fixed Point Theorem:

> Let $\mathbb{P} = (P, \leqslant)$ be a complete poset and $f : \mathbb{P} \to \mathbb{P}$ be a monotone map. Show that $\mathrm{Fix}(f) = \{x \in P \,:\, f(x) = x\}$ is again a complete poset.

Can the other fixed point theorems be strengthened in a similar way?

(E2.2)

(i) Use the Knaster-Tarski Fixed Point Theorem to prove Banach's Decomposition Theorem:

> Let X and Y be sets and let $f : X \to Y$ and $g : Y \to X$ be maps. Then there exist disjoint subsets X_1 and X_2 of X and disjoint subsets Y_1 and Y_2 of Y such that $f(X_1) = Y_1$ and $g(Y_2) = X_2$, $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$.

(ii) Use (i) to obtain the Schröder-Bernstein Theorem.

(E2.3)

Let $\mathcal{G} = (G, \to)$ be a directed graph. We call a set $T \subseteq G$ *transitive*, if

$$x \to y \in T \Rightarrow x \in T.$$

(i) Show that

$$x_0 \to x_1 \to \ldots \to x_n \in T \Rightarrow x_0 \in T$$

for any transitive T.

(ii) Use the Knaster-Tarski Fixed Point Theorem to show that any set $X \subseteq G$ there is a least (in the sense of $\subseteq$) transitive subset $T \subseteq G$ such that $X \subseteq T$. This set T is of course unique and called the *transitive closure* of X.

(E2.4)

(i) Show that there is a unique binary operation $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the following recursion equations hold:

$$\begin{aligned} n + 0 &= n \\ n + Sm &= S(n + m) \end{aligned}$$

(ii) Show that $+$ is associative, commutative and satisfies the cancellation law:

$$m + x = n + x \Rightarrow m = n.$$

(iii) Write down recursion equations for multiplication and exponentiation and prove the obvious things.

(E2.5)

Show that $\mathbb{N}$ is infinite (and make sure that your argument can be formalised in **BST** + **Infinity**).