Fachbereich Mathematik
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October 22, 2008

1st exercise sheet Set Theory
Winter Term 2008/2009

(E1.1)

Prove the following equinumerosities:

(a) $\mathbb{N}^{\mathbb{N}} =_c \mathcal{P}\mathbb{N}$.

(b) $\mathbb{R}^{\mathbb{N}} =_c \mathbb{R}$.

(c) $\{f : \mathbb{R} \to \mathbb{R} \, : \, f \text{ continuous}\} =_c \mathbb{R}$.

(d) $\{f : [0,1] \to \mathbb{R} \, : \, f \text{ monotone}\} =_c \mathbb{R}$.

(E1.2)

Show that the class of all singletons is proper.

(E1.3)

Consider Kuratowski's definition of the pair (x, y):

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

Show that

$$(x, y) = (x', y') \Leftrightarrow x = x' \text{ and } y = y'.$$

(E1.4)

Show that cardinal numbers κ, λ, μ satisfy the following high school equalities:

(a) $\kappa + 0 =_c \kappa$, $\kappa + (\lambda + \mu) =_c (\kappa + \lambda) + \mu$, $\kappa + \lambda =_c \lambda + \kappa$.

(b) $\kappa \cdot 0 =_0$, $\kappa \cdot 1 = \kappa$, $\kappa \cdot 2 = \kappa + \kappa$.

(c) $\kappa \cdot (\lambda \cdot \mu) =_c (\kappa \cdot \lambda) \cdot \mu$, $\kappa \cdot \lambda =_c \lambda \cdot \kappa$, $\kappa \cdot (\lambda + \mu) =_c \kappa \cdot \lambda + \kappa \cdot \mu$.

(d) $\kappa^0 =_c 1$, $\kappa^1 =_c \kappa$, $\kappa^2 =_c \kappa \cdot \kappa$.

(e) $(\kappa \cdot \lambda)^\mu =_c \kappa^\mu \cdot \lambda^\mu$, $\kappa^{(\lambda+\mu)} =_c \kappa^\lambda \cdot \kappa^\mu$, $(\kappa^\lambda)^\mu =_c \kappa^{\lambda \cdot \mu}$.

Why does the cancellation law

$$\kappa + \mu =_c \lambda + \mu \Rightarrow \kappa =_c \lambda$$

fail?

(E1.5)

Show the following implications for all cardinal numbers κ, λ, μ.

$$\begin{array}{lll} \kappa \leqslant_c \mu & \Rightarrow & \kappa + \lambda \leqslant_c \mu + \lambda \\ \kappa \leqslant_c \mu & \Rightarrow & \kappa \cdot \lambda \leqslant_c \mu \cdot \lambda \\ \lambda \leqslant_c \mu & \Rightarrow & \kappa^\lambda \leqslant_c \kappa^\mu \qquad (\kappa \neq 0) \\ \kappa \leqslant_c \lambda & \Rightarrow & \kappa^\mu \leqslant_c \lambda^\mu \end{array}$$

For what values of λ, μ does the third implication fail when $\kappa = 0$?

(E1.6)

(i) Use the Knaster-Tarski Fixed Point Theorem to prove Banach's Decomposition Theorem:

> Let X and Y be sets and let $f : X \to Y$ and $g : Y \to X$ be maps. Then there exist disjoint subsets X_1 and X_2 of X and disjoint subsets Y_1 and Y_2 of Y such that $f(X_1) = Y_1$ and $g(Y_2) = X_2$, $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$.

(ii) Use (i) to obtain the Schröder-Bernstein Theorem.