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Introduction to Compact Groups

We introduced the bidual $\widehat{\widehat{A}}$ (resp. $\widehat{\widehat{G}}$) of an abelian (resp. compact abelian) group and the evaluation morphisms

$$\eta_A\colon A \to \widehat{\widehat{A}} \qquad \eta_G\colon G \to \widehat{\widehat{G}}\,.$$

The goal is to show that these are isomorphisms. We proved that η_A is injective and that η_G is injective once one accepts the fundamental theorem that every compact group has enough finite dimensional continuous representations to separate the points.

We verified the bijectivity of η_A for finite A, for $\mathbb{Z}$ and had an unfinished proof of the bijectivity of $\eta_{\mathbb{T}}$ which we finish now:

The character group $\widehat{\mathbb{T}}$ consists of the multiples $n{\cdot}\epsilon$ for the identity character $\epsilon\colon \mathbb{T} \to \mathbb{T}$ which form an infinite cyclic group, freely generated by ϵ. Thus the character group $\widehat{\langle\epsilon\rangle} = \mathrm{Hom}(\langle\epsilon\rangle, \mathbb{T})$ is isomorphic to $\mathbb{T}$ by the map $\Omega \mapsto \Omega(\epsilon)$.

Let $t \in \mathbb{T}$, and $\chi = n{\cdot}\epsilon$ a character of $\mathbb{T}$. Then by the definition of η_T, we have $\eta_{\mathbb{T}}(t)(\chi) = \eta_{\mathbb{T}}(t)(n{\cdot}\epsilon) = (n.\epsilon)(t) = n{\cdot}(\epsilon(t)) = n{\cdot}t$. Now $t \in \ker \eta_{\mathbb{T}}$ if $n{\cdot}t = 0$ for all n, notably for $n = 1$ and so $\eta_{\mathbb{T}}$ is injective. If $\Omega\colon \langle\epsilon\rangle \to \mathbb{T}$ is given we set $t = \Omega(\epsilon)$ and have $\eta_T(t)(\chi) = n{\cdot}t = n{\cdot}\Omega(\epsilon) = \Omega(n{\cdot}\epsilon) = \Omega(\chi)$ and thus $\Omega = \eta_{\mathbb{T}}(t)$. This shows that η_T is surjective.

Alternative proof using extra info: By the Fundamental Theorem (info!), $\eta_{\mathbb{T}}$ is injective and thus $\eta_T(\mathbb{T})$ is a nonsingleton compact connected subgroup of $\widehat{\widehat{T}} \cong \widehat{\mathbb{Z}} \cong \mathbb{T}$. But the only compact connected subgroups of $\mathbb{T}$ are $\{0\}$ and $\mathbb{T}$ (info!) and so $\eta_{\mathbb{T}}(\mathbb{T}) = \widehat{\widehat{\mathbb{T}}}$. [Dahmen]

Program for today.

Projective Limits. Character groups of abelian groups as projective limits.