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Introduction to Compact Groups

A *character* on a compact (or, more generally, on a topological) group G is a continuous morphism $\chi: G \to \mathbb{T}$. The set $\mathrm{Hom}(G, \mathbb{T})$ of all characters is an abelian group $\hat{G}$, the *character group* of the dual of G, denoted $\widehat{G}$. We calculated $\widehat{\mathbb{T}} \cong \mathbb{Z}$ by attaching to each endomorphism $\chi: \mathbb{T} \to \mathbb{T}$ a "winding number n which we identified with the endomorphism $r \mapsto n{\cdot}r : \mathbb{R} \to \mathbb{R}$ in such a fashion that $\chi(r + \mathbb{Z}) = n{\cdot}r + \mathbb{Z}$ which amounts to the commuting of the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow[r \mapsto n\cdot r]{} & \mathbb{R} \\ {\scriptstyle \mathrm{quot}}\downarrow & & \downarrow{\scriptstyle \mathrm{quot}} \\ \mathbb{T} & \xrightarrow{\chi} & \mathbb{T}. \end{array}$$

This required the proof of the following

Extension Lemma. *Let U be a nondegenerate interval on R containing O and $f: U \to G$ a function into a group such that $r, s, r + s \in U$ implies $f(r + s) = f(r)f(s)$. Then there is a unique morphism $F: \mathbb{R} \to G$ such that $F|U = f$.*

A continuous morphism $F: \mathbb{R} \to G$ into a topological group is called a *one parameter subgroup of G*. This is an example where a morphism is called a "subgroup".

We discussed the fact that passing to the dual is an operation that also acts on morphisms, reversing arrows: If $f\colon A \to B$ is a morphism of abelian groups, then $\widehat{f}\colon \widehat{B} \to \widehat{A}$, and $\widehat{fg} = \widehat{g} \circ \widehat{f}$. In other words, $\widehat{\ }$ is a *contravariant functor*, mapping the category $\mathbb{AB}$ of abelian groups into the category $\mathbb{CAB}$ of compact abelian groups—and vice versa.

Program for today.
Biduals. Projective Limits. Character groups of abelian groups as projective limits.