Introduction to General Topology

An Introductory Course
[Wahlpflichtbereich]
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Chapter 1
Topological spaces

Topological spaces generalize metric spaces. One uses metric spaces in analysis to work with continuous functions on what appears to be the "right" level of generality. But even in this context one notices that many important concepts, such as the continuity of functions between metric spaces itself, can be expressed in the language of *open sets* alone. This observation has caused mathematicians, first Felix Hausdorff, next Paul Alexandroff and Heinz Hopf, to use the idea of *open sets* as the basis for a general theory of continuity in an axiomatic approach. In fact Hausdorff's definition was based on the concept of systems of *neighborhoods* for each point.

We shall begin by defining topological spaces and continuous functions in both ways and by showing that they are equivalent.

The *objects* of our study are the "spaces"; the *transformations* between them are the "continuous functions". One should always treat them in a parallel approach. This is what has become known as "category theoretical" procedure, but we shall not be very formal in this regard.

1. Topological spaces and continuous functions

Some basic set theoretical notation

Consider a set X and a subset $A \subseteq X$. We define

(1) $$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A \end{cases}$$

and call the function χ_A the *characteristic function* of the subset A of X. We let $\mathfrak{P}(X)$ denote the set $\{A : A \subseteq X\}$ of all subsets of X an call it the *power set* of X. The name derives from a natural bijection

$$A \mapsto \chi_A : \mathfrak{P}(X) \to \{0,1\}^X.$$

The two element set $\{0,1\}$ is often abbreviated by $\mathbf{2}$ and thus $\mathbf{2}^X = \{0,1\}^X$. A power set is never empty, because $\emptyset \in \mathfrak{P}(X)$ and $X \in \mathfrak{P}(X)$ for any set X.

The set theoretical operations of arbitrary unions and intersections are well defined on $\mathfrak{P}(X)$. If $\mathcal{A} = \big(A_j : j \in J\big)$, $A_j \subseteq X$ is a family of subsets of X, then

$$\bigcup \mathcal{A} = \bigcup_{j\in J} A_j \stackrel{\text{def}}{=} \{x \in X : (\exists j \in J)\, x \in A_j\}, \tag{2}$$

$$\bigcap \mathcal{A} = \bigcap_{j\in J} A_j \stackrel{\text{def}}{=} \{x \in X : (\forall j \in J)\, x \in A_j\}. \tag{3}$$

Exercise E1.1. (i) Verify that the function $A \mapsto \chi_A$ defined in (1) above is a bijection by exhibiting its inverse function $\mathbf{2}^X \to \mathfrak{P}(X)$.

(ii) Let $\mathcal{A}$ denote the empty set of subsets of a set X. Compute $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$, using (2) and (3).

[Hint. Regarding (i), in very explicit terms, we have for instance $\bigcap \mathcal{A} = \{x \in X : (\forall A)\,(A \in \mathcal{A}) \Rightarrow (x \in A)\}$. So what?]

(iii) Verify the following distributive law for a subset A and a family $\{A_j : j \in J\}$ of subsets A_j of a set X:

$$A \cap \bigcup_{j\in J} A_j = \bigcup_{j\in J} (A \cap A_j). \tag{4}$$

□

In order to understand all concepts accurately, we should recall what the difference is between a subset S of a set M and a family $(s_j : j \in J)$ of elements of M. A subset $S \subseteq M$ is a set (we assume familiarity with *that* concept) such that $s \in S$ implies $s \in M$. A family $(s_j : j \in J)$ of elements of M is a function $j \mapsto s_j : J \to M$. If I have a family $(s_j : j \in J)$ then I have a set, namely $\{s_j : j \in J\}$, the image of the function. In fact for many purposes of set theory a family is even denoted by $\{s_j : j \in J\}$ which, strictly speaking, is not exact. Conversely, if I have a subset S of M then I can form a family $(s : s \in S)$ of elements of M, namely the inclusion function $s \mapsto s : S \to M$. Notice that we can have an empty family $(s_j : j \in \emptyset)$, namely the empty function $\emptyset\colon \emptyset \to M$, whose graph is the empty set, a subset of $\emptyset \times M = \emptyset$. (What we cannot have is a function $X \to \emptyset$ for for $X \neq \emptyset$! Check the definition of a function!)

A *function* $f\colon X \to Y$ is a triple $f = (G, X, Y)$ of sets such that $G \subseteq X \times Y$ satisfying the following conditions:
(i) $(\forall x)\ x \in X \Rightarrow (\exists y)(y \in Y \text{ and } (x, y) \in G)$.
(ii) $(\forall x, y, y')\big((x, y) \in G \text{ and } (x, y') \in G\big) \Rightarrow y = y'$.
Instead of $(x, y) \in G$ we write $y = f(x)$. The set G is called the graph of the function f.

Topological spaces

Definition of topology and topological space

Definition 1.1. A *topology* $\mathfrak{O}$ on a set X is a subset of $\mathfrak{P}(X)$ which is closed under the formation of arbitrary unions and finite intersections.

A *topological space* is a pair $(X, \mathfrak{O})$ consisting of a set X and a topology $\mathfrak{O}$ on it. If no confusion is likely to arise one also calls X a topological space. □

Let's be a bit more explicit:
A subset $\mathfrak{O} \subseteq \mathfrak{P}(X)$ is a topology iff

(i) *For any family of sets $U_j \in \mathfrak{O}$, $j \in J$, we have $\bigcup_{j \in J} U_j \in \mathfrak{O}$.*
(ii) *For any* finite *family of sets $U_j \in \mathfrak{O}$, $j \in J$, (J finite), we have $\bigcap_{j \in J} U_j \in \mathfrak{O}$.*
(iii) *$\emptyset \in \mathfrak{O}$ and $X \in \mathfrak{O}$.* □

By Exercise E1.1(ii) these statements are not independent: Proposition (iii) is a consequence of Propositions (i) and (ii).

The following set of axioms is equivalent to (i), (ii), (iii):
A subset $\mathfrak{O}$ of $\mathfrak{P}(X)$ is a topology iff

(I) *For each subset $\mathcal{U}$ of $\mathfrak{O}$ one has $\bigcup \mathcal{U} \in \mathfrak{O}$.*
(II) *For each $U_1, U_2 \in \mathfrak{O}$ we have $U_1 \cap U_2 \in \mathfrak{O}$,*
(III) *$X \in \mathfrak{O}$.*

Notation 1.1.1. If $(X, \mathfrak{O})$ is a topological space, then the sets $U \in \mathfrak{O}$ are called *open*. A subset A of X is called *closed*, if $X \setminus A$ is open. □

The subsets $\emptyset$ and X are both open and closed.

Examples 1.2. (i) For any set X, the power set $\mathfrak{P}(X)$ is a topology, called the *discrete topology*. A space equipped with its discrete topology is called a *discrete space*.

(ii) For any set X, the set $\{\emptyset, X\}$ is a topology called the *indiscrete topology*. A space equipped with its discrete topology is called an *indiscrete space*.

(iii) For any set X, the set $\{\emptyset\} \cup \{Y \subseteq X : \operatorname{card}(X \setminus Y) < \infty\}$ is a topology, called *cofinite topology*. □

Definition 1.3. A binary relation $\leq$ on a set X is called a *quasiorder* if it is transitive and reflexive, and it is a *partial order* if in addition it is antisymmetric. A *partially ordered set* or in short *poset* is a set $(X, \leq)$ endowed with a partial order.

For a subset Y in a quasiordered set $(X, \leq)$ we write

$$\uparrow Y \stackrel{\text{def}}{=} \{x \in X : (\exists y \in Y)\, y \leq x\};$$

a set satisfying $\uparrow Y = Y$ is called an *upper set*. We also write $\uparrow x$ instead of $\uparrow\{x\}$.

A quasiordered set D is *directed* if it is not empty and for each $x, y \in D$ there is a $z \in D$ such that $x \leq z$ and $y \leq z$. A poset $(X, \leq)$ is called a *directed complete poset* or **dcpo** if every directed subset has a least upper bound. □

Example 1.4. (i) For each quasiordered set $(X, \leq)$ the set $\{Y \subseteq X : \uparrow Y = Y\}$ of *all* upper sets is a topology, called the *Alexandroff discrete topology* of the quasiordered set.

(ii) In a **dcpo** the set $\sigma(X) =$

$$\{U \subseteq X : \uparrow U = U \text{ and } (\forall D \subseteq X)\,(D \text{ is directed and } \sup D \in U) \Rightarrow D \cap U \neq \emptyset\}$$

is a topology, called the *Scott topology* of the poset. □

(iii) On the set $\mathbb{R}$ of real numbers, the set

$$\mathfrak{O}(\mathbb{R}) = \{U \subseteq \mathbb{R} : (\forall u \in U)(\exists a, b \in \mathbb{R})\, a < u < b \text{ and }]a, b[\subseteq U\}$$

is a topology on $\mathbb{R}$, called the *natural topology of* $\mathbb{R}$. □

As an exercise, determine the Scott topology on $(\mathbb{R}, \leq)$ for the natural order on $\mathbb{R}$.

We recall from basic analysis the concept of a metric and a metric space.

Definition 1.5. A *metric* of on a set X is a function $d\colon X \times X \to \mathbb{R}$ satisfying the following conditions:

(i) $(\forall x, y \in X)\, d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$.

(ii) $(\forall x, y \in X)\, d(x, y) = d(y, x)$.

(iii) $(\forall x, y, z \in X)\, d(x, z) \leq d(x, y) + d(y, z)$.

Property (i) is called *positive definiteness*, Property (ii) *symmetry*, and property (iii) the *triangle inequality*.

If a set X is equipped with a metric d, then (X, d) is a *metric space*.

If $r > 0$ and $x \in X$, then $U_r(x) \stackrel{\text{def}}{=} \{u \in X : d(x, u) < r\}$ is called the *open ball of radius* r with *center* x. □

Proposition 1.6. *For a metric space* (X, d)*, the set*

$$\mathfrak{O}(X) = \{U \subseteq X : (\forall u \in U)(\exists \varepsilon > 0)\, U_\varepsilon(u) \subseteq U\}$$

is a topology. Every open ball $U_r(x)$ *belongs to* $\mathfrak{O}(X)$. □

Definition 1.7. The topology $\mathfrak{O}(X)$ of 1.6 on a metric space is called the *metric topology* for d or the *topology induced by* d. □

Thus any metric space is automatically a topological space. The natural topology of $\mathbb{R}$ is the metric topology for the metric on $\mathbb{R}$ given by $d(x, y) = |y - x|$. Given an arbitrary set, the function $d\colon X \times X \to \mathbb{R}$ such that $d(x, y) = 1$ if $x \neq y$ and

$d(x,x) = 0$ is a metric whose metric topology is the discrete topology. Therefore it is called the *discrete metric.*

Proposition 1.8. *Assume that* $(X, \mathfrak{O})$ *is a topological space, and that* $Y \subseteq X$*. Then*

$$\mathfrak{O}|Y \stackrel{\text{def}}{=} \{Y \cap U : U \in \mathfrak{O}\}$$

is a topology of Y. □

Definition 1.9. The topology $\mathfrak{O}|Y$ is called the *induced topology.* The topological space $(Y, \mathfrak{O}|Y)$ is called the *subspace* Y of X.

With the concepts introduced so far we have an immense supply of interesting topological spaces. The absolute value of complex numbers makes the complex plane $\mathbb{C}$ into a metric space via $d(u,v) = |v-u|$ and thus into a toplogical space. The space $\mathbb{S}^1 \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| = 1\}$ is called the *unit circle*, or the *one-sphere.* More generally, if one considers on $\mathbb{R}^n$ the *norm* $\|(x_1, \ldots, x_n)\| \stackrel{\text{def}}{=} \sqrt{x_1^2 + \ldots + x_n^2}$, then the metric space determined by the metric $d(x,y) = \|y - x\|$ is called *euclidean space.* The space $B^n \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is called the closed *n-cell* or *unit ball* in n dimensions. The subspace $\mathbb{S}^n = \{x \in B^{n+1} : \|x\| = 1\}$ is called the *n-sphere.*

Continuous functions

Definition of continuous function

Definition 1.10. (i) A function $f: X \to Y$ between topological spaces is called *continuous*, if $f^{-1}(V)$ is open in X for every open $V \subseteq Y$. The set of all continuous functions $f: X \to Y$ is often denoted by $C(X, Y)$.

(ii) The function f is called *open* if $f(U)$ is open in Y for every open $U \subseteq X$. □

Exercise E1.2. (i) Every function from a discrete space into a topological space is continuous.

(ii) Every function from a topological space into an indiscrete space is continuous.

Before we move on to the first simple proposition on continuous functions we review some purely set theoretical aspects of functions between sets. If $g: X \to Y$ and $f: Y \to Z$ are functions, then the *composition* $f \circ g: X \to Z$ is the function with domain X and range Z which is well-defined by the prescription $(f \circ g)(x) = f\big(g(x)\big)$ for all $x \in X$. The *identity function* $\mathrm{id}_X: X \to X$ of a set X is defined by the rule $\mathrm{id}_X(x) = x$.

If $f: X \to Y$ is a function and $A \subseteq X$ a subset of the domain, then the *restriction* $f|A: A \to Y$ of f to A is defined by $(f|A)(a) = f(a)$ for all $a \in A$.

If $A \subseteq X$ then the *inclusion function* $\text{incl}_A: A \to X$ of A into X is defined by $\text{incl}_A(a) = a$ for all $a \in A$. If $f: X \to A$ is a function, then $f|A = f \circ \text{incl}_A$; thus a restriction is a special case of a composition.

A function $f: X \to Y$ is called *constant* if $(\exists b \in Y)(\forall x \in X)\ f(x) = b$ iff $(\forall x, x' \in X)\ f(x) = f(x')$ iff $\text{card}\, f(X) = 1$ iff there is a decomposition $f = c_b \circ n$ where $n: X \to \{0\}$ is the unique *null function* given by $n(x) = 0$ for all $x \in X$ and $c_b(0) = b$. If X and Y are sets and $b \in Y$ then the constant function with value b is denoted $\text{const}_b: X \to Y$, $\text{const}_b = c_b \circ n$.

Proposition 1.11. (i) *The identity function of any topological space is continuous, and if $g: X \to Y$ and $f: Y \to Z$ are continuous functions, then $f \circ g: X \to Z$ is continuous,*

(ii) *If $A \subseteq X$ is a subspace of a topological space, then the inclusion function* $\text{incl}_A: A \to X$ *is continuous.*

(iii) *If $f: X \to Y$ is a continuous function between topological spaces and $A \subseteq X$ is a subspace, then $f|A: A \to Y$ is continuous.*

(iv) *Every constant function is continuous.* □

In short: *Compositions of continuous functions are continuous. Restrictions of continuous functions are continuous.*

In the proofs of (iii) and (iv) one should utilize 1.11(i) and E1.2 by writing $f|A = f \circ \text{incl}_A$, respectively, $\text{const}_n = c_b \circ n$. Frontal attack proofs are likewise easy.

A function $f: (X, \leq) \to (Y, \leq)$ between two posets is called *monotone* or *order preserving* if $(\forall x, x' \in X)\ x \leq x' \Rightarrow f(x) \leq f(x')$.

Exercise E1.3. *Let $f: (X, \leq) \to (Y, \leq)$ be a function between two posets.*

(i) *Assume that Y is a topological space such that every open subset is an upper set, that X has the Alexandroff discrete topology, and that f is monotone. Then f is continuous.*

(ii) *Assume that both X and Y are* **dcpo***s. Then the following statements are equivalent:*

(a) *f is Scott continuous, that is, is continuous with respect to the Scott topologies on X and Y.*

(b) *f preserves directed sups, i.e.* $\sup f(D) = f(\sup D)$ *for all directed subsets D of X.* □

We shall characterize continuity between metric spaces shortly.

One might have expected that we distinguish functions $f: X \to Y$ between topological spaces for which $f(U) \in \mathfrak{O}(Y)$ for all $U \in \mathfrak{O}(X)$. We called such function open; they do play a role but not one that is equally important to the role played by continuous functions. If $\mathbb{R}_d$ is the space of real numbers with the discrete topology and $\mathbb{R}$ the space of real numbers with the natural topology, then the identity function $f: \mathbb{R}_d \to \mathbb{R}$, $f(x) = x$, is continuous, but not open, and the inverse function $f^{-1}: \mathbb{R} \to \mathbb{R}_d$ is open but not continuous.

Neighborhoods

Definition 1.12. If $(X, \mathfrak{O})$ is a topological space and $x \in X$, then a set $U \in \mathfrak{P}(X)$ is called a *neighborhood* of x iff

$$(5) \qquad (\exists V)\, V \in \mathfrak{O} \text{ and } x \in V \subseteq U.$$

We write

$$(6) \qquad \mathfrak{U}(x) = \{U \in \mathfrak{P}(X) : U \text{ is a neighborhood of } x\}.$$

□

As a first thought let us observe, that for a subset U of a topological space $(X, \mathfrak{O})$, the following statements are equivalent:

(a) *U is open, that is $U \in \mathfrak{O}$.*

(b) *U is a neighborhood of each of its points, that is $(\forall u \in U)\, U \in \mathfrak{U}(u)$.* Indeed, (a)⇒(b) is obvious from Definition 1.12. Conversely, assume (b). Then by 1.12, for each $u \in U$ there is a $U_u \in \mathfrak{O}$ containing u and being contained in U. One verifies at once that $U = \bigcup_{u \in U} U_u$, and thus U is open by 1.1.(i).

Thus openness and being a neighborhood of a point are intimately linked conceptually. We shall pursue this further.

Observation. *The set $\mathfrak{U}(x)$ satisfies the following conditions*

(i) $\big(\forall U \in \mathfrak{U}(x)\big)\, U \neq \emptyset$

(ii) $\big(\forall U, V \in \mathfrak{U}(x)\big)\, U \cap V \in \mathfrak{U}(x)$.

(iii) $\big(\forall U, V\big)\, \big(U \in \mathfrak{U}(x)$ *and* $U \subseteq V\big) \Rightarrow V \in \mathfrak{U}(x)$. □

This observation calls for the introduction of a new concept.

Definition 1.13. Assume that X is a set. A set $\mathfrak{F} \subseteq \mathfrak{P}(X)$ of subsets of X is called a *filter*, if it is nonempty and satisfies the following conditions

(i) $\big(\forall A \in \mathfrak{F}\big)\, A \neq \emptyset$.

(ii) $\big(\forall A, B \in \mathfrak{F}\big)\, A \cap B \in \mathfrak{F}$.

(iii) $(\forall A, B)\, (A \in \mathfrak{F} \text{ and } A \subseteq B) \Rightarrow B \in \mathfrak{F}$.

A set $\mathfrak{B} \subseteq \mathfrak{P}(X)$ of subsets of X is called a *filter basis*, if it is nonempty and satisfies the following conditions

(i) $\big(\forall A \in \mathfrak{B}\big)\, A \neq \emptyset$.

(ii) $(\forall A, B \in \mathfrak{B})(\exists C \in \mathfrak{B})\, C \subseteq A \cap B$. □

Proposition 1.14. *A subset $\mathfrak{B}$ of $\mathfrak{P}(X)$ is a filter basis iff the set*

$$\mathfrak{F} \stackrel{\text{def}}{=} \{A \in \mathfrak{P}(X) : (\exists B \in \mathfrak{B})\, B \subseteq A\}$$

is a filter. □

We shall say that $\mathfrak{F}$ is the *filter generated by* $\mathfrak{B}$, and that $\mathfrak{B}$ is a *basis of* $\mathfrak{F}$. The set of all neighborhoods of a point is a filter, the set of open neighborhoods is a filter basis. In a metric space the set of all open balls $U_r(x)$, $r > 0$ is a filter basis, and indeed the set of all open balls $U_{1/n}(x)$, $n = 1, 2, \dots$ is a filter basis as well, generating $\mathfrak{U}(x)$.

If $(X, \mathfrak{O})$ is a topological space, then

$$x \mapsto \mathfrak{U}(x) : X \to \mathfrak{P}\big(\mathfrak{P}(X)\big)$$

is a function satisfying the following conditions:
(i) Each $\mathfrak{U}(x)$ is a filter.
(ii) $(\forall x \in X,\, U \in \mathfrak{U}(x))\, x \in U$.

Now we want proceed in the reverse direction, start from such a function, and create a topology:

Hausdorff characterisation of a topological space

Theorem 1.15. *Let X be a set and*

$$x \mapsto \mathfrak{U}(x) : X \to \mathfrak{P}\big(\mathfrak{P}(X)\big)$$

a function satisfying the following conditions:
(i) *Each $\mathfrak{U}(x)$ is a filter.*
(ii) $(\forall x \in X,\, U \in \mathfrak{U}(x))\, x \in U$.

Define $\mathfrak{O}$ to be a set of subsets U of X defined by

$$U \in \mathfrak{O} \Leftrightarrow (\forall u \in U)\, U \in \mathfrak{U}(u). \qquad (*)$$

Then $\mathfrak{O}$ is a topology such that for each x, the filter $\mathfrak{U}_{\mathfrak{O}}(x)$ of $\mathfrak{O}$-neighborhoods of x is contained in $\mathfrak{U}(x)$, and that the following statements are equivalent:

(A) *$\mathcal{O}$ is the unique topology for which each $\mathfrak{U}_{\mathfrak{O}}(x) = \mathfrak{U}(x)$ for each $x \in X$.*

(B) $\big((\forall x \in X,\, U \in \mathfrak{U}(x)\big)(\exists V \subseteq U, x \in V)(\forall v \in V)\, V \in \mathfrak{U}(v)$.

(C) $\big(\forall x \in X, U \in \mathfrak{U}(x)\big)\big(\exists V \in \mathfrak{U}(x)\big)(\forall v \in V) U \in \mathfrak{U}(v)$.

Proof . The set $\mathfrak{O}$ is readily seen to be closed under arbitrary unions and finite intersections thus is a topology. In order to show $\mathfrak{U}_{\mathfrak{O}}(x) \subseteq \mathfrak{U}(x)$, let U be an $\mathfrak{O}$–neighborhood of x. Then $x \in V \subseteq U$ for some $V \in \mathfrak{O}$ by Definition 1.12. According to Definition $(*)$ of $\mathfrak{O}$, we have $V \in \mathfrak{U}(x)$, and since $\mathfrak{U}(x)$ is a filter, $U \in \mathfrak{U}(x)$ follows.

(A)⇒(B): Let $U \in \mathfrak{U}(x)$. By (A), $U \in \mathfrak{U}_{\mathfrak{O}}(x)$. By Definition 1.12, there is a $V \in \mathfrak{O}$ such that $x \in V \subseteq U$. Since $V \in \mathfrak{O}$ we have $(\forall v \in V)\; V \in \mathfrak{U}(v)$.

(B)⇒(A): Let $U \in \mathfrak{U}(x)$ and determine V according to (B). Then $V \in \mathfrak{O}$ by definition of $\mathfrak{O}$. Hence U is an $\mathfrak{O}$-neighborhood by Definition 1.12, that is, $\mathfrak{U}(x) \subseteq \mathfrak{U}_{\mathfrak{O}}(x)$. Thus (A) and (B) are equivalent.

(B)⇒(C): Let $U \in \mathfrak{U}(x)$ and determine V according to (B). Then $V \in \mathfrak{O}$ by Definition of $\mathfrak{O}$ and thus V is an $\mathfrak{O}$-neighborhood of x, and so, by (A), $V \in \mathfrak{U}(x)$. If $v \in V$, then $v \in V \subseteq U$, and since $\mathfrak{U}(v)$ is a filter, $U \in \mathfrak{U}(v)$ follows.

(C)⇒(A): Take $U \in \mathfrak{U}(x)$. We must show that U is an $\mathcal{O}$-neighborhood of x. We set

$$(**) \qquad V_0 = \{y \in X : U \in \mathfrak{U}(y)\} \subseteq U.$$

Since $U \in \mathfrak{U}(x)$ we have $x \in V_0$. Now let $v \in V_0$; then $U \in \mathfrak{U}(v)$ by $(**)$. Now by (C), $\big(\exists V \in \mathfrak{U}(v)\big)(\forall w \in V)\, U \in \mathfrak{U}(w)$. Thus $V \subseteq V_0$ by $(**)$ and $V \in \mathfrak{U}(v)$ by $(**)$, and this implies $V_0 \in \mathfrak{U}(v)$ since $\mathfrak{U}(v)$ is a filter. Now we know that $(\forall v \in V_0)\ V_0 \in \mathfrak{U}(v)$, and thus we have $V_0 \in \mathcal{O}$ by $(*)$. Hence U is an $\mathcal{O}$-neighborhood of x by 1.12, that is, $U \in \mathfrak{U}_{\mathfrak{O}}(x)$. □

Theorem 1.16. (Characterization of continuity of functions) *A function $f\colon X \to Y$ between topological spaces is continuous if and only if for each $x \in X$ and each $V \in \mathfrak{U}\big(f(x)\big)$ there is a $U \in \mathfrak{U}(x)$ such that $f(U) \subseteq V$.* □

Corollary 1.17. *A function $f\colon X \to Y$ between two metric spaces is continuous iff for each $x \in X$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that $f(U_\delta(x)) \subseteq U_\varepsilon\big(f(x)\big)$.* □

Expressed more explicitly, f is continuous if for each x and each positive number ε there is a positive number δ such that the relation $d_X(x,y) < \delta$ implies $d_Y\big(f(x), f(y)\big) < \varepsilon$.

This is the famous ε-δ definition of continuity between metric spaces. The topological descriptions of continuity are less technical.

On the other hand, the neighborhood concept allows us to define continuity at a point of a topological spaces:

Definition. Let X and Y be topological spaces and $x \in X$. Then a function $f\colon X \to Y$ is said to be *continuous at* x, if for every neighborhood $V \in \mathfrak{U}\big(f(x)\big)$ there is a neighborhood $U \in \mathfrak{U}(x)$ such that $f(U) \subseteq V$. □

Clearly f is continuous if and only if it is continuous at each point $x \in X$.

Example 1.18. The two element space $\mathbf{2}$ is a topological space with respect to the discrete topology, but also with respect to the Scott topology $\sigma(\mathbf{2}) = \{\emptyset, \{1\}, \mathbf{2}\}$. Let us denote with $\mathbf{2}_\sigma$ the two element space with respect to this topology. This space is sometimes called the *Sierpinski space.*

If $(X, \mathfrak{O})$ is a topological space and A a subset of X, then the characteristic function $\chi_A\colon X \to \mathbf{2}$ (with the discrete topology on $\mathbf{2}$) is continuous iff A is open and closed, i.e. $A, X \setminus A \in \mathfrak{O}$.

The characteristic function $\chi_A\colon X \to \mathbf{2}_\sigma$ is continuous iff A is open, i.e. $A \in \mathfrak{O}$. The function

$$A \mapsto \chi_A : \mathfrak{O} \to C(X, \mathbf{2}_\sigma)$$

is a bijection. □

Exercise E1.4. Verify the assertions made in the discussion of Example 1.18.

The interior and the closure of a set

Definition 1.19. Consider a topological space $(X, \mathfrak{O})$ and $Y \subseteq X$. Define Y° or $\operatorname{int} Y$ to be the union of all open subsets $U \subseteq Y$, that is, $Y^\circ = \bigcup\{U \in \mathfrak{O} : U \subseteq Y\}$. This set is the largest open subset contained in Y and is called the *interior* of Y.

The intersection of all closed supersets $A \supseteq Y$ is the smallest closed set containing Y. It is called the *closure* of Y, written $\overline{Y}$ or $\operatorname{cl} Y$. Then complement $\overline{Y} \setminus Y^\circ$ is called the *boundary* of Y, sometimes written ∂Y or $\operatorname{bdry} Y$. □

Proposition 1.20. *In a topological space $(X, \mathfrak{O})$ we have the following conclusions:*

(i) $Y^{\circ\circ} = Y^\circ$ and $Y \subseteq Z \subseteq X$ implies $Y^\circ \subseteq Z^\circ$.
(ii) $\overline{\overline{Y}} = \overline{Y}$ and $Y \subseteq Z \subseteq X$ implies $\overline{Y} \subseteq \overline{Z}$.
(iii) $Y^\circ = X \setminus \overline{X \setminus Y}$ and $\overline{Y} = X \setminus (X \setminus Y)^\circ$. □

Proposition 1.21. *Let $(X, \mathfrak{O})$ denote a topological space and Y a subset. Let $x \in X$. Then the following assertions are equivalent:*
(i) $x \in Y^\circ$.
(ii) $(\exists U \in \mathfrak{O})\, x \in U \subseteq Y$.
(iii) $(\exists N \in \mathfrak{U}(x))\, N \subseteq Y$.
(iv) $Y \in \mathfrak{U}(x)$.
Also, the following statements are equivalent:
(i) $x \in \overline{Y}$.
(ii) $(\forall U \in \mathfrak{O})\, x \in U \Rightarrow U \cap Y \neq \emptyset$.
(iii) *Every neighborhood of x meets Y.*
(iv) $X \setminus Y$ *is not a neighborhood of x.*

One says that a subspace D of a topological space X is *dense* if $\overline{D} = X$. The set $\mathbb{Q}$ of rational numbers is dense in the space $\mathbb{R}$ or real numbers in its natural topology.

Basis and subbasis of a topology

Often we shall define a topology by starting from a certain set of open sets which generate all open sets in a suitable sense.

Definition 1.22. A set $\mathfrak{B} \subseteq \mathfrak{P}(X)$ is called a *basis of a topology* if $X = \bigcup \mathfrak{B}$ and

$$(\forall A, B \in \mathfrak{B})(\forall x \in A \cap B)(\exists C \in \mathfrak{B})\, x \in C \subseteq A \cap B. \tag{7}$$
□

Proposition 1.23. *For a subset $\mathfrak{B} \subseteq \mathfrak{P}(X)$, the following conditions are equivalent:*

(i) $\mathfrak{B}$ *is a basis of a topology.*
(ii) $\mathfrak{O} \stackrel{\text{def}}{=} \{U : (\forall u \in U)(\exists B \in \mathfrak{B})\, u \in B \subseteq U\}$ *is a topology.*
(iii) *The set of all unions of sets of members of* $\mathfrak{B}$ *is a topology.* □

In the circumstances of 1.23 we say that $\mathfrak{B}$ *is a basis of* $\mathfrak{O}$. The discrete topology $\mathfrak{P}(X)$ of a set has a unique smallest basis, namely, $\big\{\{x\} : x \in X\big\}$.

Example 1.24. (i) Let (X, d) be a metric space. The set $\mathfrak{B}$ of all open balls $U_{1/n}(x)$, $n \in \mathbb{N}$, $x \in X$ is a basis for the metric topology $\mathfrak{O}(X)$.

(ii) The natural topology of $\mathbb{R}$ has a countable basis

$$\left\{ \left]q - \frac{1}{n}, q + \frac{1}{n}\right[: q \in \mathbb{Q},\, n \in \mathbb{N} \right\}. \tag{8}$$

Definition 1.25. One says that a topological space $(X, \mathfrak{O})$ satisfies the *First Axiom of Countability*, if every neighborhoodfilter $\mathfrak{U}(x)$ has a countable basis. It satisfies the *Second Axiom of Countability* if $\mathfrak{O}$ has a countable basis. A space $(X, \mathfrak{O})$ is said to be *separable* if it contains a countable dense subset.

Exercise E1.5. Every space satisfying the Second Axiom of Countability satisfies the first axiom of countability. The discrete topology of a set satisfies the first axiom of countability; but if it fails to be countable, it does not satisfy the Second Axiom of Countability. A a separable metric space satisfies the Second Axiom of Countability.

Every set of cardinals has a smallest element. Given this piece of information we can attach to a topological space $(X, \mathfrak{O})$ a cardinal, called its *weight* :

$$w(X) = \min\{\text{card}\, \mathfrak{B} : \mathfrak{B} \text{ is a basis of } \mathfrak{O}\}. \tag{9}$$

The weight of a topological space is countable iff it satisfies the Second Axiom of Countability.

Definition 1.26. A set $\mathfrak{B}$ of subsets of a topological space is said to be a *basis for the closed sets* if every closed subset is an intersection of subsets taken from $\mathfrak{B}$. □

The set of complements of the sets of a basis of a topology is a basis for the closed sets of this topology and vice versa.

Proposition 1.27. *Let* $\mathcal{T} \subseteq \mathfrak{P}\big(\mathfrak{P}(X)\big)$ *be a set of topologies. Then* $\bigcap \mathcal{T} \subseteq \mathfrak{P}(X)$ *is a topology.* □

By Proposition 1.27, every set $\mathfrak{M}$ of subsets of a set X is contained in a unique smallest topology $\mathfrak{O}$, called the *topology generated by* $\mathfrak{M}$. Under these circumstances, $\mathfrak{M}$ is called a *subbasis of* $\mathfrak{O}$.

Proposition 1.28. *The topology generated by a set* $\mathfrak{M}$ *of a set* X *consists of all unions of finite intersections of sets taken from* $\mathfrak{M}$. □

Definition 1.29. Let $(X, \leq)$ be a totally ordered set, i.e. a poset for which every two elements are comparable w.r.t. $\leq$. Then the set of all subsets X, $\uparrow a$, $a \in X$, and $\downarrow a$, $a \in X$ is a subbasis for the closed sets of a topology, called the *order topology* of $(X, \leq)$. □

[It should be understood that by $\downarrow a$ in a poset we mean the set of all $x \in X$ with $x \leq a$.]

Example 1.30. In $\mathbb{R}$ the set of all $]q, \infty[$, $q \in \mathbb{Q}$ and $]-\infty, q[$, $q \in \mathbb{Q}$ form a subbasis of the natural topology. □

Exercise E1.6. Show that the order topology on $\mathbb{R}$ is the natural topology of $\mathbb{R}$.

The Lower Separation Axioms

Lemma 1.31. *The relation $\preceq$ in a topological space defined by*

$$x \preceq y \text{ if and ony if } (\forall U \in \mathfrak{O})\, x \in U \Rightarrow y \in U. \tag{10}$$

is reflexive and transitive. □

We have $x \preceq y$ if every neighborhood of x is a neighborhood of y. If $(X, \leq)$ is a quasiordered set and $\mathfrak{O}$ is the Alexandroff discrete topology, then $x \preceq y$ iff $x \leq y$.

Definition 1.32. The quasiorder $\preceq$ on a topological space is called the *specialisation quasiorder.* □

While this is not relevant here, let us mention that the name arises from algebraic geometry.

Exercise E1.7. Set $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ and show that on $\big(\mathbb{R}^*, \sigma(\mathbb{R}^*)\big)$ with the Scott topology, one has $x \preceq y$ iff $x \leq y$. □

Notice that the Alexandroff discrete topology $\alpha(\mathbb{R}^*)$ has a basis $\{\uparrow r : r \in \mathbb{R}^*\}$, and thus $\sigma(\mathbb{R}^*) \subseteq \alpha(\mathbb{R}^*)$ but $\sigma(\mathbb{R}^*) \neq \alpha(\mathbb{R}^*)$. So two different topologies can produce the same specialisation quasiorder.

The specialisation quasiorder with respect to the indiscrete topology is the trivial quasiorder that holds always between two elements. The specialisation order with respect to the discrete topology is equality.

Proposition 1.33. *In a topological space, the point closure $\overline{\{a\}}$ is the lower set $\downarrow a$ w.r.t. the specialisation order.* □

Definition 1.34. A topological space $(X, \mathfrak{O})$ is said to satisfy the Axiom (T_0), or is said to be a T_0*-space* if and only if the specialisation quasiorder is a partial order. Under these conditions, the topology $\mathfrak{O}$ is called a T_0*-topology.* □

Sometimes (following Alexandroff and Hopf), the Axiom (T_0) is called *Kolmogoroff's Axiom.*

The Axiom (T_0) is equivalent to the following statement:

(T_0') The function $x \mapsto \mathfrak{U}(x): X \to \mathfrak{P}\big(\mathfrak{P}(X)\big)$ which assigns to an element its neighborhood filter is injective.

In other words: "Different points have different neighborhood filters."

Proof of (T_0) $\Leftrightarrow$ (T_0'): $\preceq$ is a partial order iff $x \preceq y$ and $y \preceq x$ implies $x = y$. Now $x \preceq y \Leftrightarrow \mathfrak{U}(x) \subseteq \mathfrak{U}(y)$. Hence $\preceq$ is a partial order iff $\mathfrak{U}(x) = \mathfrak{U}(y)$ implies $x = y$, and this is exactly (T_0').

Now $\mathfrak{U}(x) = \mathfrak{U}(y)$ means that for all $U \in \mathfrak{O}$, the relation $x \in U$ holds iff the relation $y \in U$ holds, that is $(\forall U \in \mathfrak{O})\, x \in \mathfrak{U} \Leftrightarrow y \in \mathfrak{U}$ and so (T_0') is equivalent to saying that $x \neq y \Rightarrow (\exists U \in \mathfrak{O})(x \in \mathfrak{U}$ and $y \notin U)$ or ($y \in U$ and $x \notin U$), and this shows that (T_0) is also equivalent to

(T_0'') For two different elements x and y in X, there is an open set such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

In other words, for two different points there is an open set containing precisely one of the two points.

Definition 1.35. The space X is said to satisfy the separation axiom (T_1) (or to be a T_1*-space*), and its topology $\mathfrak{O}$ is called a T_1*-topology*, if the specialisation quasiorder is discrete, i.e., is equality.

A topological space is a T_1 space if and only if

- every singleton subset is closed.
  That is $\overline{\{a\}} = \{a\}$ for all $a \in X$.
  Another equivalent formulation of the Axiom (T_1) is
- If x and y are two different points then there is an open set U containing x but not y.

Example 1.36. The cofinite topology is always a T_1-topology. □

The Alexandroff-discrete topology of a nontrivial poset is a T_0-topology but not a T_1-topology. For instance, the Sierpinski space $\mathbf{2}_\sigma$ is a T_0-space which is not a T_1-space.

The terminology for the hierarchy (T_n) of separation axioms appears to have entered the literature 1935 through the influential book by Alexandroff and Hopf in a section of the book called „**T**rennungsaxiome" (pp. 58 ff.). Alexandroff and Hopf call the Axiom (T_1) „das erste Frechetsche Trennungsaxiom", p. 58, 59), and they attach with the higher separation axioms the names of Hausdorff, Vietoris, and Tietze. In due time we shall face these axioms.

In Bourbaki T_0-spaces are called ≪espaces de Kolmogoroff≫ (s. §1, Ex. 2, p. 89). Alexandroff and Hopf appear to have had access to an unpublished manuscript by Kolmogoroff which appears to have dealt with quotient spaces (see Alexandroff and Hopf p. 61 and p. 619) and which is likely to have been the origin of calling

(T_0) Kolomogoroff's Axiom; Alexandroff continues to refer to it under this name in later papers. Fréchet calls T_1-spaces ≪espaces accessibles≫.

Definition 1.37. The space X is said to satisfy the *Hausdorff separation axiom* (T_2) (or to be a *T_2-space*), and its topology $\mathfrak{O}$ is called a *Hausdorff topology*, respectively, *T_2-topology*, if the following condition is satisfied:

$$(T_2) \qquad (\forall x, y \in X)\, x \neq y \Rightarrow \big(\exists\, U \in \mathfrak{U}(x),\, V \in \mathfrak{U}(y)\big)\; U \cap V = \emptyset.$$

In other words, two different points have disjoint neighborhoods.

Exercise E1.8. Let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be two topologies on a set such that $\mathfrak{O}_1 \subseteq \mathfrak{O}_2$. If $\mathfrak{O}_1$ is a T_n-topology for $n = 0, 1, 2$, then $\mathfrak{O}_2$ is a T_n-topology.

Definition 1.38. The space X is said to be *regular*, and its topology $\mathfrak{O}$ is called a *regular topology* if the following condition is satisfied:

$$(\forall x \in X)\big(\forall U \in \mathfrak{U}(x)\big)\big(\exists A \in \mathfrak{U}(x)\big)\; \overline{A} = A \text{ and } A \subseteq U.$$

It is said to satisfy the axiom (T_3) (or to be a *T_3-space*), if it is a regular T_0-space. In other words:
(T_3) X is a T_0-space and every neighborhood filter has a basis of closed sets. □

For a T_0-space X, the axiom (T_3) is also equivalent to the following statement:
($*$) *For any $x \in X$ and any neighborhood $U \in \mathfrak{U}(x)$, there are open sets V and W such that $x \in V$, $V \cap W = \emptyset$, and $U \cup W = X$.*

Exercise E1.9. (a) Show that (T_3) is equivalent to (T_0) and ($*$).
(b) Prove the following propositions:
(i) Every metric space is regular. In particular, the natural topology of $\mathbb{R}$ is regular.
(ii) Every metric space is a Hausdorff space.
(iii) On $\mathbb{R}$ let $\mathfrak{O}^*$ be the collection of all sets $U \setminus C$ where U is open in the natural topology $\mathfrak{O}$ of $\mathbb{R}$ and C is a countable set. Then $\mathfrak{O}^*$ is a topology which is properly finer than the natural topology of $\mathbb{R}$, that is, the identity function $\mathrm{id}_{\mathbb{R}}\colon (\mathbb{R}, \mathfrak{O}^*) \to (\mathbb{R}, \mathfrak{O})$ is continuous, but its inverse function is not continuous. The topology $\mathfrak{O}^*$ is not regular. A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous as a function $(\mathbb{R}, \mathfrak{O}_c) \to (\mathbb{R}, \mathfrak{O})$ if and only if it is continuous as a function $(\mathbb{R}, \mathfrak{O}) \to (\mathbb{R}, \mathfrak{O})$.
(iv) (T_3)⇒(T_2)⇒(T_1)⇒(T_0) and (T_0) $\not\Rightarrow$(T_1) $\not\Rightarrow$(T_2) $\not\Rightarrow$(T_3).

Comments on Exercise E1.9.(iii)

For a proof of the assertions in Exercise E1.9.(iii) we have to show, in the first place, that $\mathfrak{O}^*$ is a topology. This seems less than obvious. We prove a few lemmas first.

Lemma 1. (i) *Let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be topologies on a set X. Then there is a unique smallest topology $\mathfrak{O}_1 \vee \mathfrak{O}_2$ containing both $\mathfrak{O}_1$ and $\mathfrak{O}_2$.*

(ii) If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are bases of $\mathfrak{O}_1$ and $\mathfrak{O}_2$, respectively, then

$$\mathfrak{B} \stackrel{\text{def}}{=} \{U \cap V : U \in \mathfrak{B}_1 \text{ and } V \in B_2\}$$

is a basis of $\mathfrak{O}_1 \vee \mathfrak{O}_2$.

Proof . (i) The set $\mathfrak{O}_1 \cup \mathfrak{O}_2$ is a subbasis for the topology $\mathfrak{O}_1 \vee \mathfrak{O}_2$ according to Proposition 1.28.

(ii) Let $U_j \in \mathfrak{B}_1$ and $V_j \in \mathfrak{B}_2$, $j = 1, 2$ and $x \in (U_1 \cap V_1) \cap (U_2 \cap V_2)$. Find $U_3 \in \mathfrak{B}_1$ and $V_3 \in \mathfrak{B}_2$ such that $x \in U_3 \subseteq U_1 \cap U_2$ and $x \in V_3 \subseteq V_1 \cap V_2$. Then $x \in U_3 \cap V_3 \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2) = (U_1 \cap V_1) \cap (U_2 \cap V_2)$. It follows from Proposition 1.23 that $\mathfrak{B}$ is a basis of a topology which contains $\mathfrak{O}_1$ and $\mathfrak{O}_2$, and since $U \cap V$, for $U \in \mathfrak{O}_1$ and $V \in \mathfrak{O}_2$, is contained in $\mathfrak{O}_1 \vee \mathfrak{O}_2$ we know that $\mathfrak{B}$ is a basis of $\mathfrak{O}_1 \vee \mathfrak{O}_2$. □

Lemma 2. *If $\mathfrak{O}_1$ has a countable basis $\{U_1, U_2, \ldots\}$, then for every $W \in \mathfrak{O}_1 \vee \mathfrak{O}_2$ there is a sequence of sets $(V_n)_{n\in\mathbb{N}}$, $V_n \in \mathfrak{O}_2$ such that $W = \bigcup_{n\in\mathbb{N}} U_n \cap V_n$.*

Proof . By Lemma 1 there is a family $(U_{n_j} \cap V_j : j \in J\}$, $V_j \in \mathfrak{O}_2$, such that $W = \bigcup_{j\in J}(U_{n_j} \cap V_j)$. For each natural number n let $J_n = \{j \in J : n_j = n\}$. Then $W = \bigcup_{n\in\mathbb{N}} \bigcup_{j\in J_n}(U_n \cap V_j)$. We have

$$\bigcup_{j\in J_n}(U_n \cap V_j) = U_n \cap \bigcup_{j\in J_n} V_j.$$

Set $V_n = \bigcup_{j\in J_n} V_j$. Then $V_n \in \mathfrak{O}_2$ and $W = \bigcup_{n\in\mathbb{N}} U_n \cap V_n$. □

In general, many of the V_n will be empty. Set

$$U \stackrel{\text{def}}{=} \bigcup\{U_n : V_n \neq \emptyset\}$$

Lemma 3. *In the circumstances of* Lemma 2*,*

$$(*) \qquad U \cap \bigcap_{n\in N} V_n \subseteq W \subseteq U$$

Proof . The left containment follows from $U_n \cap \bigcap_{m\in\mathbb{N}} V_m \subseteq U_n \cap V_n$, and the right containment from $U_n \cap V_n \subseteq U_n$, $n = 1, 2, \ldots$. □

Lemma 4. *Let X be a set. Then the set of subsets of X consisting of the empty set and all complements of countable subsets of X is a topology that is closed under countable intersections.*

Proof. The collection of countable subsets, plus the whole set X, is closed under arbitrary intersections and countable unions. The complements therefore yield the desired topology. □

The topology of Lemma 4 is called the *cocountable topology*. If X is countable, it agrees with the discrete topology. More generally, on each countable subset of a set X, the cocountable topology induces the discrete topology.

Proposition 5. *Let $(X, \mathfrak{O})$ be a topological space satisfying the Second Axiom of Countability, and let $\mathfrak{O}_{\text{cc}}$ be the cocountable topology on X. Then $\mathfrak{O} \vee \mathfrak{O}_{\text{cc}}$ consists of all sets of the form $U \setminus C$ where $U \in \mathfrak{O}$ and C is countable.*

Proof. The sets V_n in Lemma 3 are either empty or are complements of countable sets. Then $\bigcap_{n\in\mathbb{N}} V_n$ is either empty or is a complement of a countable set. Thus for each $W \in \mathfrak{O} \vee \mathfrak{O}_{\text{cc}}$ there is a countable set D in X and there is an open set $U \in \mathfrak{O}$ such that

$$(**) \qquad U \setminus D \subseteq W \subseteq U$$

Then $C \stackrel{\text{def}}{=} W \setminus (U \setminus D) \subseteq U \setminus (U \setminus D) = D$. Therefore $W = (U \setminus D) \cup C = U \setminus (D \setminus C)$. Since $D \setminus C$ is countable, this is what we had to show. □

We apply this to $X = \mathbb{R}$, and the natural topology $\mathfrak{O}$ on $\mathbb{R}$. We see that the set of all $U \setminus C$ with open subsets U of $\mathbb{R}$ and countable subsets C of R is a topology $\mathfrak{O}^*$ of $\mathbb{R}$. If $U \setminus C \in \mathfrak{O}^*$, then the $\mathfrak{O}^*$-closure U^* of $U \setminus C$ agrees with $\overline{U}$, the $\mathfrak{O}$-closure of U and the interior $\neg\neg U$ of $\overline{U}$ is also the $\mathfrak{O}^*$-interior of U^*. These facts imply further properties $(X, \mathfrak{O}^*)$ for example, that the topology is Hausdorff but not regular.

Quotient Spaces

An equivalence relation R on a set X is a reflexive, symmetric, and transitive relation. Recall that a binary relation is a subset of $X \times X$; in place of $(x, y) \in R$ one frequently writes $x\,R\,y$.

Every equivalence relation R on a set X gives rise to a new set X/R, the set of all equivalence classes $R(x) = \{x' \in X : (x, x') \in R\}$. Note $x \in R(x)$. If A and B are R-equivalence classes, then either $A \cap B = \emptyset$ or $A = B$. Thus X is a disjoint union of all R equivalence classes. One calls a set $\mathcal{P} \subseteq \mathfrak{P}(X)$ of subsets a *partition* of X if two different members of $\mathcal{P}$ are disjoint and $\bigcup \mathcal{P} = X$. We have seen that every equivalence relation on a set X provides us with a partition of X. Conversely, if $\mathcal{P}$ is a partition of X, then $R \stackrel{\text{def}}{=} \{(x, x') \in X \times X : (\exists A \in \mathcal{P}) x,\ x' \in A\}$ is an equivalence relation whose partition is the given one. There is a bijection between equivalence relations and partitions.

The new set X/R is called the *quotient set modulo* R. The function $p_R\colon X \to X/R$, $p_R(x) = R(x)$ is called the *quotient map*.

One of the primary occurrences of equivalence relations is the kernel relation of a function, as follows. Let $f: X \to Y$ be a function. Define $R_f = \{(x, x') : f(x) = f(x')\}$. Then there is a bijective function $f': X/R_f \to f(X)$ which is unabiguously defined by $f'\big(R_f(x)\big) = f(x)$. If $\mathrm{inc}: f(X) \to Y$ is the inclusion map $y \mapsto y: f(X) \to Y$ then we have the so-called *canonical decomposition* $f = \mathrm{inc} \circ f' \circ p_{R_f}$ of the given function:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle p_{R_f}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{inc}} \\ X/R_f & \xrightarrow[\quad f' \quad]{} & f(X). \end{array}$$

In this decomposition of f the quotient map p_{R_f} is surjective, the induced function f' is bijective, the inclusion map is injective.

The objective of this subsection is to endow the quotient space X/R of a topological space X with a topology in a natural way so that the quotient map is continuous and that, if R is the kernel relation of a continuous function, the induced bijective function $f': X/R_f \to f(X)$ is continuous.

If R is an equivalence relation on a set X we shall call a subset $Y \subseteq X$ *saturated* (with respect to R) iff for each $y \in Y$ the entire equivalence class $R(y)$ of y is contained in Y. Thus saturated subsets of X are exactly the subsets $Y \subseteq X$ which satisfy $Y = \bigcup_{y \in Y} R(y)$. If Y is a saturated subset, we let Y/R denote the partition $\{R(y) : y \in Y\}$ of Y induced by R.

Now let R be an equivalence relation on a topological space $(X, \mathfrak{O})$. We let $\mathfrak{O}_R$ denote the set of all saturated open subsets of X, that is $\mathfrak{O}_R = \{U \in \mathfrak{O} : U = \bigcup_{u \in U} R(u)\}$. and we set $\mathfrak{O}(X/R) = \{U/R : U \in \mathfrak{O}_R\} \subseteq \mathfrak{P}(X/R)$. Obviously the function $U \mapsto U/R : \mathfrak{O}_R \to \mathfrak{O}(X/R)$ is a bijection preserving arebitrary sups and infs. Since $\mathfrak{O}_R$ is closed under the formation of arbitrary unions and intersections, and since $X, \emptyset \in \mathfrak{O}_R$ and $X/R, \emptyset \in \mathfrak{O}(X/R)$ we see that $\mathfrak{O}(X/R)$ is a topology on X/R.

Definition 1.39. The topological space $\big(X/R, \mathfrak{O}(X/R)\big)$ is called the quotient space of X modulo R.

Proposition 1.40. *The quotient map $p_{R_f}: X \to X/R$ is continuous.*

The quotient space X/R is a T_1 space if and only if all R-equivalence classes are closed.

The quotient space X/R is a Hausdorff space if and only if for two disjoint R-classes A and B there are disjoint saturated open sets U and V containing A and B, respectively. □

The following is a recall from 1.10:

Definition 1.41. A function $f: X \to Y$ between topological spaces is called *open* if $f(U)$ is open for each open set U, that is, if open sets have open images.

Exercise E1.10. Show that the function $x \mapsto x^2 : \mathbb{R} \to \mathbb{R}$ fails to be open. □

Proposition 1.42. *For an equivalence relation R on a topological space X, the following statements are equivalent:*

(i) The quotient map $p_R: X \to X/R$ is open.
(ii) For each open subset U of X the *saturation* $\bigcup_{u \in U} R(u)$ is open. □

Group Actions

There is a prominent situation for which quotient maps are open.

Definition 1.43. A continuous function $f: X \to Y$ between topological spaces is called a *homeomorphism*, if it is bijective and its inverse function $f^{-1}: Y \to X$ is continuous. Two spaces X and Y are called *homeomorphic* if there exists a homeomorphism between them. □

A function $f: (X, \mathfrak{O}_X) \to (Y, \mathfrak{O}_Y)$ between topological spaces is a homeomorphism if and only if the function f implements a bijection $U \mapsto f(U) : \mathfrak{O}_X \to \mathfrak{O}_Y$.

Exercise E1.11. (i) For any topological space X, the set H of homeomorphisms $f: X \to X$ is a group.

(ii) Let G be a subgroup of H. Let us write $g{\cdot}x = g(x)$ for $g \in G$ and $x \in X$. Then the set $X/G \stackrel{\text{def}}{=} \{G{\cdot}x | x \in X\}$ is a partition of X. The corresponding equivalence relation is given by $x \sim y$ iff $(\exists g \in G)\, y = g{\cdot}x$.

(iii) We let $p: X \to X/G$ denote the quotient map defined by $p(x) = G{\cdot}x$ and endow X/G with the quotient topology. Then p is an open map. □

The set $G{\cdot}x$ is called the *orbit* of x under the action of G, or simply the G*-orbit* of x. The quotient space X/G is called the *orbit space*.

Exercise E1.12. (i) Let X be the space $\mathbb{R}$ of real numbers with its natural topology. Every $r \in \mathbb{R}$ defines a function $T_r: X \to X$, via $T_r(x) = r + x$, the translation by r. Every such translation is a homeomorphism of $\mathbb{R}$.

(ii) Let G be the group of all homeomorphisms T_r with $r \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers.

Describe the orbits $G{\cdot}x$ of the action of G on X.

Describe the orbit space X/G. Show that it is homeomorphic to the one-sphere $\mathbb{S}^1$.

(iii) Now let X be as before, but take $G = \{T_r : r \in \mathbb{Q}\}$ where $\mathbb{Q}$ is the set of rational numbers. Discuss orbits and orbit space.

(iv) Test these orbit spaces for the validity of separation axioms. □

A Universal Construction

Let us consider another useful application of quotient spaces.

On any topological space X with topology $\mathfrak{O}_X$, the binary relation defined by $x \equiv y$ iff $x \le y$ and $y \le x$ (with respect to the specialisation quasiorder $\le$) is an equivalence relation. The quotient space $X/\equiv$ endowed with its quotient topology $\mathfrak{O}_{X/\equiv}$ will be denoted by $T_0(X)$.

Proposition 1.44. *For any topological space X, the space $T_0(X)$ is a T_0-space, and if $q_X: X \to T_0(X) = X/\equiv$ denotes the quotient map which assigns to each point its equivalence class, then the function $U \mapsto q_X^{-1}(U): \mathfrak{O}_{T_0(X)} \to \mathfrak{O}_X$ is a bijection. Moreover, if $f: X \to Y$ is any continuous function into a T_0-space, then there is a unique continuous function $f': X/\equiv \to Y$ such that $f = f' \circ q_X$.* □

As a consequence of these remarks, for most purposes it is no restriction of generality to assume that a topological space under consideration satisfies at least the separation axiom (T_0).

The Canonical Decomposition

It is satisfying to know that the quotient topology provides the quotient space modulo a kernel relation with that topology which allows the canonical decomposition of a *continuous* function between topological spaces to work correctly.

Theorem 1.45. (The Canonical Decomposition of Continuous Functions) *Let $f: X \to Y$ be a continuous function between topological spaces and let*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle p_{R_f}}\downarrow & & \uparrow{\scriptstyle \mathrm{inc}} \\ X/R_f & \xrightarrow[\quad f' \quad]{} & f(X). \end{array}$$

be its canonical decomposition, where R_f denotes the kernel relation of f. Then $f': X/R_f \to f(X)$ is a continuous bijection, the quotiend map p_{R_f} is a continuous surjection, the inclusion map is an embedding, i.e., a homeomorphism onto its image. □

If the space Y is a Hausdorff space, then so is the subspace $f(X)$; then the continuous bijection f' in the canonical decomposition theorem tells us at once that the quotients space X/R_f is a Hausdorff space—whether X itself is a Hausdorf space or not.

Corollary 1.46. *If $f: X \to Y$ is a continuous function into a Hausdorff space, then the quotient space X/R_f is a Hausdorff space.* □

Naturally one wishes to understand when f' is a homeomorphism.

Proposition 1.47. *Let $f\colon X \to Y$ be a continuous function between topological spaces. Then the following conditions are equivalent:*
(i) *The corestriction $x \mapsto f(x)\colon X \to f(X)$ is open.*
(ii) *The quotient morphism $p_f\colon X \to X/R_f$ is open and $f'\colon X/R_f \to f(X)$ is a homeomorphism.* □

Products

Definition 1.48. Let $(X_j : j \in J)$ be a family of sets. The *cartesian product* or simply *product* of this family, written $\prod_{j\in J} X_j$, is the set of all functions $f\colon J \to \bigcup_{j\in J} X_j$ such that $(\forall j \in J)\, f(j) \in X_j$. These functions are also written $(x_j)_{j\in J}$ with $x_j = f(j)$ and are called *J-tuples.* The function $\mathrm{pr}_k\colon \prod_{j\in J} X_j \to X_k$, $\mathrm{pr}_k\big((x_j)_{j\in J}\big) = x_k$ is called the *projection* of the product onto the factor X_k. □

The following statement looks innocent, but it is an axiom:
Axiom 1.49. (Axiom of Choice) *For each set J and each family of nonempty sets $(X_j : j \in J)$ the product $\prod_{j\in J} X_j$ is not empty.* □

Proposition 1.50. *If the product $P \overset{\text{def}}{=} \prod_{j\in J} X_j$ is not empty, then for each $k \in J$, the projection $pr_k\colon P \to X_k$ is surjective, and there is an injection $s_k\colon X_k \to P$ such that* $\mathrm{pr}_k \circ s_k = \mathrm{id}_{X_k}$. □

Now we wish to consider families of topological spaces and to endow their products with suitable topologies. For this purpose let us consider a family $(X_j : j \in J)$ of topological spaces. Let us call a family $(U_j : j \in J)$ of open subsets U_j of X_j a *basic family of open subsets*, if there is a finite subset F of J such that $U_j = X_j$ for all $j \in J \setminus F$. Thus for a basic family of open subsets only a finite number of them consists of *proper* subsets.

Lemma 1.51. *The set $\mathfrak{B}$ of all products*

$$U \overset{\text{def}}{=} \prod_{j\in J} U_j,$$

where $(U_j : j \in J)$ ranges through the set of all basic open subfamilies of $(U_j : j \in J)$ is a basis for a topology on

$$P \overset{\text{def}}{=} \prod_{j\in J} X_j,$$

and $\mathfrak{B}$ is closed under finite intersections. The set of all unions of members of $\mathfrak{B}$ is a topology $\mathfrak{O}$ on P. □

Definition 1.52. The topology $\mathfrak{O}$ on P is called the *product topology* or the *Tychonoff topology*. The topological space $(P, \mathfrak{O})$ is called the *product space* of the family $(X_j : j \in J)$ of topological spaces. □

Proposition 1.53. *Let $(X_j : j \in J)$ be a family of topological spaces and let $P \stackrel{\text{def}}{=} \prod_{j \in J} X_j$ the product.*

(i) *Each projection $\text{pr}_k \colon P \to X_k$ is continuous and open.*

(ii) *A function $f \colon X \to P$ from a topological space into the product P is continuous if and only if for all $k \in J$ the functions $\text{pr}_k \circ f \colon X \to X_k$ are continuous.*

(iii) *The product topology of P is the smallest topolog such that all projections $\text{pr}_k \colon P \to X_k$ are continuous.* □

Proposition 1.54. *The product $\prod_{j \in J} X_j$ is a Hausdorff space if all factors X_j are Hausdorff. If $(b_j)_{j \in J} \in \prod_{j \in J} X_j$ and the product is Hausdorff, then all factors are Hausdorff.*

Proof. Assume all factors are T_2 and consider $(x_j)_{j \in J} \neq (y_j)_{j \in J}$. There is at least one $k \in J$ such that $x_k \neq y_k$. Use that X_k is Hausdorff and complete the proof that $\prod_{j \in J}$ is Hausdorff. Now assume that the product is Hausdorff and contains $(b_j)_{j \in J}$. Let $k \in J$ and $x \neq y$ in X_k. Define

$$x_j \begin{cases} x & \text{if } j = k, \\ b_j & \text{if } j \neq k, \end{cases} \quad \text{and} \quad y_j \begin{cases} y & \text{if } j = k, \\ b_j & \text{if } j \neq k. \end{cases}$$

Then $(x_j)_{j \in J}$ and $(y_j)_{j \in J}$ are different and thus have disjoint neighborhoods which we may just as well assume to be two basic neighborhoods $U \stackrel{\text{def}}{=} \prod_{j \in J} U_j$ and $V \stackrel{\text{def}}{=} \prod_{j \in J} V_j$. For $j \neq k$ we have $b_j \in U_j \cap V_j$. For U and V to be disjoint it is therefore necessary that U_k and V_k are disjoint. Finish the proof! □

Chapter 2
Compactness

We proceed to special properties of topological spaces. From basic analysis we know that compactness is one of these.

Definition 2.1. Let $(X, \mathfrak{O})$ be a topological space. An *open cover* is a subset $\mathcal{C} \subseteq \mathfrak{O}$ such that $X = \bigcup \mathcal{C}$ or a family $(U_j : j \in J)$ of open sets $U_j \in \mathfrak{O}$ such that $X = \bigcup_{j \in J} U_j$. The cover is said to be *finite* if $\mathcal{C}$, respectively, J is finite. A subset $\mathcal{C}' \subseteq \mathcal{C}$ which is itself a cover is called a *subcover*. A subcover of an open cover $(U_j : j \in J)$ is a subfamily $(U_j : j \in K)$, $K \subseteq J$ which is itself a cover.

Definition 2.2. A topological space $(X, \mathfrak{O})$ is said to be *compact* if every open cover has a finite subcover. □

Proposition 2.3. *For a topological space $(X, \mathfrak{O})$ the following statements are equivalent:*
(i) *X is compact.*
(ii) *Every filterbasis of closed subsets has a nonempty intersection.* □

Exercise E2.1. Prove the following assertions:
(i) A closed subspace of a compact space is compact.
(ii) If X is a compact subspace of a Hausdorff space Y, then X is closed in Y.
(iii) Every finite space is compact.
(iv) In the Sierpinski space $\mathbf{2}_s$ the subset $\{1\}$ is compact but not closed.
(v) Every set is compact in the cofinite topology.
(vi) Every compact and discrete space is finite. □

Definition 2.4. An element x of a topological space is said to be an *accumulation point* or a *cluster point* of a sequence $(x_n)_{n \in \mathbb{N}}$ of X if for each $U \in \mathfrak{U}(x)$ the set $\{n \in \mathbb{N} : x_n \in U\}$ is infinite. □

A point x in a topological space is an accumulation point of the sequence $(x_n)_{n \in \mathbb{N}}$ iff for each natural number n and each $U \in \mathfrak{U}(x)$ there is an $m \geq n$ such that $x_m \in U$.

Lemma 2.5. *Assume that $(x_n)_{n \in \mathbb{N}}$ is a sequence in a topological space X. Let $\mathcal{F}$ be the set of all sets*

$$F_n \stackrel{\text{def}}{=} \overline{\{x_m : n \leq m\}} = \overline{\{x_n, x_{n+1}, x_{n+2}, \dots\}}.$$

Then the following conclusions hold:

(i) A point $x \in X$ is an accumulation point of the sequence iff

$$x \in \overline{\{x_n, x_{n+1}, x_{n+2}, \ldots\}} \text{ for all } n \in \mathbb{N}.$$

(ii) $\mathcal{F}$ is a filter basis, and $\bigcap \mathcal{F}$ is the set of all accumulation points of $(x_n)_{n\in\mathbb{N}}$.□

Proposition 2.6. *Let X be a compact topological space. Then every sequence $(x_n)_{n\in\mathbb{N}}$ in X has at least one accumulation point.*

Proof. For a given sequence $(x_n)_{n\in\mathbb{N}}$ let $\mathcal{F}$ be the filterbasis of compact sets whose members are $F_n \stackrel{\text{def}}{=} \overline{\{x_m : n \leq m\}}$. By (i) and Proposition 2.3 we know $\bigcap \mathcal{F} \neq \emptyset$. In view of Lemma 2.5(ii), this proves the claim. □

The reverse implication is not true in general, however we shall see that it is true for metric spaces. These matters are more involved. We first establish two lemmas which are of independent interest. A topological space in which every sequence has an accumulation point is called *sequentially compact*

Lemma 2.7. (Lebesgue's Lemma) *Let (X, d) be a sequentially compact metric space and let $\mathcal{U}$ be an arbitrary open cover of X. Then there is a number $r > 0$ such that for each $x \in X$ there is a $U \in \mathcal{U}$ such that the open ball $U_r(x)$ of radius r around x is contained in U.*

Proof . Suppose that the Lemma is false; then there is an open cover $\mathcal{U}$ such that for each $m \in \mathbb{N}$ there is an $x_m \in X$ such that $U_{1/m}(x_m)$ is contained in no $U \in \mathcal{U}$. Since X is sequentially compact, the sequence $(x_m)_{m\in\mathbb{N}}$ has at least one accumulation point x. Since $X = \bigcup \mathcal{U}$ there is a $U \in \mathcal{U}$ with $x \in U$. Since U is open, there is an $s > 0$ such that $U_s(x) \subseteq U$. Now let $n \in \mathbb{N}$ be such that $2/n < s$. Then $U_{1/n}(x)$ contains at least one x_m with $m \geq n$. Then $U_{1/m}(x_m) \subseteq U_{2/n}(x) \subseteq U$, and that is a contradiction to the choice of x_m. □

A number $r > 0$ as in the conclusion of Lemma 2.7 is called a *Lebesgue number* of the cover $\mathcal{U}$.

Lemma 2.8. *Let (X, d) be a sequentially compact metric space and let $r > 0$. Then there is a finite subset $F \subseteq X$ such that for each $x \in X$ there is an element $y \in F$ with $d(x, y) < r$. That is, $\{U_r(y) : y \in X\}$ is a cover of X.*

Proof. Suppose that the Lemma is false. Then there is a number $r > 0$ such that for each finite subset $F \subseteq X$ one finds an $x \in X$ such that $d(x, y) \geq r$ for all $y \in F$. Pick an arbitrary $x_1 \in X$ and assume that we have found elements $x_1, \ldots, x_m$ in such a fashion that $d(x_j, x_k) \geq r$ for all $j \neq k$ in $\{1, \ldots, m\}$. By hypothesis we find an $x_{m+1} \in X$ such that $d(x_j, x_{m+1}) \geq r$ for all $j = 1, \ldots, m$. Recursively we thus find a sequence $x_1, x_2, \ldots,$ in X. Since X is sequentially compact, this sequence has an accumulation point $x \in X$. By the definition of accumulation point the set $\{n \in \mathbb{N} : x_n \in U_{r/2}(x)\}$ is infinite. Thus we find two different indices $h \neq k$ in $\mathbb{N}$

such that x_j, $x_k \in U_{r/2}(x)$, whence $d(x_j, x_k) \le d(x_j, x) + d(x, x_k) < r/2 + r/2 = r$. This is a contradiction to the construction of $(x_n)_{n\in\mathbb{N}}$. □

Definition 2.9. A metric space (X, d) is said to be *precompact* or *totally bounded* if for each number $r > 0$ there is a finite subset $F \subseteq X$ such that $X \subseteq \bigcup_{x\in F} U_r(x)$.□

We have observed in Lemma 2.8 that every compact metric space is precompact. The space $\mathbb{Q} \cap [0,1]$ is precompact with its natural metric but not compact.

Lemma 2.10. *Assume that X is a precompact metric space, and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Then there is an increasing sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers such that the equations $y_n \stackrel{\text{def}}{=} x_{m_n}$ define a Cauchy sequence $(y_n)_{n\in\mathbb{N}}$ in X.*

Proof . Assume that we had found a descending sequence $V_1 \supseteq V_2 \supseteq \cdots$ of subsets of X such that the diameters δ_n of V_n exist and converge to 0, and that moreover $\{m \in \mathbb{N} : x_m \in V_n\}$ is infinite for all $n \in \mathbb{N}$. Then we let $m_1 \in \mathbb{N}$ be such that $x_{m_1} \in V_1$ and assume that $m_1 < m_2 < \cdots < m_n$ have been selected so that $x_{m_k} \in V_k$ for $k = 1, \ldots, n$. Since $\{m \in \mathbb{N} : x_m \in V_{n+1}\}$ is infinite we find an $m_{n+1} > m_n$ such that $x_{m_{n+1}} \in V_{n+1}$. We set $y_n = x_{m_n}$, notice that $y_n \in V_n$ and show that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $V_{n+k} \subseteq V_n$ we have $d(y_n, y_{n+k}) \le \delta_n$. Thus for any $\varepsilon > 0$ we find an $N \in \mathbb{N}$ such that $n > N$ implies $\delta_n < \varepsilon$ and thus $d(y_n, y_{n+k}) < \varepsilon$ for all $k \in \mathbb{N}$. So $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

It therefore remains to construct the sets V_n. For each natural number k there is a finite number $F_k \subseteq X$ such that

$$(*) \qquad X = \bigcup_{x\in F_k} U_{1/k}(x).$$

We shall use a simple fact in the proof:

(∗∗) If $(p_n)_{n\in\mathbb{N}}$ is a sequence in a set M such that for finitely many subsets $M_k \subseteq M$, the set $\{m \in \mathbb{N} : p_m \in M_1 \cup \cdots \cup M_n\}$ is infinite, then there is at least one index k such that $\{m \in \mathbb{N} : p_m \in M_k\}$ is infinite.

From $(*)$ with $k = 1$ and $(**)$ we find $z_1 \in F_1$ such that $\{m \in \mathbb{N} : x_m \in U_1(z_1)\}$ is infinite. Set $V_1 = U_1(z_1)$. Assume that $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n$ have been found such that $\{m \in \mathbb{N} : x_m \in V_n\}$ is infinite and the diameter δ_k of V_k is $\le 2/k$. Now $V_n \subseteq X = \bigcup_{z\in F_{n+1}} U_{1/n+1}(z)$ by $(*)$. Apply $(**)$ to $V_n = \bigcup_{z\in F_{n+1}} \big(V_n \cap U_{1/n+1}(z)\big)$ and find a $z_{n+1} \in F_{n+1}$ such that $\{m \in \mathbb{N} : x_m \in V_n \cap U_{1/n+1}(z_{n+1})\}$ is infinite. Set $V_{n+1} = V_n \cap U_{1/n+1}(z_{n+1})$. This completes the recursive construction of V_n with $\delta_n \le 2/n$ and thereby completes the proof of the lemma □

Recall that a metric space is said to be *complete*, if every Cauchy-sequence converges.

Theorem 2.11. *For a metric space (X, d) with the metric topology $\mathcal{O}$, the following statements are equivalent:*

(i) *$(X, \mathfrak{O})$ is compact.*

(ii) $(X, \mathfrak{O})$ *is sequentially compact.*
(iii) (X, d) *is precompact and complete.*

Proof . (i)⇒(ii): This was shown in Proposition 2.6.

(ii)⇒(iii): A sequentially compact metric space is precompact by Lemma 2.8. We verify completeness: Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Since X is sequentially compact by (ii), this sequence has a cluster point x. We claim that $x = \lim_{n\to\infty} x_n$. Indeed let $\varepsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that $m, n > N$ implies $d(x_m, x_n) < \varepsilon/2$. Since x is an accumulation point, there is an $m > N$ such that $d(x, x_m) < \varepsilon/2$. Thus for all $n > N$ we have $d(x, x_n) < d(x, x_m) + d(x_m, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This proves the assertion.

(iii)⇒(ii): Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. By Lemma 2.10 there are natural numbers $m_1 < m_2 < \cdots$ such that $(x_{m_n})_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is complete, this sequence has a limit x. If $\varepsilon > 0$ then there is an N such that $n > N$ implies $x_{m_n} \in U_\varepsilon(x)$. Since $m_n < m_{n+1}$ we conclude that $\{m \in \mathbb{N} : x_m \in U_\varepsilon(x)\}$ is infinite. Hence x is an accumulation point of $(x_n)_{n\in\mathbb{N}}$.

We have shown that (ii) and (iii) are equivalent.

(ii)⇒(i): Let $\mathcal{U}$ be an open cover. Let $r > 0$ be a Lebesgue number for this cover according to Lemma 2.7. Since X is precompact by what we know we find a finite set $F \subseteq X$ such that $X = \bigcup_{x\in F} U_r(x)$. For each $x \in F$ we find an $U_x \in \mathcal{U}$ such that $U_r(x) \subseteq U_x$ by Lemma 2.7. Then $X = \bigcup_{x\in F} U_r(x) \subseteq \bigcup_{x\in F} U_x \subseteq X$ and thus $\{U_x : x \in X\}$ is a finite subcover of $\mathcal{U}$. □

Theorem 2.11 is remarkable in so far as the three statements (i), (ii), and (iii) have very little to do with each other on the surface.

Theorem 2.11 links our general concept of compactness with elementary analysis where compactness is defined as sequential compactness.

Exercise E2.2. (i) Show that a compact subspace X of a metric space is always bounded, i.e. that there is a number R such that $d(x, y) \leq R$ for all $x, y \in X$

(ii) Give an example of an unbounded metric on $\mathbb{R}$ which is compatible with the natural topology.

(iii) Show that a closed subset X of $\mathbb{R}$ which is bounded in the sense that it is contained in an interval $[a, b]$ is compact.

(iv) Prove the following result from First Year Analysis. (Theorem of Bolzano-Weierstrass).

A subset of $\mathbb{R}^n$ *is compact if an only if it is closed and bounded with respect to the norm given by* $\|(x_1, \ldots, x_n)\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$.

(v) Show that the Theorem of Bolzano-Weierstrass holds for any norm on $\mathbb{R}^n$.□

Exercise E2.3. Use Theorem 2.11 for proving that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and norm bounded. □

In this spirit, Theorem 2.11 is the "right" generalisation of the Bolzano-Weierstrass Theorem.

There are some central results concerning compact spaces which involve the Axiom of Choice. Therefore we must have an Interlude on the Axiom of Choice.

Chapter AC
An Interlude on the Axiom of Choice.

We need some concepts from order theory.

Definition AC.1. A poset $(X, \leq)$ as well as the partial order $\leq$ are said to be *inductive*, if each totally ordered subset (that is, a *chain* or a *tower*) $T \subseteq X$ has an upper bound b in X (i.e. $t \leq b$ for all $b \in T$).

Example AC.2. Let V be a vector space over any field K and let $X \subseteq \mathfrak{P}(X)$ be the set of all linearly independent subsets. On X we consider the partial order $\subseteq$. If T is a totally ordered set of linearly independent subsets of V then $b \stackrel{\text{def}}{=} \bigcup T$ is a linearly independent set due to the fact, that linear independence of a set F of vectors is a property involving only finite subsets of F. Also, b contains all members of T. Hence $(X, \subseteq)$ is inductive. □

Definition AC.3. A binary relation $\leq$ is called a *well-order*, and $(X, \leq)$ is called a *well-ordered set* if $\leq$ is a total order (i.e. a partial order for which every pair of elements is comparable) such that every nonempty subset has a smallest element.□

Example AC.4. Every finite totally ordered set is well-ordered. The set $\mathbb{N}$ of natural numbrs with its natural order is well-ordered. The set $\mathbb{N} \cup \{n - \frac{1}{m} : n, m \in \mathbb{N}\}$ is well ordered with the natural order. □

We begin by formulating a couple of statement concerning sets, posets, topological spaces.

AC: The Axiom of Choice. *For every family of nonempty sets $(X_j : j \in J)$ the product $\prod_{j \in J} X_j$ is not empty.* □

ZL: Zorn's Lemma. *Every inductive set has maximal elements.* □

WOP: The Well-Ordering Principle. *Every set can be well-ordered.* □

TPT: The Tychonov Product Theorem. *Each product of compact spaces is compact.* □

The point of this interlude is to prove that these four statements are equivalent. Let us begin with a couple of simple implications

TPT⇒AC: Let $(X_j : j \in J)$, $X_j \neq \emptyset$ for all $j \in J$ for some set J. Then $Y_j = X_j \cup \{X_j\}$ is a set and $X_j \notin X_j$. The product $\prod_{j\in J} Y_j$ is not empty because it contains the element $(X_j)_{j\in J}$. Let $\mathrm{Fin}(J)$ denote the set of finite subsets of J. For each finite subset $F \in \mathrm{Fin}(J)$ we set $S_F = \prod_{j\in J} Z_j$ where

$$Z_j = \begin{cases} X_j & \text{if } j \in F, \\ Y_j & \text{if } j \in J \setminus F. \end{cases}$$

Since $X_j \neq \emptyset$ for all $j \in J$ we have $\prod_{j\in F} X_j \neq \emptyset$ since we accept the "finite" Axiom of Choice. Now we topologize Y_j by declaring $\mathfrak{O}_j = \{\emptyset, \{X_i\}, X_j\}$ to be its topology. Then X_j is a closed subset of Y_j, and Y_j is compact. Now $\mathcal{S} = \{S_F : F \in \mathrm{Fin}(J)\}$ is a filter basis of closed subsets of $P = \prod_{j\in J} Y_j$ and P is compact by TPT. Thus $\prod_{j\in J} X_j = \bigcap_{F\in\mathrm{Fin}(J)} S_F \neq \emptyset$ by 2.3. □

ZL⇒WOP: Let X be a set, pick $x_0 \in X$ and consider the set $(\mathcal{X}, \preceq)$ be the set of all well ordered subsets $(A, \leq_A)$ with x_0 as minimal element, where
$(A, \leq_A) \preceq (B, \leq_B)$ if

(1) $A \subseteq B$,
(2) $\leq_B |(A \times A) = \leq_A$, and
(3) A is an initial segment of B.

We claim that the poset $(\mathcal{X}, \preceq)$ is inductive: Let $\mathcal{T}$ be a totally ordered subset. Then we form the subset $T = \bigcup \mathcal{T}$ and define a binary relation $\leq$ on T as follows: Let $x, y \in T$. Then there is an $S \in \mathcal{X}$ containing x and y. Since S is totally ordered we have $x \leq_S y$ or $y \leq_S x$, and we set $x \leq_T y$ in the first case and $y \leq_T y$ in the second. It is readily seen that this definition is independent of the choice of S. It is seen that $S \in \mathcal{T}$ implies that S is an initial segment of T. If $\emptyset \neq A \subseteq T$, then there is an $a \in A$ and then $a \in \bigcup \mathcal{T}$; thus $a \in S$ for some $S \in \mathcal{T}$. Since S is well-ordered, $m = \min(A \cap S)$ exists. Since S is an initial segment of T we conclude $m = \min A$. Thus $(\mathcal{X}, \preceq)$ is inductive. By Zorn's Lemma ZL we find a maximal element $(M, \leq_M)$. We claim $M = X$. Suppose not. Then there is an $x \in X \setminus M$. We extend $\leq_M$ to $M' = M \cup \{x\}$ by making x bigger than all elements of M. Then $(M', \leq_{M'})$ is a well-ordered set with M as an initial segment. This contradicts the maximality of $(M, \leq_M)$. Thus $X = M$ and $(X, \leq_X)$ is well-ordered. □

WOP⇒AC: Let $(X_j : j \in J)$ be a family of nonempty sets; set $X \overset{\text{def}}{=} \bigcup_{j\in J} X_j$. Let $\leq$ be a well-order of X. Then $(\min X_j)_{j\in J} \in \prod_{j\in J} X_j$. □

We still have to show AC⇒ZL and ZL⇒TPT. First we shall show that AC implies ZL; then we shall conclude the interlude and prove ZL⇒TPT in the course of our discussion of compact spaces.

For a proof of AC⇒ZL we prove a Lemma of independent interest in oder theory.

Theorem AC.5. (Tarski's Fixed Point Theorem) *Let* $(X, \leq)$ *be a poset such that every totally ordered subset has a least upper bound. Assume that the function* $f: X \to X$ *satisfies* $(\forall x \in X)\, x \leq f(x)$. *Then* f *has a fixed point, that is, there is an* $x_0 \in X$ *such that* $f(x_0) = x_0$.

Let us note the parallel to Banach's Fixed Point Theorem: *Let* (X, d) *be a metric space such that every Cauchy sequence converges. Assume that the function* $f: X \to X$ *satisfies* $(\exists \lambda) \cdot 0 \leq \lambda < 1$ *and* $(\forall x, y \in X) d\big(f(x), f(y)\big) \leq \lambda d(x, y)$. *Then* f *has a fixed point.*

That is, X satisfies some completeness hypothesis and f satisfies some contraction hypothesis. But the proofs proceed quite differently.

Yet before we prove Tarski's Fixed Point Theorem we shall show how with its aid one can use AC to prove ZL. Thus let $(X, \leq)$ be a inductive poset. Let $\mathcal{X}$ denote the set of all totally ordered subsets of X; then $(\mathcal{X}, \subseteq)$ is a poset in which every totally ordered subset $\mathcal{T}$ has a least upper bound, namely, $T \stackrel{\text{def}}{=} \bigcup \mathcal{T}$. We claim that $\mathcal{X}$ has maximal elements. If not, then for any $Y \in \mathcal{X}$ the set $\mathcal{M}_Y = \{Z \in \mathcal{X} : Y \subset Z, Y \neq Z\}$ is not empty. By the Axiom of Choice there is an element $f \in \prod_{Y \in \mathcal{X}} \mathcal{M}_Y$, that is f is a selfmap of $\mathcal{X}$ such that $Y \subset f(Y)$, $Y \neq f(Y)$, and this contradicts the Tarski Fixed Point Theorem AC.5. Thus we find a maximal chain M. Since X is inductive, M has an upper bound b. But now b is a maximal element of X because otherwise there would have to be an element $c \in X$ such that $b < c$, yielding a chain $M \cup \{c\}$ properly containing M in contradiction with the maximality of M. Therefore X has maximal elements and this is what Zorn's Lemma asserts.

Let us also notice that with the aid of ZL a proof of the Tarski Fixed Point Theorem is trivial: If X is an inductive set, then by ZL it contains a maximal element c. If now f is a self map of X with $x \leq f(x)$, then $c \leq f(c)$ implies $c = f(c)$ by the maximality of c, and thus c is a fixed point of f.

The entire point now is to prove the Tarski Fixed Point Theorem without AC or ZL.

For a proof of Tarski's Fixed Point Theorem let $(X, \leq)$ be a poset such that $\sup C$ exists for each chain $C \subseteq X$. Let us call a subset $A \subseteq X$ *closed* if for each chain $C \subseteq A$ we have $\sup C \in A$ and $f(A) \subseteq A$. The empty set is a chain and thus the set X has a smallest element $\min X = \sup \emptyset$. Moreover, if A is closed, then $\emptyset$ is a chain contained in A, and thus $\min X = \sup \emptyset \in A$.

Let $X' = \bigcap \{A \subseteq X : A \text{ is a closed subset of } X\}$. Then X' is the smallest losed subset of X. It suffices to prove the Fixed Point Theorem for X' and $f|X'$. We shall therefore assume from now on that X has no proper closed nonempty subsets. (Notice that $\emptyset$ is closed.)

Definition A. We say that $x \in X$ *decomposes* X if, for any $y \in X$, either $y \leq x$ or $f(x) \leq y$, that is, $X = {\downarrow} x \cup {\uparrow} f(x)$.

Definition B. Let us call an element $x \in X$ a *roof* if $y < x$ always implies $f(y) \leq x$.

Lemma C. *Any roof decomposes X.*

Proof . Let x be a roof. Set

$$Z = \{y \in X : y \leq x \text{ or } f(x) \leq y\} = {\downarrow}x \cup {\uparrow}f(x).$$

We claim that Z is a closed set; once this claim is proved, we are done: Since X has no proper closed subsets, this implies $Z = X$.

To prove the claim, let C be a chain in Z and write $z = \sup C$. We must show that $z \in Z$. If $C \subseteq {\downarrow}x$, then $z \in {\downarrow}x \subseteq Z$; if $C \not\subseteq {\downarrow}x$, then there is $c \in C$ such that $f(x) \leq c$, and then $f(x) \leq \sup C = z$. Hence $z \in Z$.

Next let $z \in Z$. We must also show that $f(z) \in Z$. If $z < x$, then $f(z) \leq x$ since x is a roof. Thus $f(z) \in {\downarrow}x \subseteq Z$. If $z = x$, then $f(z) = f(x) \in {\uparrow}f(x) \subseteq Z$. If, finally, $f(x) \leq z$, then $f(x) \leq z \leq f(z)$ and thus $f(z) \in {\uparrow}f(x) \subseteq Z$. This completes the proof of Lemma C. □

Lemma D. *Every element in X is a roof.*

Proof . Set

$$D = \{y \in X : y \ \text{ is a roof}\}.$$

We claim that D is a closed subset. Again, since X has no proper closed subsets, this will prove $X = D$.

For a proof that D is closed, let $C \subseteq D$ be a chain and $d = \sup C$; we must show that d is a roof. If $d \in C$ then we are done, thus we assume that $c < d$ for all $c \in C$. Let $x < d$; we must show $f(x) \leq d$. We assert that there is a $c \in C$ such that $x < c$. Suppose this is not the case. Then $x \not< c$ for all $c \in C$; since c is a roof, $x \not< c$ means $f(c) \leq x$ by Lemma C. But then $c \leq f(c) \leq x$ for all $c \in C$, and so $d \leq x < d$, a contradiction. Thus $x < c$ for some $c \in C$ as asserted. Since c is a roof, $f(x) \leq c \leq d$ and so $f(x) \leq d$. This shows that d is a roof as asserted and thus $d \in D$.

Next let $d \in D$, that is, d is a roof; we must show that $f(d)$ is a roof as well. So let $x < f(d)$. Since the element d is a roof it separates X by Lemma C, that is $X = {\downarrow}d \cup {\uparrow}f(d)$. Then $x \leq d$. If $x < d$, then $f(x) \leq d$ since d is a roof, and so $f(x) < f(d)$. If $x = d$ then $f(x) = f(d) \leq f(d)$. So $f(d)$ is a roof as was asserted. □

At this point it follows that X is a chain. Let $x_0 = \max X$. Then $f(x_0) \leq x_0$. By hypothesis on f we have $x_0 \leq f(x_0)$. Thus $f(x_0) = x_0$ and thus x_0 is the desired fixed point of f. This completes the proof of the Tarski Fixed Point Theorem. □

Now we know that (AC), (ZL), and (WOP) are equivalent.

More information on filters

Lemma AC.6. *Let X be a set. The set* $\mathrm{Filt}(X)$ *of all filters on X, as a subset of $\mathfrak{P}\big(\mathfrak{P}(X)\big)$ is an inductive poset.* □

Definition AC.7. Any maximal element in $\mathrm{Filt}(X)$ is called an *ultrafilter.*

If X is a set and $x \in X$, then $\mathcal{U} = \{F \subseteq X : x \in F\}$ is an ultrafilter. Such an ultrafilters is called a *fixed ultrafilter.*

Proposition AC.8. (AC) *Every filter on a set is contained in an ultrafilter. Every filter basis is contained in an ultrafilter.*

Proof . Let $\mathcal{F}$ be a filter. The set of all filters containing $\mathcal{F}$ is inductive an thus by Zorn's Lemma contains maximal elements. If $\mathcal{B}$ is a filter basis, then the set $\mathcal{F}$ of all supersets of members of $\mathcal{B}$ is a filter which is contained in an ultrafilter by the preceding. □

The preceding proposition is also called the *Ultrafilter Theorem* (UT). We saw that (AC)⇒(UT). The reverse implication is not true; however, for the working mathematician this is a subtlety which we do not dwell on here.

Exercise EAC.3. Prove directly the following proposition:

Let X be a set and $\mathcal{U}$ an ultrafilter in X. If $f\colon X \to Y$ is a surjective function, then $\{f(U) : U \in \mathcal{U}\}$ is an ultrafilter on Y. If $A \in \mathcal{U}$, then $\{A \cap U : U \in \mathcal{U}\}$ is an ultrafilter on A. □

Definition AC.9. A filter basis is called an *ultrafilter basis* if the filter of all of of its supersets is an ultrafilter.

Proposition AC.10. *The following statements are equivalent for a filter $\mathcal{F}$ on a set X:*

(i) *$\mathcal{F}$ is an ultrafilter.*
(ii) *Whenever $X = A \cup B$ and $A \cap B = \emptyset$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*
(ii′) *If X is the disjoint union of finitely many subsets $A_1, \ldots, A_n$, then $A_j \in \mathcal{F}$ for some $j \in \{1, \ldots, n\}$.*
(ii″) *Whenever $X = A_1 \cup A_2 \cup \cdots \cup A_n$, then $A_j \in \mathcal{F}$ for some $j \in \{1, \ldots, n\}$.*
(iii) *Whenever $X = A \cup B$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

The following statements are equivalent for a filter basis $\mathcal{B}$ on a set X:

(I) *$\mathcal{B}$ is an ultrafilter basis.*
(II) *Whenever $X = A \cup B$ and $A \cap B = \emptyset$, then there is a $C \in \mathcal{B}$ such that $C \subseteq A$ or $C \subseteq B$.*
(III) *Whenever $X = A \cup B$, then there is a $C \in \mathcal{B}$ such that $C \subseteq A$ or $C \subseteq B$.*

Proof . (i)⇒(ii): Assume (i) and $X = A \cup B$ and $A \cap B = \emptyset$. If the assertion fails, then $\mathcal{F}_A = \{S \subseteq X : (\exists F \in \mathcal{F})\, A \cap F \subseteq S\}$ and $\mathcal{F}_B = \{S \subseteq X : (\exists F \in \mathcal{F})\, B \cap F \subseteq S\}$ are two filters such that $\mathcal{F} \subseteq \mathcal{F}_A \cap \mathcal{F}_B$. Since $\mathcal{F}$ is an ultrafilter, $\mathcal{F}_A = \mathcal{F} = \mathcal{F}_B$;, but then $A \in \mathcal{F}$ and $B \in \mathcal{F}$ whence $\emptyset = A \cap B \in \mathcal{F}$, a contradiction.

(ii)⇒(i). Assume (ii) and consider $\mathcal{F} \subseteq \mathcal{G}$ for some filter $\mathcal{G}$ on X. Suppose that $\mathcal{G}$ is properly larger than $\mathcal{F}$; then there is a set $A \in \mathcal{G} \setminus \mathcal{F}$. Set $B = X \setminus A$. From (ii) we conclude $B \in \mathcal{F}$. But then $B \in \mathcal{G}$ and thus $\emptyset = A \cap B \in \mathcal{G}$, a contradiction.

(ii)⇒(ii′). Using (i), by induction we see that if X is a disjoint finite union of $n \geq 2$ subsets, then, given an ultrafilter on X, one of these subsets belongs to the ultrafilter.

(ii′)⇒(ii″) Assume that $X = A_1 \cup A_2 \cup \cdots \cup A_n$. The set of all finite intersections and all unions of these is a finite set of subsets of X closed under the formation of intersection and union: It is a finite topology. The set of minimal elements of this topology is a partition of X. (see Proposition 1.44). Then by (*ii*'), for each ultrafilter $X\mathcal{F}$ there is (exactly) one $J \in \mathcal{J}$ such that $X_J \in \mathcal{F}$. Then for all $j \in J$ we have $A_j \in \mathcal{F}$, since $\mathcal{F}$ is a filter.

(ii″)⇒(iii)⇒(ii) is trivial.

(I)⇔(II)⇔(III). Apply the preceding to $\mathcal{F} = \langle \mathcal{B} \rangle$, the filter generated by $\mathcal{B}$. □

Definition AC.11. Let $f\colon X \to Y$ be a function and $\mathcal{F}$ and $\mathcal{G}$ be filter bases on X and Y, respectively. Set

$$f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\} \text{ and } f^{-1}(\mathcal{G}) = \{f^{-1}(G) : G \in \mathcal{G}\}.$$ □

Proposition AC.12. (i) *$f(\mathcal{F})$ is a filterbasis, and if $\mathcal{F}$ is a filter and f is surjective, then $f(\mathcal{F})$ is a filter as well.*

(ii) *$f^{-1}(\mathcal{G})$ is a filter basis, and if $\mathcal{G}$ is a filter and f is injective, then $f^{-1}(\mathcal{G})$ is a filter.*

Proof . (i) Let $F_1, F_2 \in \mathcal{F}$; then $\mathcal{F}$ contains an F such that $F \subseteq F_1 \cap F_2$ since $\mathcal{F}$ is a filter basis. Then $f(F) \subseteq f(F_1 \cap F_2) \subseteq f(F_1) \cap f(F_2)$. Thus $f(\mathcal{F})$ is a filter basis.

Now assume that f is surjective and that $\mathcal{F}$ is a filter. Let $F \in \mathcal{F}$ and $f(F) \subseteq B$. Then $F \subseteq f^{-1}(B)$, and since $\mathcal{F}$ is a filter, $f^{-1}(B) \in \mathcal{F}$. Since f is surjective, $B = f\big(f^{-1}(B)\big) \in f(\mathcal{F})$.

(ii) Let $G_1, G_2 \in \mathcal{G}$. Then there is a $G \in \mathcal{G}$ with $G \subseteq G_1 \cap G_2$. Then $f^{-1}(G) \subseteq f^{-1}(G_1 \cap G_2) = f^{-1}(G_1) \cap f^{-1}(G_2)$. Thus $f^{-1}(\mathcal{G})$ is a filter basis.

Now assume that f is injective and that $\mathcal{G}$ is a filter. Let $G \in \mathcal{G}$ and $f^{-1}(G) \subseteq A$. Then $G \subseteq f(A) \cup \big(Y \setminus f(X)\big)$ since f is injective and $f(A) \cup \big(Y \setminus f(X)\big) \in \mathcal{G}$ since $\mathcal{G}$ is a filter. Now $A = f^{-1}\big[f(A) \cup \big(Y \setminus f(X)\big)\big]$ is in $f^{-1}(\mathcal{G})$. □

Proposition AC.13. *Let $\mathcal{U}$ be an ultrafilter basis in X and $f\colon X \to Y$ any function. Then $f(\mathcal{U})$ is an ultrafilter basis.*

In particular, if f is surjective and $\mathcal{U}$ is an ultrafilter, then $f(\mathcal{U})$ is an ultrafilter as well.

Proof. We prove this by using the equivalence of (I) and (II) in Proposition AC.10. So let $Y = A \cup B$, $A \cap B = \emptyset$. Then $X = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Now, since $\mathcal{U}$ is an ultrafilter basis, by (I)⇒(II) in Proposition AC.8, there is a $U \in \mathcal{U}$ such that either $U \subseteq f^{-1}(A)$ or $U \subseteq f^{-1}(B)$. In the first case, $f(U) \subseteq A$, in the second case, $f(U) \subseteq B$. Thus by (II)⇒(I) in Proposition AC.8 we see that $f(\mathcal{U})$ is an ultrafilter basis.

Finally, AC.10(i) proves the remainder. □

Chapter 2
Compactness Continued

Now we shall show that Zorn's Lemma implies Tychonov's Product Theorem. We need the concept of convergence for filters.

Definition 2.12. We say that a filter $\mathcal{F}$ on X *converges* to $x \in X$ if $\mathfrak{U}(x) \subseteq \mathcal{F}$. A filter basis $\mathcal{B}$ converges to x if the filter generated by $\mathcal{B}$ converges to x.

A point to which a filter, respectively, filter basis $\mathcal{F}$ converges is also called a *limit point* of $\mathcal{F}$ □

It is immediate that a filter basis $\mathcal{B}$ converges to x iff for each neighborhood U of x there is a member $B \in \mathcal{B}$ such that $B \subseteq U$.

A sequence $(x_n)_{n\in\mathbb{N}}$ is said to converge to x if for every neighborhood U of x there is an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$. Once we are given these definitions it is an easy exercise to show that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x iff the filter basis $\mathcal{B} = \big\{\{x_n, x_{n+1}, \ldots\} : n \in \mathbb{N}\big\}$ converges to x.

For an ultratilter $\mathcal{U}$ on a topological space X, a point $x \in X$ is a limit point iff for all $U \in \mathfrak{U}(x)$ and each $F \in \mathcal{U}$ we have $U \cap F \neq \emptyset$ iff $x \in \bigcap_{F\in\mathcal{U}} \overline{F}$.

Theorem 2.13. (UT) *For a topological space $(X, \mathfrak{O})$ the following statements are equivalent:*
(i) *X is compact.*
(ii) *Every ultrafilter converges.*

Proof. (i)⇒(ii): Let $\mathcal{U}$ be an ultrafilter. By (i) there is an x such that $x \in \overline{V}$ for all $V \in \mathcal{U}$ (see 2.3). This means that $U \cap V \neq \emptyset$ for all $U \in \mathfrak{U}(x)$ and all $V \in \mathcal{U}$. Then $\mathcal{F} \overset{\text{def}}{=} \{F : U \cap V \subseteq F, U \in \mathfrak{U}(x), V \in \mathcal{U}\}$ is a filter containing $\mathcal{U}$. Since

$\mathcal{U}$ is maximal among all filters, we have $\mathcal{F} = \mathcal{U}$ and thus $\mathcal{U} = \mathcal{F} \supseteq \mathfrak{U}(x)$, i.e., $\mathcal{U}$ converges to x.

(ii)⇒(i). (UT) Let $\mathcal{B}$ be a filter basis of closed sets; we must show that $\bigcap \mathcal{B} \neq \emptyset$. By the Ultrafilter Theorem (UT), the filter basis $\mathcal{B}$ is contained in an ultrafilter $\mathcal{U}$ which by (ii) converges to some element x. Let U be a neighborhood of x. Then $U \in \mathcal{U}$. Now let $B \in \mathcal{B}$; since $\mathcal{B} \subseteq \mathcal{U}$ we have $B \in \mathcal{U}$, and thus $B \cap U \in \mathcal{U}$; in particular, $B \cap U \neq \emptyset$. Therefore $x \in \overline{B} = B$ for all $B \in \mathcal{B}$. □

The proof of the preceding characterisation theorem for compactness required actually the Ultrafilter Theorem (UT). This axiom allows the proof of other characterisation theorems for compactness as is exemplified in the following exercise.

Exercise E2.4. Recall that a subbasis for a topology $\mathfrak{O}$ is any subset $\mathcal{S}$, such that $\mathfrak{O}$ is the smallest topology containing $\mathcal{S}$. This means that for each open set $U \in \mathfrak{O}$ and each $x \in U$, there are finitely many subbasic open sets $S_1, \ldots, S_n$ such that $x \in S_1 \cap \cdots \cap S_n \subseteq U$.

A set $\mathfrak{A}$ of closed sets is called a subbasis for the closed sets if $\{X \setminus A : A \in \mathfrak{A}\}$ is a subbasis for the topology, that is iff for each closed set A and a point $x \notin A$ there are finitely many subbasic closed sets $S_1, \ldots, S_n$ such that $A \subseteq S_1 \cup \cdots \cup S_n \subseteq X \setminus \{x\}$.

Prove the following theorem: **Alexander Subbasis Theorem.** *Let $\mathcal{S}$ be a subbasis of the topology of a topological space X. Then X is compact if and only if any open cover taken from $\mathcal{S}$ has a finite subcover.*

[Hint. By definition of compactness if X is a compact space then *any* open cover has a finite subcover. We have to assume that every open cover of **subbasic** open sets has a finite subcover and then conclude that **any** open cover has a finite subcover.

We might just as well assume that for a suitable subbasis $\mathcal{S}$ of closed sets every filterbasis generated by subbasic closed sets has a nonempty intersection, and prove that each ultrafilter converges (Theorem 2.13.) So let $\mathcal{U}$ be an ultrafilter. We must show that $\bigcap_{F \in \mathcal{U}} \overline{F} \neq \emptyset$ because this intersection is the set of all limit points of $\mathcal{U}$.

Let $F \in \mathcal{U}$ and $x \in X \setminus F$. Since $\mathcal{S}$ is a subbasis for the set of closed sets, there are finitely many $S_1, \ldots, S_n \in \mathcal{S}$ such that $F \subseteq \overline{F} \subseteq S_1 \cup \cdots \cup S_n \subseteq X \setminus \{x\}$ Since $\mathcal{U}$ is an ultrafilter, one of the S_j, denoted $S \in \mathcal{S}$, is contained in $\mathcal{U}$. (See Notes, Proposition AC.10). Therefore

(1) $$\mathcal{S} \cap \mathcal{U} \neq \emptyset, \text{ and}$$

(2) $$(\forall F \in \mathcal{U}, x \notin F)(\exists S \in \mathcal{S})\, F \cap S \in \mathcal{U} \text{ and } x \notin S.$$

Thus $\mathcal{F}_0 \stackrel{\text{def}}{=} \mathcal{S} \cap \mathcal{U}$ is not empty by (1). A finite collection of elements of $\mathcal{F}_0$ is contained in the filter $\mathcal{U}$ and therefore has a nonempty intersection; the set of all finite intersections of elements of $\mathcal{F}_0$ is a filter basis $\mathcal{B}$ of closed subbasic sets. It therefore satisfies $\bigcap \mathcal{F}_0 = \bigcap \mathcal{B} \neq \emptyset$ by hypothesis. We claim

(3) $$\bigcap_{F \in \mathcal{F}} \overline{F} = \bigcap \mathcal{F}_0.$$

The relation "$\subseteq$" is clear (why?). For a proof of the reverse containment, assume $x \notin \bigcap_{F\in\mathcal{U}} \overline{F}$ and show that $x \notin \bigcap \mathcal{F}_0$. Under this assumption there is an $F \in \mathcal{U}$ such that $x \notin \overline{F}$. Then by (2) there is an $S \in \mathcal{F}_0$ such that $x \notin S$. Hence $x \notin \bigcap \mathcal{F}_0$. This shows "$\supseteq$". Thus (3) holds and the left side is nonempty. This is what we had to show.]

Notice that the Alexander Subbasis Theorem requires the Ultrafilter Theorem (UT) which is secured by the Axiom of choice (AC).

The next theorem is the crucial one. It will prove that AC$\Rightarrow$TPT.

Theorem 2.14. (AC) *The product of any family of compact spaces is compact.*

Proof. Let $(X_j : j \in J)$ be a family of compact spaces. Let $P \stackrel{\text{def}}{=} \prod_{j\in J} X_j$. If one X_j is empty, then $P = \emptyset$ and thus P is compact. Assume now that $X_j \neq \emptyset$. We prove compactness of P by considering an ultrafilter $\mathcal{U}$ on P and showing that it converges.

For each $j \in J$ the projection $\mathrm{pr}_j(\mathcal{U})$ is an ultrafilter. Let $L_j \subseteq X_j$ be the set of points to which it converges. Since X_j is compact, $L_j \neq \emptyset$. By the Axiom of Choice $L \stackrel{\text{def}}{=} \prod_{j\in J} L_j \neq \emptyset$. Let $(x_j)_{j\in J} \in L$.

Now let U be a neighborhood of $x \stackrel{\text{def}}{=} (x_j)_{j\in J}$. We may assume that U is a basic neighborhood of the form $U = \prod_{j\in J} U_j$, where $U_j = X_j$ for all $j \in J \setminus F$ for some finite subset of J. Then we find a member $M \in \mathcal{U}$ such that $\mathrm{pr}_j(M) \subseteq U_j$ for $j \in F$. Thus $M \subseteq \mathrm{pr}_j^{-1}(U_j)$ and so $\mathrm{pr}_j^{-1}(U_j) \in \mathcal{U}$. Accordingly $U = \prod_{j\in J} U_j = \bigcap_{j\in F} \mathrm{pr}_j^{-1}(U_j) \in \mathcal{U}$. □

Notice that we have used the Axiom of Choice by applying the Ultrafilter Theorem **and** by selecting $(x_j)_{j\in J}$.

Exercise E2.5. Prove:

In a Hausdorff space, a filter $\mathcal{F}$ converges to at most one point..

Thus in a Hausdorff space a converging filter converges to exactly one point x, called the *limit point* and written $x = \lim \mathcal{F}$.

Corollary 2.15. (UT) *The product of a family of compact Hausdorff spaces is a compact Hausdorff space.* □

The Ultrafilter Theorem (UT) indeed suffices for a proof of this theorem.

Example. (Cubes) Let $\mathbb{I}$ denote the unit interval $[0, 1]$ and $\mathbb{D}$ the complex unit disc. For each set J the products $\mathbb{I}^J$ and $\mathbb{D}^J$ are compact spaces. □

We have made good use of the concept of a filter and its convergence. In passing we mention the concept of a Cauchy-filter on a metric space. Let us first recall that in a metric space (X, d) a subset $B \subseteq X$ is *bounded* if there is a number C such that $d(b, c) \leq C$ for all $b, c \in B$. For a bounded subset B, the number $\sup\{d(b, c) : b, c \in B\}$ exists and is called the *diameter* of B. When we speak of the diameter of a subset, we imply that we assume that the subset is bounded.

Definition 2.16. A filter $\mathcal{F}$ on a metric space (X, d) is called a *Cauchy-filter* if for each $\varepsilon > 0$ it contains a set of diameter less than ε. A filter basis is a *Cauchy-filter basis* if it contains a set of diameter less than ε. □

Clearly, a filter basis is a Cauchy-filter basis if and only if the filter of all super sets of its members is a Cauchy-filter.

Exercise E2.6. Show that a sequence $(x_n)_{n\in\mathbb{N}}$ is a a Cauchy-sequence iff the filter basis of all $\{x_n, x_{n+1}, \ldots\}$, $n \in \mathbb{N}$ is a Cauchy filter basis.

Lemma 2.17. (i) *Let $\mathcal{F}$ be a Cauchy-filter in metric space. Then there is a countable Cauchy-filter basis $\mathcal{C}$, $C_1 \supseteq C_2 \supseteq \cdots$ such that the diameter of C_n is less than $\frac{1}{n}$ and $\mathcal{C} \subseteq \mathcal{F}$.*

(ii) *If $\mathcal{F} \subseteq \mathcal{G}$ are two filters in a metric space such that $\mathcal{F}$ is a Cauchy-filter and $\mathcal{G}$ converges to x then $\mathcal{F}$ converges to x.* □

If $\mathcal{C}$ converges, and thus the filter $\langle\mathcal{C}\rangle$ of all supersets of the C_n converges, that is, contains some neighborhood filter $\mathfrak{U}(x)$, then the given filter $\mathcal{F}$ converges. If now we select in each set C_n an element c_n, then $(c_n)_{n\in N}$ is a Cauchy-sequence. Then $\big\{\{c_n, c_{n+1}, \ldots\} : n \in \mathbb{N}\big\}$ is a Cauchy-filter basis $\mathcal{B}$ which converges iff $(c_n)_{n\in\mathbb{N}}$ converges. Moreover, $\langle\mathcal{C}\rangle \subseteq \langle\mathcal{B}\rangle$.

Proposition 2.18. *A metric space (X, d) is complete if and only if every Cauchy-filter converges.* □

For a given ε, a precompact metric space is covered by finite number of open ε-balls. Thus any ultrafilter contains one of them. Hence every ultrafilter on a precompact spaces is a Cauchy-filter. Thus on a complete precompact metric space every ultrafilter converges. This is an alternative proof that a metric space is compact iff it is complete and precompact. This approach has the potential of being generalized beyond the metric situation.

Exercise E2.7. Fill in the details of this argument.

Compact spaces and continuous functions

Proposition 2.19. *Let $f\colon X \to Y$ be a continuous surjective function of topological spaces and assume that X is compact. Then Y is compact. Let*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle q_f}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{id}_Y} \\ X/R_f & \xrightarrow[\quad f' \quad]{} & Y \end{array}$$

be the canonial decomposition of f. If Y is a Hausdorff space, then f' is a homeomorphism. □

In short: Continuous images of compact spaces are compact, and as easy consequence we know that a bijective continuous map between Hausdorff spaces is a homeomorphism.

Corollary 2.20. *If $\mathfrak{O} \subseteq \mathfrak{O}'$ are Hausdorff topologies on a set and $\mathfrak{O}'$ is compact, then $\mathfrak{O} = O'$.* □

Among Hausdorff topologies, compact ones are minimal.

A totally ordered set is defined to be *order complete* iff every subset has a least upper bound.

If $A \subseteq X$ and $(X, \leq)$ is order complete, let L be the set of lower bounds of A. Then $\sup L = \inf A$.

If $A \subseteq X$ is closed w.r.t. the order topology, then $\sup A = \max A$ and $\inf A = \min A$.

Exercise E2.8. Prove these claims.

Lemma 2.21. *A totally ordered space $(X, \leq)$ is compact w.r.t. the order topology if and only if X is order complete.*

Proof . If $X = \emptyset$, then X is complete by default. Assume that X is compact and $A \subseteq X$. Show that $\max \overline{A} = \sup A$

Now assume that X is order complete. $\mathcal{B}$ be a filter basis of closed subsets. Let $M = \{\min B : B \in \mathcal{B}\}$. Show that $\sup M \in \bigcap \mathcal{B}$. □

Exercise E2.9. Fill in the details of the proof of Lemma 2.21.

Proposition 2.22. (Theorem of the Maximum) *Let $f\colon X \to Y$ be a continuous function from a compact space into a totally ordered space. Then f attains its maximum and it s minimum, i.e. there are elements $x, y \in X$ such that $f(x) = \max f(X)$ and $f(y) = \min f(X)$.* □

Uniform Continuity, Uniform Convergence, Equicontinuity

Compactness has substantial applications in Analysis; we sample some of them

Definition 2.23. A function $f\colon X \to Y$ between metric spaces is called *uniformly continuous*, if

$$(1) \qquad (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X)\, f(U_\delta(x)) \subseteq U_\varepsilon\big(f(x)\big). \qquad \square$$

Recall that f if continuous if

$$(2) \qquad (\forall \varepsilon > 0)(\forall x \in X)(\exists \delta > 0)\, f(U_\delta(x)) \subseteq U_\varepsilon\big(f(x)\big).$$

Proposition 2.24. *A continuous function $f\colon X \to Y$ from a compact metric space into a metric space is uniformly continuous.* $\square$

[For each $\varepsilon > 0$ and each $x \in X$ and pick $d(x) > 0$ so that $f\big(U_d(x)\big) \subseteq U_{\varepsilon/2}\big(f(x)\big)$ and let δ be a Lebesgue number of the open cover $\{U_{d(x)}(x) : x \in X\}$.]

Definition 2.25. Let X be a set and Y a metric space. Define *B(X, Y)* to be the set of all bounded functions, i.e. functions $f\colon X \to Y$ such that $\{d\big(f(x), f(x')\big) : x, x' \in X\}$ is a bounded subset of $\mathbb{R}$.

If X and Y are topological spaces, then *C(X, Y)* denotes the set of all continuous functions from X to Y. $\square$

Proposition 2.26. (i) *Let X be a set and Y a metric space. Then $B(X, Y)$ is a metric space with respect to the metric $d(f, g) = \sup\{d\big(f(x), g(x)\big) : x \in X\}$*

(ii) *If X is a compact topological space and Y is a metric space, then $C(X, Y) \subseteq B(X, Y)$, and $C(X, Y)$ is a closed subset.* $\square$

We say that $B(X, Y)$ carries the (metric) topology $\mathfrak{O}_{\mathrm{u}}$ of *uniform convergence*. The topology induced on $B(X, Y)$ by the product topology of Y^X is called the topology of *pointwise convergence*, denoted $\mathfrak{O}_{\mathrm{p}}$. Clearly $\mathfrak{O}_{\mathrm{p}} \subseteq \mathfrak{O}_{\mathrm{u}}$.

Let $\mathbf{F} \subseteq C(X, Y)$ be a set of functions. Can we give conditions such that $\mathbf{F}|\mathfrak{O}_{\mathrm{p}} = \mathbf{F}|\mathfrak{O}_{\mathrm{u}}$?

Definition 2.27. A set $\mathbf{F}$ of functions $X \to Y$ from a topological space to a metric space is called *equicontinuous* if

$$(3) \qquad (\forall x \in X)(\forall \varepsilon > 0)\big(\exists U_{x,\varepsilon} \in \mathfrak{U}(x)\big)(\forall f \in \mathbf{F})\, f(U_{x,\varepsilon}) \subseteq U_\varepsilon\big(f(x)\big). \qquad \square$$

By comparison, the statement that all functions in $\mathbf{F}$ are continuous reads as follows:

$$(4) \qquad (\forall f \in \mathbf{F})(\forall x \in X)(\forall \varepsilon > 0)\big(\exists U_{f,x,\varepsilon} \in \mathfrak{U}(x)\big)\, f(U_{f,x,\varepsilon}) \subseteq U_\varepsilon\big(f(x)\big).$$

If X is compact and $\mathbf{F} \subseteq Y^X$ is equicontinuous, then $\mathbf{F} \subseteq C(X,Y) \subseteq B(X,Y)$. Thus on $\mathbf{F}$ we can consider the topologies of pointwise and of uniform convergence.

Proposition 2.28. *Let X be a compact space, Y a metric space and $\mathbf{F}$ an equicontinuous set of functions $X \to Y$. Then $\mathbf{F}|\mathfrak{O}_{\mathrm{p}} = \mathbf{F}|\mathfrak{O}_{\mathrm{u}}$, that is the topologies of pointwise and of uniform convergence agree on $\mathbf{F}$.*

Proof. Let $f \in \mathbf{F}$, and $\varepsilon > 0$. We must find a $\delta > 0$ and $E \subseteq X$ finite such that

$$(5) \qquad (\forall g \in \mathbf{F})\big[(\forall e \in E)\, d\big(f(e), g(e)\big) < \delta\big] \Rightarrow (\forall x \in X)\, d\big(f(x), g(x)\big) < \varepsilon.$$

For each $x \in X$ we find an open neighborhood V_x of x in X such that $(\forall f \in \mathbf{F})\, f(V_x) \subseteq U_{\varepsilon/3}\big(f(x)\big)$ Since X is compact, there is a finite set $E \subseteq X$ such that $X = \bigcup_{e\in E} V_e$. Set $\delta = \varepsilon/3$ and assume that $g \in \mathbf{F}$ satisfies $d\big(f(e), g(e)\big) < \varepsilon/3$ for $e \in E$. Now let $x \in X$ arbitrary. Then there is an $e \in E$ such that $x \in V_e$. Accordingly, $d\big(f(x), g(x)\big) \le d\big(f(x), f(e)\big) + d\big(f(e), g(e)\big) + d\big(g(e), g(x)\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. The required condition is now satisfied with $\delta = \varepsilon/3$. □

Lemma 2.29. *If $\mathbf{F}$ is an equicontinuous subset of $B(X,Y)$ for a topological space X and a metric space Y, then the closure $\overline{\mathbf{F}}$ of $\mathbf{F}$ in Y^X is equicontinuous. As a consequence, if X is compact, then the closures of $\mathbf{F}$ in the topologies of uniform convergence and that of pointwise convergence agree and are contained in $C(X,Y)$.*

Proof. Let $\varepsilon > 0$ and $x \in X$. Find a $U \in \mathfrak{U}(x)$ such that $f(U) \in \mathfrak{U}_{\varepsilon/3}\big(f(x)\big)$ for all f. Now let g be in the closure of $\mathcal{F}$ with respect to the pointwise topology and let $u \in U$. Then there is an $f \in \mathcal{F}$ such that $d\big(f(u), g(u)\big) < \varepsilon/3$ and $d\big(f(x), g(x)\big) < \varepsilon/3$. Now $d\big(g(u), g(x)\big) \le d\big(g(u), f(u)\big) + d\big(f(u), f(x)\big) + d\big(f(x), g(x)\big) < 3\cdot\frac{\varepsilon}{3} = \varepsilon$. This proves the first claim.

Now let $\mathbf{G}$ be the closure of $\mathbf{F}$ with respect to the uniform topology in $B(X,Y)$. Then $\mathbf{G} \subseteq \overline{\mathbf{F}}$. If X is compact, Proposition 2.28 applies and shows $\overline{\mathbf{F}}|\mathfrak{O}_u = \overline{\mathbf{F}}|\mathfrak{O}_p$. Therefore, $\mathbf{G} = \overline{\mathbf{F}}$. By Proposition 2.26, $\mathbf{G} \subseteq C(X,Y)$. □

Definition 2.30. A subset R of a topological space Y is called *relatively compact* in Y if its closure $\overline{R}$ is compact in Y.

Let X be a set and $\mathbf{F} \subseteq Y^X$ a set of functions $X \to Y$ into a topological space. Then $\mathbf{F}$ is called *pointwise relatively compact* if the set $\mathbf{F}(x) \stackrel{\text{def}}{=} \{f(x) : f \in \mathbf{F}\}$ is relatively compact in Y for each $x \in X$. □

Relative compactness is not a property or R as a sspace in its own right, but depends on Y as well. Let $Y = \mathbb{Q}$ with the natural metric. Then $R =]0,1[\cap\mathbb{Q}$ is precompact as a metric space but is not relatively compact in $\mathbb{Q}$ as its closure $[0,1] \cap \mathbb{Q}$ in $\mathbb{Q}$ is not compact. If, however, $Y = \mathbb{R}$ in the natural topology, then $]0,1[\cap\mathbb{Q}$ is relatively compact in Y. A relatively compact subspace of a metric space is always precompact.

Lemma 2.31. *A pointwise relatively compact subset* $\mathbf{F} \subseteq Y^X$ *for any set* X *and a Hausdorff topological space* Y *is relatively compact in* Y^X *in the topology* $\mathfrak{O}_p$ *of pointwise convergence.*

Proof . For $x \in X$ let $K_x \stackrel{\text{def}}{=} \overline{\mathbf{F}(x)}$. Then $\prod_{x\in X} K_x \subseteq Y^X$ is a compact subspace of the product space Y^X by the Tychonov Product Theorem (TPT) 2.14. Now $\mathbf{F}(x) \subseteq \prod_{x\in X} K_x$ and so $\overline{\mathbf{F}} \subseteq \overline{\prod_{x\in X} K_x} \prod_{x\in X} K_x$ since Y^X is Hausdorff by 1.54 and compact subsets of a Hausdorff space are closed by E2.1(ii). Hence $\overline{\mathbf{F}}$ is compact since closed subsets of a compact space are compact by 2E.1(i). Thus $\mathbf{F}$ is relatively compact. □

As a corollary of the previous proposition we get

Corollary 2.32. *Let* X *be a compact space,* Y *a metric space and* $\mathbf{F}$ *an equicontinuous pointwise relatively compact set of functions* $X \to Y$*. Then* $(\mathbf{F}, \mathfrak{O}_{\mathrm{u}})$*, that is,* $\mathbf{F}$ *with the (metric) topology of uniform convergence, is relatively compact.*

Proof . Let $\mathbf{G}$ be the closure of $\mathbf{F}$ in Y^X for the topology $\mathfrak{O}_p$ of pointwise convergence. By Lemma 2.29, $\mathfrak{O}_u|\mathbf{G} = \mathfrak{O}_p|\mathbf{G}$ and $\mathbf{G}$ is the closure of $\mathbf{F}$ in $C(X,Y)$ in the topology of uniform convergence. By Lemma 2.31, $\mathbf{G}$ is compact in Y^X for $\mathfrak{O}_p$ and thus in $C(X,Y)$ with respect to the topology of uniform convergence. □

In the circumstances of Corollary 2.32, $\mathbf{F}$ is, in particular, precompact.

If X and Y are sets then the function $\mathrm{ev}\colon Y^X \times X \to Y$, $\mathrm{ev}(f,x) = f(x)$ is called the *evaluation function.* Re call that for a compact space X and a metric space Y, on $C(X,Y)$ we consider the topology $\mathfrak{O}_u$ of uniform convergence.

Lemma 2.33. *If* X *is compact and* Y *is metric, then* $\mathrm{ev}\colon C(X,Y) \times X \to Y$ *is continuous.*

Proof . Exercise. □

We retain the hypotheses of 2.33.

Lemma 2.34. *If* $\mathbf{F}$ *is a compact subset of* $C(X,Y)$*, then* $\mathbf{F}$ *is equicontinuous.*

Proof . Exercise. □

Exercise E2.10. Prove Lemmas 2.33 and 2.34. [Hint. 2.33: Let $f \in C(X,Y)$ and $x \in X$ and $\varepsilon > 0$ Let D be the sup metric of $C(X,Y)$ and pick an open neighborhood U of x so that $f(U) \subseteq U_{\varepsilon/2}\big(f(x)\big)$. Now let $g \in C(X,Y)$ satisfy $D(g,f) < \varepsilon/2$ and take $u \in U$. Then $d_Y(f(x), g(u)) \leq d_Y\big(f(x), f(u)\big) + d_Y\big(f(u), g(u)\big) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus $\mathrm{ev}(U_{\varepsilon/2}(f) \times U) \subseteq U_\varepsilon\big(f(x)\big)$.

2.34: Let $\varepsilon > 0$ and $x \in X$ be given. By 2.33, for each $g \in \mathbf{F}$ find a neighborhood $\mathbf{W}_g$ of g in $\mathbf{F}$ and a neighborhood U_g of x in X such that $\mathrm{ev}(\mathbf{W}_g \times$

$U_g) \subseteq U_{\varepsilon/2}\big(g(x)\big)$. Use compactness of $\mathbf{F}$ to find a finite set $\mathbf{E} \subseteq \mathbf{F}$ such that $\mathbf{F} = \bigcup_{g\in\mathbf{E}} \mathbf{W}_g$. Set $U = \bigcap_{g\in\mathbf{E}} U_g$. Then U is a neighborhood of x in X. Show that for every $f \in \mathbf{F}$ and every $u \in U$ we get $d_Y\big(f(u), f(x)\big) < \varepsilon$.]

All these pieces of information, taken together lead to the following theorem which plays an important role in analysis.

Theorem 2.35. (Ascoli Theorem) *Let X be a compact space, Y a metric space, and $\mathbf{F} \subseteq C(X,Y)$ be a set of continuous functions $X \to Y$. Endow $\mathbf{F}$ with the metric topology of uniform convergence. Then the following statements are equivalent:*

(i) $\mathbf{F}$ *is compact (as a subspace of $C(X,Y)$).*

(ii) $\mathbf{F}$ *is equicontinuous, pointwise relatively compact, and closed.* □

Under the circumstances of an equicontinuous set closedness of $\mathbf{F}$ *means closedness in either $B(X,Y)$ with respect to uniform convergence or in Y^X with respect to the product topology.*

The Ascoli Theorem has variants which generalize what is said in 2.35, but they are not different in principle. The Ascoli Theorem is the only way to verify that a space of continuous functions is compact.

Example. Let $(E, \|\cdot\|)$ be a Banach space. Let I be a compact real interval, $K \geq 0$ a nonnegative number, and let $\mathbf{F}_K \subseteq C(I,E)$ be the set of of all differentiable functions such that $\|f'(t)\| \leq K$ for all $f \in \mathbf{F}$. Then $\mathbf{F}_K$ is equicontinuous.

Let $a = \min I$, $b = \min I$, and let $x_0 \in E$. Define $\mathbf{F}_{K,x_0}$ to be the set of all $f \in \mathbf{F}_K$ satisfying $f(a) = x_0$. Then $f(I) \subseteq B_{K(b-a)}\big(f(t)\big) \subseteq B_{2K(b-a)}(x_0)$.

If $\dim E < \infty$ then $B_{2K(b-a)}(x_0)$ is compact, and thus by the Ascoli Theorem, $\mathbf{F}_{K,x_0}$ is compact.

This permits a very quick proof of a basic theorem in the theory of ordinary differential equations stating the existence of local solutions of the initial value problem $\dot{u}(t) = f\big(t, u(t)\big)$, $f(t_0) = x_0$.

Chapter 3
Connectivity

We proceed to further special properties of topological spaces. From basic analysis we know that, next to compactness, connectivity is another important property of topological spaces.

A subset S of a topological space X is called *open-closed* or *clopen* if it is at the same time open and closed. The empty set and the whole space are clopen. [The adjective "clopen" is artificial. It is convenient, but stylistically it is far from being a brilliant creation.]

We shall say that S is a *proper clopen subset* if is is a clopen subset which is neither $\emptyset$ nor X.

Definition 3.1. A topological space $(X, \mathfrak{O})$ is said to be *disconnected*, if it has a proper clopen subset. Otherwise it is called *connected*. □

That is, X is connected iff $\emptyset$ and X are the only clopen subsets of X.

Exercise E3.1. Let $(X, \leq)$ be a totally ordered set and consider the order topology on it. Prove:

If X has a nonempty subset Y which has an upper bound but does not have a least upper bound, then X is disconnected.

If X contains two elements $a < b$ such that $X = {\downarrow}a \cup {\uparrow}b$, then X is disconnected.

If S is a clopen subset of X then ${\downarrow}S$ is clopen.

A subset $\{a, b\} \subseteq X$ of a totally ordered set such that $a < b$ and $X = {\downarrow}a \cup {\uparrow}b$ is called a *gap*. We say that a totally ordered set X *satisfies the Least Upper Bound Axiom* (LUB for short) if ever nonempty subset which has an upper bound has a least upper bound.

The set $\mathbb{R}$ or real numbers with its natural order satisfies LUB. The set $\mathbb{Q}$ of rational numbers in its natural order does not satisfy LUB. Neither of these totally ordered sets has gaps. The Cantor set has gaps and satisfies LUB (indeed it is complete).

Theorem 3.2. *For a totally ordered set X, the following two statements are equivalent:*

(1) *X is connected.*

(2) *X satisfies the Least Upper Bound Axiom and has no gaps.*

If X is connected and $Y \subseteq X$, then the following statements are equivalent:

(3) Y is connected in the induced topology.

(4) Y is an interval in D.

Finally, (4) implies

(5) The order topology of X induces on Y the order topology of Y.

Proof . (1)⇒(2): Exercise E3.1.

(2)⇒(1): Assume that X satisfies the Least Upper Bound Axiom and has no9 gaps. We claim that X is connected and suppose, by way of cfontradiction, that A be a proper clopen set. Let $B = X \setminus A$. W.l.o.g. we find an $a \in A$ and a $b \in B$ such that $a < b$. Since A is open, there is a largest interval $I \subseteq A$ with $a = minI$. Then $s \stackrel{\text{def}}{=} \sup I \leq b$. Since A is closed, $s \in A$. Since A is open, there is a $t > s$ such that $[s, t[\in A$. Then $[s, t]$ is a gap. Contradiction.

(4)⇒(5): If Y is a subset of a totally ordered set $(X, \leq)$ then the order topology of Y (generated by the sets $\uparrow y \setminus \{y\}$ and $\downarrow y \setminus \{y\}$, $y \in Y$ is contained in the topology induced on Y by the order topology of X (generated by the sets $\uparrow x \setminus \{x\}$ and $\downarrow x \setminus \{x\}$, $x \in X$).

Now assume that Y is an interval. In view of the preceding paragraph, we have to show that $\mathfrak{O}_X|Y \subseteq \mathfrak{O}_Y$. It suffices to show that for any subbasic set $S = \uparrow x \setminus \{x\}$, $\downarrow x \setminus \{x\}$, $x \in X$ of $\mathfrak{O}_X$ we have $S \cap Y \in \mathfrak{O}_Y$. So Let $x \in X$ and $S = \uparrow x \setminus \{x\}$. Then either $x \in Y$ or $x \notin Y$. If $x \in Y$, then, since X is an interval, $S \cap Y = (\uparrow_Y x \setminus \{z\} \in \mathfrak{O}_Y$. If $x \notin Y$, then, again since Y is an interval, either $Y \subseteq S$ (if x is a lower bound of Y), or $S \cap Y = \emptyset$ (if x is an upper bound of Y). If S is a subbasic downset, the proof is analogous.

(4)⇒(3). Assume again that Y is an interval. We claim that Y satisfies the Least Upper Bound Axiom and has no gaps. So let $a \in A \subseteq Y$ and let $b \in Y$ be an upper bound of A. Then $s \sup A$ exists in X since X satisfies the Least Upper Bound Axiom. As $a \leq s \leq b$ and Y is an interval, $s \in Y$, and so s is the least upper bound of A in Y. Secondly, if $y < y'$ in Y then, since X has no gaps, there is an $x \in X$ such that $y < x < y'$; since Y is an interval, $x \in Y$ and so $\{y, y'\}$ is not a gap in Y. Thus the claim is verified. Now by 3.2.A, Y is connected in its order topology $\mathfrak{O}_Y$. Since (4) implies (5), $\mathfrak{O}_Y = \mathfrak{O}_X|Y$, and so Y is connected in the induced topology.

¬(4)⇒¬(3). Let $Y \subseteq X$ and assume that Y fails to be an interval. Then there are elements $y < x < y'$ such that $y, y' \in Y$, $x \in X \setminus Y$. Then $\downarrow x \cap Y$ is a proper clopen subset of Y. □

Corollary 3.3. *A set of real numbers is connected in the induced topology if and only if it is an interval.*

Proof . Since $\mathbb{R}$ satisfies the least upper bound axiom and has no gaps, this is immediate from Theorem 3.3. □

There is a subtle point concerning the induced and the order topology of a subset. The subset $X \stackrel{\text{def}}{=} [0, 1] \cup]3, 4]$ is disconnected in the induced topology but is connected in its own order topology. On the subset $\{0, 1\} \subseteq \mathbb{R}$, the induced and the order topology agree.

Recall our convention $\mathbb{I} = [0, 1]$.

Definition 3.4. A topological space is called *arcwise connected* or *path-connected* if for all $(x,y)\in X\times X$ there is a $\gamma\in C(\mathbb{I},X)$ such that $\gamma(0)=x$ and $\gamma(1)=y$. □

Proposition 3.5. *An arcwise connected space is connected.* □

Exercise E3.2. Set $\mathbb{R}^+=\{r\in\mathbb{R}:0\le r\}$. In $\mathbb{R}\times\mathbb{C}$ consider the following subspace

$$S\stackrel{\text{def}}{=}\{(x,z):\big((\exists r\in\mathbb{R}^+)\,x=e^{-r},\ z=e^{2\pi ir}\big)\text{ or }x=0,\ |z|=1\}.$$

Draw a sketch of this set. Prove that it is connected but not arcwise connected.

Prove that $\mathbb{R}\times\mathbb{C}$ has a continuous commutative and associative multiplication given by

$$(r,c)(r',c')=(rr',cc'),\quad (r,c),\ (r',c')\in\mathbb{R}\times\mathbb{C}.$$

A topological space with a continuous associative multiplication is called a *topological semigroup*. If it has an identity, one also calls it a *topological monoid*.

Show that S is a compact subset satisfying $SS\subseteq S$. Thus S is a compact topological monoid.

Does it contain a subset which is a topological monoid and a group?

Theorem 3.6. *Let $f\colon X\to Y$ be a continuous surjective function between topological spaces. If X is connected, then Y is connected. If X is arcwise connected, then Y is arcwise connected.* □

One may express this result in the form: *A continuous image of a connected space is connected; a continuus image of an arcwise connected space is arcwise connected.*

Corollary 3.7. *A continuous image of a compact connected space is compact and connected.* □

Corollary 3.8. *A continuous image of a real interval is arcwise connected. A continuous image of a compact interval is compact and connected.*

In the basic courses on Analysis one learns about the Peano-Hilbrt curve which is a surjective continuous function $f\colon[0,1]\to[0,1]^2$.

Corollary 3.9. (The Intermediate Value Theorem of Real Calculus) *Let $f\colon[a,b]\to\mathbb{R}$ be a continuous function and $f(a)\le y\le f(b)$ or $f(b)\le y\le f(a)$. Then there is an $x\in[a,b]$ such that $y=f(x)$.* □

The Intermediate Value Theorem gives us a solution x of the equation $y=f(x)$ for given y.

Corollary 3.10. *A continuous self-map of $[0,1]$ has a fixed point.* □

Lemma 3.11. *If Y is a connected subspace of a topological space X, then the closure $\overline{Y}$ is connected as well.* □

Proposition 3.12. (a) *Let X be a topological space. The relation R given by*

$$R = \{(x,y) \in X \times X : (\exists Y)\, Y \text{ is a connected subspace of } X \text{ and } x, y \in Y\}$$

is an equivalence relation. (b) *All cosets $R(x)$ are closed.* (c) *The quotient space X/R is a T_1-space.* □

For (b)⇒(c) see 1.40.

Definition 3.13. The equivalence relation R of Lemma 3.12 is called the *connectivity relation*, and its equivalence classes are called the *connected components* or *components* of the space X.

Exercise E3.3. Prove the following analog of Proposition 3.13:
Let X be a topological space. Recall that a curve from p to q in a topological space X is a continuous function $f\colon \mathbb{I} \to X$, $\mathbb{I} = [0,1]$ such that $f(0) = p$ and $f(1) = q$. The relation $R_{\rm arc}$ on X given by

$$R_{\rm arc} = \{(x,y) \in X \times X : \text{ there is a curve from } x \text{ to } y\}$$

is an equivalence relation.

Give an example of a space such that the equivalence classes of this relation fail to be closed.

Each connected component of a space is the intersection of its open neighborhoods: Indeed, if $y \notin R(x)$ then $R(y) \cap R(x) = \emptyset$, and thus $R(x)$ is the intersection of the open sets $X \setminus R(y)$, $y \in X \setminus R(x)$.

Definition 3.14. A topological space in which all components are singletons is called *totally disconnected.*

Exercise E3.4. (i) Show that every discrete space is totally disconnected.

(ii) Show that the space of rational numbers, the space of irrational numbers, the Cantor set are all totally disconnected but nondiscrete spaces.

Theorem 3.15. (i) *If A is a connected subspace of a space X and $\{B_j : j \in J\}$ is a family of connected subspaces of a topological space X such that $A \cap B_j \neq \emptyset$ for all $j \in J$, then $A \cup \bigcup_{j\in J} B_j$ is connected.*

(ii) *Let $\{X_j : j \in J\}$ be a family of topological spaces and let $X \stackrel{\rm def}{=} \prod_{j\in J} X_j$ be its product. If all X_j are connected, respectively, arcwise connected, then X is connected, respectively, arcwise connected.*

(iii) *For any family of topological spaces $\{X_j : j \in J\}$, if R is the connectivity relation of $X \stackrel{\rm def}{=} \prod_{j\in J} X_j$ and R_j the connectivity relation of X_j for $j \in J$, then*

$$R = \{\big((x_j)_{j\in J}, (y_j)_{j\in J}\big) : (\forall j \in J)\, (x_j, y_j) \in R_j\}.$$

Equivalently, $R\big((x_j)_{j\in J}\big) = \prod_{j\in J} R(x_j)$ *for all* $j \in J$.

(iv) *If all* X_j *are totally disconnected, then* X *is totally disconnected.* □

The proof of (ii) is easy for arc connectivity, but is less obvious for connectivity.

Lemma 3.16. *If* X *is a space such that for each pair* $x, y \in X$ *of different points there is a clopen subset* U *with* $x \in U$ *and* $y \notin U$, *then* X *is totally disconnected.* □

Proposition 3.17. *Let* R *be the connectivity relation on* X. *Then* X/R *is totally disconnected* T_1*-space.*

Proof. If U and V are open and $U \cup V = X$ and $U \cap V = \emptyset$, then any R-class is entirely contained in either U or V. Hence U and V are R-saturated, i.e. are unions of R-equivalence classes. Thus by the definition of the quotient topology, the sets U/R and V/R are open; morover, $X/R = (U/R) \cup (V/R)$ and $(U/R) \cap (V/R) = \emptyset$. Suppose that C is a component of X/R. Then we consider $X' = \bigcup C$ (i.e., $X' = q_R^{-1}(C)$ where $q_R\colon X \to X/R$ is the quotient map. Then $R' \stackrel{\text{def}}{=} R \cap (X' \times X')$ is the connectivity relation of X' and $C = X'/R'$. By replacing X by X' and renaming, if necessary, let us assume that X/R is connected. We claim that X/R is singleton, i.e. that X is connected. So let $X = U \cup V$, $U \cap V = \emptyset$ for open subsets U and V of X. By what we have seen this implies $X/R = (U/R) \cup (V/R)$ and $(U/R) \cap (V/R) = \emptyset$. Since X/R is connected, one of U/R or V/R is empty. Hence one of U and V is empty, showing that X is connected.

This shows that X/R is totally disconnected. Since all connected components $R(x)$ are closed by 3.12, the singletons in X/R are closed by the definition of the quotient topology. Hence X/R satisfies the Frechet separation axiom T_1. □

Recall $\mathbb{I} = [0,1]$ and let $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with the topology induced from that of of $\mathbb{R}$ and set $X = (S \times \mathbb{I}) \setminus (\{0\} \times]0,1[$ with the topology induced from that of $\mathbb{R}^2$. Let R be the connectivity relation on X. Then $T \stackrel{\text{def}}{=} X/R$ is a totally disconnected compact T_1-space which is not Hausdorff. Each equivalence class of R is closed, but $R \subseteq X \times X$ is not closed.

Proposition 3.18. *Any continuous function* $f\colon X \to Y$ *into a totally disconnected space factors through* $q_R\colon X \to X/R$ *where* R *is the connectivity relation on* X. *That is, there is a continuous function* $\varphi\colon X/R \to Y$ *such that* $f = \varphi \circ q_R$.

Proof. If $x \in X$ then the image $f\big(R(x)\big)$ of the component $R(x)$ of x is connected by 3.6. On the other hand, as a subspace of the totally disconnected space Y it is totally disconnected. Hence it is singleton. Set $\varphi\big(R(x)\big) = f(x)$. If V is open in Y, then $q_R^{-1}\big(\varphi^{-1}(V)\big) = f^{-1}(U)$ is an open R-saturated set. But then $\varphi^{-1}(U) = q_R\big(f^{-1}(U)\big)$ is open by the definition of the quotient topology. Thus φ is continuous. □

Corollary 3.19. *For a topological space X the following conditions are equivalent:*

(i) *X is a connected.*

(ii) *All continuous functions $f: X \to Y$ into a totally disconnected space are constant.*

Proof. Let R be the connectivity relation on X and $q_R: X \to X/R$ the quotient map.

(i)⇒(ii): Every continuous function $f: X \to Y$ into a totally disconnected space Y factors through $q_R: X \to X/R$ by 3.18. But since X is connected by (i), X/R is singleton, and thus f is constant.

(ii)⇒(i): $q: X \to X/R$ is a continuous surjective function into a totally disconnected space by 3.17; since such a function is constant by (ii), X/R is singleton. i.e. X is connected. □

We saw that a connected component C of a space X does have clopen neighborhoods. It is not true in general that C is the intersection of all of its clopen neighborhoods.

Proposition 3.20. *For an arbitrary topological space X with connectivity relation R, the following conditions are equivalent:*

(i) *Every component is the intersection of its clopen neighborhoods.*

(ii) *X/R is a totally disconnected Hausdorff space in which every singleton is the intersection of its clopen neighborhoods.* □

The best situation prevails for compact spaces. We discuss this now; but we need a bit of preparation.

Lemma 3.21. (A. D. Wallace's Lemma) *Let A be a compact subspace of X and B a compact subspace of Y, and assume that there is an open subset U of $X \times Y$ containing $A \times B$. Then there are open neighborhoods V of A in X and W of B in Y such that $V \times W \subseteq U$.* □

[See Exercise Sheet no 12, Exercise 4 with hints.]

Lemma 3.22. (Normality Lemma) *Let A and B be two disjoint compact subsets of a Hausdorff space X. Then there are two disjoint open neighborhoods of A and B, respectively.*

[See Exercise Sheet no 12, Exercise 5 with hints.]

In fact the Normality Lemma shows that A and B have disjoint *closed* neighborhoods: Let U and V be open neighborhoods of A and B, respectively. Then $\overline{U} \cap V = \emptyset$ since $X \setminus V$ is a closed set containing U. Now apply the Normality Lemma to $\overline{U}$ and B and find disjoint open sets P and Q such that $\overline{U} \subseteq P$ and $B \subseteq Q$. Now $\overline{Q} \cap \overline{U} = \emptyset$. Hence $\overline{U}$ and $\overline{Q}$ are two disjoint closed neighborhoods of A and B, respectively.

Lemma 3.23. (Filter Basis Lemma). *Let $\mathfrak{B}$ be a filter basis of closed subsets in a space and assume that $\mathfrak{B}$ has a a compact member B. If U is an open set containing $\bigcap \mathfrak{B}$, then there is a $C \in \mathfrak{B}$ such that $C \subseteq U$.*

In particular, a filter basis of closed sets in a compact Hausdorff space converges to x iff $\bigcap \mathfrak{B} = \{x\}$.

[Hint. Suppose not, then $\{C \setminus U : C \in \mathfrak{B}\}$ is a filter basis of closed sets, whose members are eventually contained in the compact space B, hence there is an element $b \in \bigcap_{C\in\mathfrak{B}} C \setminus U$. Then $b \in (\bigcap \mathfrak{B}) \setminus U = \emptyset$, a contradiction.]

The Filter Basis Lemma allows us to formulate 3.20 for compact Hausdorff spaces in a sharper form

Proposition 3.20′. *For a compact Hausdorff space X with connectivity relation R the following conditions are equivalent:*

(i) *Every component has a basis of clopen neighborhoods.*

(ii) *X/R is a totally disconnected Hausdorff space in which every singleton is the intersection of its clopen neighborhoods.* □

If U is a clopen subset of a space X, then U and $X \setminus U$ are the classes of an equivalence relation with open cosets. The intersection of any family of equivalence classes is an equivalence class; a finite collection of open closed sets thus gives rise to a finite decomposition of the space into finitely many clopen sets. An equivalence relation with clopen classes on a compact space has finitely many classes.

Definition 3.24. An equivalence relation R on a topological space is *open* if R is open as a subset of $X \times X$.

Remark 3.25. The connectivity relation is contained in all open equivalence relations. □

Proposition 3.26. *Let R be an equivalence relation on a space X. Then the following conditions are equivalent:*

(i) *All equivalence classes are open.*

(ii) *R is open in $X \times X$.*

(iii) *The quotient space X/R is discrete.*

(iv) *All components are clopen.*

Proof. (i)⇔(ii): For every equivalence relation R we have $R = \bigcup_{x\in X} R(x) \times R(x)$. If each $R(x)$ is open in X, then each $R(x) \times R(x)$ is open in $X \times X$ and vice versa.

(i)⇒(iii): If $R(x)$ is open in X, then by the definition of the quotient topology, the singleton set $\{R(x)\}$ is open in X/R.

(iii)⇒(iv): In a discrete space every subset is clopen, so $\{R(x)\}$ is clopen in X/R and thus $R(x)$ is clopen by the continuity of the quotient map.

(iv)⇒(i) is trivial. □

Notice that for a compact space X, the component space X/R is a compact totally disconnected Hausdorff space regardless of any separation property of X.

Lemma 3.27. *On a topological space X, the following conditions are equivalent:*
(i) *The connectivity relation is the intersection of all open equivalence relation.*
(ii) *Every component is the intersection of its clopen neighborhoods.*

Proof . Exercise. □

[Hint for (ii)⇒(i): If U is a clopen subset of X, then U and $X \setminus U$ are the two classes of an open equivalence relation $_U$. If $(x, y) \notin R$, let U be clopen neighborhood of $R(a)$ not containing b (by (ii)). Then $(a, b) \notin R_U$.]

If $A, B \subseteq X \times X$ are binary relations on X, then

$$A \circ B \stackrel{\text{def}}{=} \{(x, z) : (\exists y \in X)\,(x, y) \in A \text{ and } (y, z) \in B\}.$$

Note that $A \circ A \subseteq A$ means that A is transitive.

Exercise E3.5. Show that
on a compact Hausdorff space the relation product $A \circ B$ of two closed binary relations is closed.

A space X is a Hausdorff space iff the diagonal is closed. Then by the Normality Lemma, Δ has a basis of closed neighborhoods.

Exercise E3.6. Show that
on a compact Hausdorff space every neighborhood U of the diagonal Δ of $X \times X$ contains a neighborhood W of Δ such that $W \circ W \subseteq U$.

[Hint. Suppose that U is an open member of $\mathfrak{U}(\Delta)$, the set of neighborhoods of the diagonal Δ in $X \times X$ such that $W \circ W \not\subseteq U$ for all $W \in \mathfrak{U}(\Delta)$. Then $\{(W \circ W) \setminus U : W = \overline{W} \in \mathfrak{U}(\Delta)\}$ is a filter basis of closed sets on the compact space $(X \times X) \setminus U$. Let (x, y) be in the intersection of this filterbasis. Then, on the one hand, $(x, y) \in \Delta$, i.e. $x = y$ and one the other $(x, y) \notin U$.]

Theorem 3.28. *Let X by a compact Hausdorff space. Then every component has a neighborhood basis of clopen subsets.*

Proof . Let X by a compact Hausdorff space. Then every component has a neighborhood basis of clopen subsets.

Proof . Let U be a neighborhood of the diagonal Δ in $X \times X$. By replacing U by $\{(u, v) : (u, v), (v, u) \in U\}$ if necessary, we may assume that U is symmetric. We define R_U to be the set of all pairs (x, y) such that there is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $(x_{j-1}, x_j) \in U$; we shall call such a sequence a U-chain. Then R_U is reflexive, symmetric, and transitive. Hence R_U is an equivalence relation. Write $U(x) = \{u \in X : (x, u) \in U\}$. Then $U(x)$ is a neighborhood of x. Since $U(x') \subseteq R_U(x)$ for each $x' \in R_U(x)$, the relation R_U is open and therefore closed as the complement of all other equivalence classes. Let S be the intersection of the clopen equivalence relations R_U as U ranges through the filterbasis $\mathfrak{U}_s(\Delta)$ of symmetric neighborhoods of Δ. Then S is an equivalence relation and S is closed in $X \times X$. Then for each $x \in X$, every pair of elements in $S(x)$ is R_U-equivalent for all $U \in \mathfrak{U}_s(\Delta)$. Let R denote the connectivity relation on X and set $C = S(x)$. The component $R(x)$ of x is contained in C. We aim to show that C is connected. Then $C = R(x)$ for all x and thus $R = S$. So $R(x) = \bigcap_{U \in \mathfrak{U}_s(\Delta)} R_U(x)$, and then, by the Filter Basis Lemma, the sets $R_U(x)$ form a basis of the neighborhoods of $C = R(x)$. This will complete the proof.

Now suppose that C is not connected. Then $C = C_1 \dot\cup C_2$ with the disjoint nonempty closed subsets of C. We claim that there is an open symmetric neighborhood $U \in \mathfrak{U}(\Delta)$ of the diagonal Δ in $X \times X$ such that the set $U(C_1) \cap C_2$ is empty. [It suffices to show that every open neighborhood W of a compact subset K of X contains one of the form $U(K)$. Proof by contradiction: If not, then for all open neighborhoods U of the diagonal in $X \times X$, $U(K) \cap (X \setminus W)$ is not empty and the collection of sets $U(K) \cap (X \setminus W)$ is a filterbasis on the compact space $X \setminus W$. Let z be in the intersection of the closures of the sets in this filterbasis. Since X is Hausdorff, the diagonal is closed in $X \times X$ and by the Normality Lemma is the intersection of its closed neighborhoods. Thus z in the intersection of all $U(K)$ for all closed U and this is K. Thus $z \in K \setminus W = \emptyset$, a contradiction!]

Recall that for two subsets $A, B \subseteq X \times X$ we set $A \circ B = \{x, z) \in X \times X : (\exists y \in X)\ (x, y) \in A$ and $(y, z) \in B\}$. Now assume that W is an open neighborhood of the diagonal such that $W \circ W \circ W \subseteq U$. and set $D = X \setminus \big(W(C_1) \cup W(C_2)\big)$. Now let $V \in \mathfrak{U}(\Delta)$, $V \subseteq W$. By replacing V by $\{(u, v) : (u, v), (v, u) \in V\}$ if necessary, we may assume that V is symmetric.

If $x \in C_1$ and $c_2 \in C_2$, then $(x, c_2) \in R_V$ since $C \in R_V(x)$. Now any V-chain $x = x_0, x_1, \ldots, x_n = c_2$ has at least one element in D. Thus $R_V(x) \cap D \neq \emptyset$. Thus the $R_V(x) \cap D$ form a filterbasis on the compact space D. Let y be in its intersection. Then $y \in \bigcap_{V \in \mathfrak{U}_s(\Delta)} R_V(x) = C$ and $y \in D$, whence $y \in C \cap D = \emptyset$: a contradiction. This shows that C is connected as asserted and completes the proof. □

The preceding theorem shows that
the connectivity relation R is the intersection of open equivalence relations.

In Theorem 3.28, compactness is sufficient, but it is not necessary.

Exercise E3.7. Show that in the space $\mathbb{Q}$ in its order topology every point has a basis of clopen neighborhoods. □

Corollary 3.28. *In a compact totally disconnected space, every point has a a neighborhood basis of clopen sets.* □

Quite generally, a space is clalled *zero-dimensional* if for all of its elements x, the neighborhood filter $\mathfrak{U}(x)$ has a basis of clopen sets.

Thus *a compact Hausdorff spaces is zero-dimensional iff it is totally disconnected.*

Chapter 4
Covering Spaces and Maps

Definition 4.1. A function $f: X \to Y$ is called a *covering map* or simply *a covering* if Y has an open cover $\{U_j \mid j \in J\}$ such that for each $j \in J$ there is a nonempty discrete space F_j and a homeomorphism $h_j: F_j \times U_j \to f^{-1}(U_j)$ such that the following diagram commutes:

$$\begin{array}{ccc} F_j \times U_j & \xrightarrow{h_j} & f^{-1}(U_j) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle f|f^{-1}(U_j)} \\ U_j & \xrightarrow[\mathrm{id}_{U_j}]{} & U_j. \end{array}$$

We will briefly say that $f^{-1}(U_j)$ *is compatibly homeomorphic to* $F_j \times U_j$. We call F_j the *fiber* over U_j and Y the *base space* of the covering.

A function $f: X \to Y$ between topological spaces is said to induce *a local homeomorphism* at x if there are open neighborhoods U of x in X and V of $f(x)$ in Y such that $f|U: U \to V$ is a homeomorphism. It is said to *induce local homeomorphisms* if it induces local homeomorphisms at all points. (Many authors say in these circumstances that f *is a local homeomorphism.*) □

Local homeomorphisms are clearly continuous and open; thus a covering is always a continuous and open map.

Remark 4.2. Every covering induces local homeomorphisms. The converse fails in general. □

The assertions in the following examples are left as an exercise.

A topological group G is a topological space and a group such that multiplication $(x, y) \mapsto xy : G \times G \to G$ and inversion $x \mapsto x^{-1}: G \to G$ are continuous functions. The additive group $\mathbb{R}$ and the multiplicative group $\mathbb{C} \setminus \{0\}$, and indeed all matrix groups are topological groups.

Examples 4.3. (i) Let G be a topological group and H a discrete subgroup. Let G/H denote the space of all cosets gH, $g \in G$ endowed with the quotient topology and let $p: G \to G/H$, $p(g) = gH$ be the quotient map. Then p is a covering.

(ii) If $f: G \to H$ is a continuous homomorphism of topological groups then f is a covering if and only if the following conditions are satisfied:

(a) $\ker f$ is discrete.
(b) f is open.

(c) f is surjective.

(iii) Let $(g,x) \mapsto g{\cdot}x : G \times X \to X$ be an action of a finite discrete group G on a Hausdorff space such that all $x \mapsto g{\cdot}x$ are continuous and that the action is free, i.e. that $g{\cdot}x = x$ implies $g = 1$. Then the orbit map $q\colon X \to X/G = \{G{\cdot}x \mid x \in X\}$ is a covering when X/G is given the quotient topology.

(iv) By (i), the homomorphism $p\colon \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $p(r) = r + \mathbb{Z}$ is a covering. Its restriction to $\left]0, 1\frac{1}{2}\right[$ induces local homeomorphisms but is not a covering. □

Exercise E4.1. Verify the claims of Examples 4.3.

[Hint for (iii). Let $x \in X$. Find an open neighborhood U of x in X such that $(\forall g \in G \setminus \{1\})\, g{\cdot}U \cap U = \emptyset$; indeed if that were not possible, then for each U there would be $x_U,\, y_U \in U$ and a $g_U \in G$ such that $g_U{\cdot}x_U = y_U$. Since G is finite, we may assume that for a basis of neighborhoods V of x we have $g_V = g \in G \setminus \{1\}$. But $x_V,\, y_V \to x$; thus $g{\cdot}x = x$ by the continuity of $z \mapsto g{\cdot}z$; a contradiction to the freeness of the action. Now the function $(g,u) \mapsto G \times U \to G{\cdot}U = q^{-1}q(U)$ is a homeomorphism.] □

One can construct new coverings from given ones as the following proposition shows.

Proposition 4.4. (i) *If $f_j\colon X_j \to Y_j$, $j = 1,2$ are coverings, then $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is a covering. In short: Finite products of coverings are coverings.*

(ii) If $p\colon E \to B$ is a covering, $f\colon X \to B$ any continuous function, and if

$$\begin{array}{ccc} P & \xrightarrow{\;f^*\;} & E \\ {\scriptstyle p^*}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[\;f\;]{} & B \end{array}$$

is a pullback diagram (i.e. $P = \{(x,e) \in X \times E \mid f(x) = p(e)\}$, $f^*(x,e) = e$, $p^*(x,e) = x$), then $p^*\colon P \to X$ is a covering. In short: Pullbacks of coverings are coverings.

(iii) If $f\colon X \to Y$ is a covering and $Y' \subseteq Y$, then $f'\colon X' \to Y'$ is a covering where $X' = f^{-1}(Y')$ and $f' = f|X'$. In short: Restrictions of coverings are coverings.

(iv) Assume that $p\colon E \to B$ is a covering, B is connected, and that B admits a cover of connected open sets U_j, $j \in J$ such that $p^{-1}(U_j)$ is compatibly homeomorphic to $F \times U_j$. Then for every connected component E' of E the restriction $p|E'\colon E' \to B$ is a covering.

Proof . The proofs are largely straightforward from the definition of a covering:

(i) Assume that $\{U_j \mid j \in J\}$ is an open cover of Y_1 such that for each $j \in J$, the space $p^{-1}(U_j)$ is compatibly homeomorphic to $F_j \times U_j$, and $\{V_k \mid k \in K\}$ is an open cover of Y_2 such that $f_2^{-1}(V_k)$ is compatibly homeomorphic to $G_k \times V_k$. Then $\{U_j \times V_k \mid (j,k) \in J \times K\}$ is an open cover of $Y_1 \times Y_2$ such that $(f_1 \times f_2)^{-1}(U_j \times V_k)$ is compatibly homeomorphic to $(F_j \times G_k) \times (U_j \times V_k)$.

(ii) Assume that $\{U_j \mid j \in J\}$ is an open cover of B such that for each $j \in J$, the space $f_1^{-1}(U_j)$ is compatibly homeomorphic to $F_j \times U_j$. Then $\{f^{-1}(U_j) \mid j \in J\}$ is an open cover of X such that for each j the space $(p^*)^{-1}\big(f^{-1}(U_j)\big) = \{(x,e) \in X \times E \mid f(x) = p(e) \in U_j\}$ is compatibly homeomorphic to $F_j \times f^{-1}(U_j)$.

In fact this proof shows that in pullbacks the fibers are pulled back.

The proof of (iii) is quite straightforward.

(iv) For each $j \in J$ there is a homeomorphism $h_j: F_j \times U_j \to p^{-1}(U_j)$ such that $ph_j(x,u) = u$. We consider $e \in p^{-1}(U_i) \cap E'$. Then $h_j\big(x, p(e)\big) = e$ for some $x \in F_j$, and $h_j(\{x\} \times U_j)$ is a connected open subset of E containing e. Hence it is contained in E'. If we set $F'_j = \{x \in F_j \mid h_j(\{x\} \times U_j) \neq \emptyset\}$, then $(p|E)^{-1}(U_j) = p^{-1}(U_j) \cap E'$ is compatibly homeomorphic to $F'_j \times U_j$. □

Even though in the context of topological groups the great generality in which coverings are defined is justified, the most viable context is that of connected spaces and of pointed spaces. A *pointed space* is a pair (X,x) of a space and a *base point* $x \in X$; a *morphism of pointed spaces* $f: (X,x) \to (Y,y)$ is a continuous function $f: X \to Y$ such that $f(x) = y$. It is also called a *base point preserving continuous map.* Often pointed spaces occur quite naturally; e.g. all topological groups have their identity as a natural base point, and homomorphisms are automatically base point preserving.

A *covering of pointed spaces* is a covering between pointed spaces which is base point preserving.

If $p: (E,e) \to (B,b)$ is a covering of pointed spaces and $f: (X,x) \to (B,b)$ is a morphism of pointed spaces, then a function $F: X \to E$ is called *a lifting of f across p* if it is a morphism of pointed spaces and $f = p \circ F$.

$$\begin{array}{ccc}
(X,x) & \xrightarrow{\quad F \quad} & (E,e) \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
(X,x) & \xrightarrow[\quad f \quad]{} & (B,b).
\end{array}$$

Proposition 4.5. (i) *Assume that X is a connected space, $x_0 \in X$ and that φ, $\psi: X \to Y$ are continuous functions such that $\varphi(x_0) = \psi(x_0)$. Assume further that for some continuous function $\rho: Y \to Z$ which induces local homeomorphisms the compositions $\rho \circ \varphi$ and $\rho \circ \psi$ agree. Then $\varphi = \psi$.*

(ii) *A lifting of a morphism f of pointed spaces across a covering of pointed spaces is unique if the domain of f is connected.* □

Exercise E4.2. Prove Proposition 4.5.

[Hint. (i) Define $X' = \{x' \in X \mid \varphi(x') = \psi(x')\}$. Since all spaces considered are assumed to be Hausdorff spaces, X' is closed. Note that $x_0 \in X'$ and prove that X' is open in X using the fact that p induces local homeomorphisms. Use the connectivity of X to conclude the assertion. Derive (ii) from (i).] □

Definition 4.6. (Defining Simple Connectivity) A topological space X is called *simply connected* if it is connected and has the following universal property: For any covering map $p: E \to B$ between topological spaces, any point $e_0 \in E$ and any continuous function $f: X \to B$ with $p(e_0) = f(x_0)$ for some $x_0 \in X$ there is a continuous map $\widetilde{f}: X \to E$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(x_0) = e_0$.

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{f}} & E \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B. \end{array}$$ □

The lifting $\widetilde{f}$, if it exists, is automatically unique by 4.5(ii).

The definition we give is particularly useful in the context of topological groups and transformation groups because it specifies directly the property one uses most often. It is noteworthy that it does not depend on arcwise connectedness.

The conventional definition in the context of arcwise connected spaces is more geometric but coincides on this class of spaces with our definition. We shall deal with the equivalence of the two concepts for arcwise connected pointed spaces in Proposition 4.10 and Exercise E4.6 below.

One notices that Definition 4.6 is best phrased in terms of the category of pointed topological spaces and base point preserving continuous maps. Then it simply says that a pointed space is simply connected if any morphism into the base space of a covering lifts across the covering. In this category, simply connected spaces are, for those who know category theoretical elementary concepts, exactly the connected relative projectives with respect to the class of epics containing exactly the coverings.

Notice also that the definition of a simply connected pointed space (X, x_0) can also be expressed as follows.

Whenever

$$\begin{array}{ccc} (P, p_0) & \xrightarrow{F} & (E, e) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ (X, x_0) & \xrightarrow[f]{} & (B, b) \end{array}$$

is a pullback, then there is a subspace (P', p_0) of (P, p_0) such that $\pi|(P', p_0)$ is bijective. Indeed, this restriction being a covering by 4.4.(ii), (iii), its inversion is continuous and gives rise to the required lifting; the necessity is clear.

This raises at once the question of the existence of simply connected spaces. We shall first give examples and later exhibit a far ranging existence theorem.

Example 4.7. Assume that X is a totally ordered space, i.e. a set with a total order and a topology generated by the set of all open intervals $]a, b[\stackrel{\text{def}}{=} \{x \in X \mid a < x < b\}$, and assume that X is connected. Then X is simply connected. Examples are the space of real numbers and all of its intervals. □

Exercise E4.3. Prove the claim in Example 4.7.

[Hint. Let $x_0 \in X$ and $f\colon X \to B$ a continuous map for a covering $p\colon E \to B$ and let $e_0 \in E$ be such that $p(e_0) = f(x_0)$. Let $\mathcal{U}$ be the set of all functions $\varphi\colon U \to E$ such that U is an open interval of X with $x \in U$ and that $\varphi(x_0) = e_0$ and $p\big(\varphi(u)\big) = f(u)$ for $u \in U$. Consider an open neighborhood V of $f(x_0)$ and a discrete set F and a homeomorphism $h\colon F \times V \to E$ such that $\operatorname{im} h = p^{-1}(V)$ and $p\big(h(y,v)\big) = v$. Let $h\big(y_0, f(x_0)\big) = e_0$. There is an open interval U around x_0 in X with $f(U) \subseteq V$ and define $\varphi\colon U \to E$ by $\varphi(u) = h\big(y_0, f(u)\big)$. Verify $\varphi \in \mathcal{U}$. Show that $\mathcal{U}$ is inductive with respect to extension of functions as partial order. Let $\widetilde{f}\colon W \to E$ be a maximal element in $\mathcal{U}$ using Zorn's Lemma. Finish the proof by showing that $W = X$; if not then there is an $x \in X$ with $u < x$ (say) and $x \notin W$. Set $W_1 = \{x \in X \mid (\exists w \in W)\, x \le w\}$. Since W is open, so is W_1. Set $W_2 = \{x \in X \mid (\forall w \in W)\, w < x\}$. For $x \in W_2$ use the covering property around $f(x) \in B$ to show that there is a whole neighborhood of x contained in W_2. Thus W_2 is open. Show that $X = W_1 \cup W_2$ and note that this is a contradiction to the connectivity of X.] □

Proposition 4.8. (i) *Assume that (X, x_0) and (Y, y_0) are simply connected pointed spaces and that $p\colon (E, e) \to (B, b)$ is a covering. Let $f\colon \big(X \times Y, (x_0, y_0)\big) \to (B, b)$ be a morphism of pointed spaces. Then f has a lifting $\widetilde{f}\colon \big(X \times Y, (x_0, y_0)\big) \to (E, e)$ across p.*

(ii) If X and Y are simply connected, then $X \times Y$ is simply connected.

(iii) All spaces $\mathbb{R}^n$, $[0,1]^n$ (i.e. all open and all closed n-cells), $n \in \mathbb{N}$, are simply connected.

(iv) Each retract of a simply connected space is simply connected. In particular, if a product of spaces is simply connected, then each factor is simply connected.

Proof. Exercise E4.4. □

Exercise E4.4. Prove Proposition 4.8.

[Hint. (i) Assume that

$$\begin{array}{ccc} (P, p_0) & \xrightarrow{\quad F \quad} & (E, e) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \big(X \times Y, (x_0, y_0)\big) & \xrightarrow[\quad f \quad]{} & (B, b) \end{array}$$

be a pullback; i.e. $P = \{(x, y, z) \in X \times Y \times E \mid f(x, y) = p(z)\}$, $p_0 = (x_0, y_0, e)$, $\pi(x, y, z) = (x, y)$ and $F(x, y, z) = z$. The restriction of f to $X \times \{y_0\}$ lifts across p to a function $\varphi\colon (X, x_0) \to (E, e)$ so that $\big(x, y_0, \varphi(x)\big) \in P$. Then the restriction of f to $\{x\} \times Y$ lifts to a function $\psi_x\colon (Y, y_0) \to \big(E, \varphi(x)\big)$ so that $\big(x, y, \psi_x(y)\big) \in P$. Now the restriction π' of π to $P' \overset{\text{def}}{=} \{(x, y, \psi_x(y)) \in P \mid (x, y) \in X \times Y\}$ is bijective. If ι is the inclusion of P' into P, then $\widetilde{f} \overset{\text{def}}{=} \iota \circ \pi'^{-1}\colon \big(X \times Y, (x_0, y_0)\big) \to (E, e)$ is the required lifting.

(ii) is a consequence of (i) and (ii) implies (iii).

(iv) Let $X \subseteq Y$ and $r: Y \to X$ be a retraction. If $p: E \to B$ is a covering and $f: X \to B$ a continuous function, then $fp: Y \to B$ has a lifting $F: Y \to E$. Then $F|X: X \to E$ is the required lifting of f.] □

Proposition 4.9. *For a connected space X, the following statements are equivalent:*

(i) X is simply connected.
(ii) Whenever $f: E \to X$ is a covering and E_0 is a connected component of E, then $f|E_0: E_0 \to X$ is a homeomorphism.

Proof. (i)⇒(ii) Let $f: E \to X$ be a covering and assume (i). Pick $e_0 \in E_0$ and set $x_0 = f(e_0)$. Then the identity map $i: X \to X$ has a lifting $\widetilde{i}: X \to E$ such that $f\big(\widetilde{i}(x_0)\big) = x_0$ and $f \circ \widetilde{i} = i$. We claim that the image $\widetilde{i}(X)$ is open in E. Indeed let $x \in X$ and find an open neighborhood U of x in X such that for some discrete set F and some homeomorphism $h: F \times U \to f^{-1}(U)$ we have $f\big(h(y,u)\big) = u$. Let $h(y_0, x) = \widetilde{i}(x)$. Then $W \stackrel{\text{def}}{=} h(\{y_0\} \times U)$ is an open neighborhood of $\widetilde{i}(x)$. Since $\widetilde{i}$ is continuous there is an open neighborhood V of x in U such that $\widetilde{i}(V) \subseteq W$. Since $f|W: W \to U$ is a homeomorphism, $\widetilde{i}(V)$ is a neighborhood of $\widetilde{i}(x)$. Hence W and thus $\widetilde{i}(X)$ is a neighborhood of $\widetilde{i}(x)$. Now $\widetilde{i} \circ f: E \to E$ is a retraction with image $\widetilde{i}(X)$ and the image of retractions in Hausdorff spaces are closed, $\widetilde{i}(X)$ is a connected open closed subset of E containing e_0. It therefore agrees with E_0 and the assertion follows.

(ii)⇒(i) Assume (ii) and consider a covering $p: E \to B$ and a continuous function $f: X \to B$ such that $f(x_0) = p(e_0)$ for suitable $(x_0, e_0) \in X \times E$. Now we consider the pullback

$$\begin{array}{ccc} P & \xrightarrow{\quad f^* \quad} & E \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\quad f \quad]{} & B. \end{array}$$

Let $p_0 \in P$ denote the unique point with $p^*(p_0) = x_0$ and $f^*(p_0) = e_0$. Then p^* is a covering by 4.3.(ii). Let P_0 denote the component of p_0 in P. Then by (ii) the restriction $p^*|P_0: P_0 \to X$ is a homeomorphism. Denote the inclusion map $P_0 \to P$ by j and set $\widetilde{f} \stackrel{\text{def}}{=} f^* \circ j \circ (p^*|P_0)^{-1} : X \to E$. Then $\widetilde{f}(x_0) = f^*(p_0) = e_0$ and $p \circ \widetilde{f} = p \circ f^* \circ j \circ (p^*|P_0)^{-1} = f \circ p^* \circ j \circ (p^*|P_0)^{-1} = f$. This completes the proof. □

Sometimes simple connectivity is defined by condition (ii).

We say that two continuous functions f, $g: (X, x_0) \to (Y, y_0)$ of pointed spaces are *homotopic* if there is a continuous function $H: [0,1] \times X \to Y$ such that $H(0,x) = f(x)$, $H(t, x_0) = y_0$ and $H(1,x) = g(x)$ for all $t \in [0,1]$ and $x \in X$. Let $\mathbb{I}$ denote the pointed unit interval $([0,1], 0)$. A continuous function $f: (\mathbb{S}^1, 1) \to (Y, y_0)$ is called *a loop* at y_0. It is said to be *contractible*, if it is homotopic to the constant morphism of pointed spaces. We note that the contractibility of loops in X at x_0 is the same as saying that every continuous function $\partial D \to X$ from the boundary of the unit square $D = [0,1]^2$ into X (mapping $(0,0)$ to x_0) extends to a continuous function $D \to X$, and that, in turn, means that two paths $\alpha, \beta: \mathbb{I} \to (X, x_0)$ starting at x_0 and ending at the same point $\alpha(1) = \beta(1)$ are homotopic. Homotopy is an equivalence relation on the set $C_0(X,Y)$ of base point preserving functions from a pointed space X to a pointed space Y.

For each point x in an arcwise connected pointed space (X, x_0) we associate a discrete set $F(x)$, namely the set of homotopy classes $[\alpha]$ of arcs $\alpha: \mathbb{I} \to X$ from $x_0 = \alpha(0)$ to $x = \alpha(1)$. Write $\widetilde{X} = \bigcup_{x \in X} F(x)$ and set $p: \widetilde{X} \to X$, $p([\alpha]) = \alpha(x)$. Now assume that X has an open cover $\{U_j \mid j \in J\}$ such that each U_j is arcwise connected and every loop in every U_j is contractible; we call such spaces *locally arcwise simply connected*. For each $j \in J$ pick a $u_j \in U_j$. For each $f = [\alpha] \in F_j \stackrel{\text{def}}{=} F(u_j)$ and $u \in U_j$ we connect u_j and u by an arc ε in U_j. Every other arc from u_j to u is homotopic to ε by assumption on U_j. Let β denote the arc obtained by going from x_0 to u_j by α and from u_j to u by ε. Write $h_j(f,u) = [\beta] \in F(u)$. Then $p\big(h_j(f,u)\big) = u$. Thus $h_j: F_j \times U_j \to p^{-1}(U_j)$ is a well-defined function. If $[\gamma] \in p^{-1}(U_j)$, then $u = p([\gamma]) = \gamma(1)$ and there is an arc η in U_j from u to u_j, unique up to homotopy. The arc δ is obtained by going from x_0 to u by γ and from $u \to u_j$ by η. Then $f \stackrel{\text{def}}{=} [\delta]$ is an element of $F(u_j) = f_j$, and $(f,u) = h^{-1}(u)$. Thus h_j is bijective. There is a unique topology on $\widetilde{X}$ which induces on $p^{-1}(U_j)$ that topology which makes $h_j: F_j \times U_j \to p^{-1}(U_j)$ a homeomorphism. Then $p: \widetilde{X} \to X$ is a covering map.

Proposition 4.10. *For an arcwise connected locally arcwise connected pointed Hausdorff space (X, x_0) consider the following conditions:*

(i) All loops at x_0 are contractible.
(ii) (X, x_0) is simply connected.

Then (i) implies (ii). If X is also locally arcwise simply connected, then both conditions are equivalent.

Proof . (i)⇒(ii) Let $p: (E, e) \to (X, x_0)$ be a covering which we assume to be connected by 4.4(iv). By the simple connectivity of $\mathbb{I}$, every arc $\alpha: \mathbb{I} \to (X, x_0)$ lifts to a unique arc $\widetilde{\alpha}: \mathbb{I} \to (E, e)$, and by the simple connectivity of D, homotopic arcs lift to homotopic arcs. Define $\sigma: (X, x_0) \to (E, e)$ by $\sigma(x) = \widetilde{\alpha}(1)$ for any member α of the class of homotopic arcs from x_0 to x. Then $p\sigma(x) = p\widetilde{\alpha}(1) = \alpha(1) = x$. Let $\{U_j \mid j \in J\}$ be an open cover of X consisting of arcwise connected open sets such that for each U_j there is a discrete space F_j and a homeomorphism $h_j: F_j \times U_j \to p^{-1}(U_j)$ such that $p\big(h_j(f,u)\big) = u$ for all $(f,u) \in F_j \times U_j$. Let $x \in U_j$. Elements y nearby in U_j can be reached by a small arc ε from x to y, giving an arc via α from x_0 to x and from there to y; call this arc β. There is a unique $f \in F_j$ such that $h_j(f, \alpha(x)) = \widetilde{\alpha}(x) = \sigma(x)$. Then $t \mapsto h_j\big(f, \varepsilon(t)\big)$ is a small arc in $h_j(\{f\} \times U_j)$ from $\sigma(x)$ to a unique point in the set above y, which is necessarily the endpoint of $\widetilde{\beta}$. This point is $\sigma(y)$. It follows that $\sigma(u) = h_j(f,u)$ for $u \in U_j$. In particular, σ is continuous, induces local homeomorphisms, and satisfies

$p\sigma = \mathrm{id}_X$. Then $\sigma(X)$ is an open subspace of E such that for all $j \in J$ the relation $h_j(\{f\} \times U_j) \cap \sigma(X) \neq \emptyset$ implies $h_j(\{f\} \times U_j) \subseteq \sigma(X)$. Hence $p|\sigma(X): \sigma(X) \to X$ is a covering map and the complement of $\sigma(X)$ in E is open, too. Since E is connected, $\sigma(X) = E$. Then $\sigma = p^{-1}$. That is, p is a homeomorphism. Then X is simply connected by 4.9.

(ii)⇒(i) Let $p: \widetilde{X} \to X$ be the covering constructed in the paragraph preceding the proposition. Since X is simply connected, p is bijective by 4.9. By the definition of $\widetilde{X}$ this means that two arcs linking x_0 with a point x in X are homotopic, and this is equivalent to (i). □

Example 4.11. (i) All continuous functions $f: (X, x_0) \to (C, c_0)$ preserving base points into a convex subset C of any real topological vector space E are contractible. Hence all convex subsets of any real topological vector space are simply connected.

(ii) All spheres $\mathbb{S}^n$ are simply connected spaces with the exception of the zero- and one-dimensional ones. In particular $\mathbb{S}^3 \cong \mathrm{SU}(3)$ is a simply connected compact topological group.

(iii) Let $\{S_j \mid j \in J\}$ be a family of simply connected, arcwise connected, locally arcwise connected and locally arcwise simply connected pointed spaces. Then the product space $\prod_{j \in J} S_j$ is simply connected.

Proof . Exercise E4.5. □

Exercise E4.5. Prove the assertions of the examples in 4.11.

[Hint. (i) Work with the function $H(r, x) = (1 - r)\cdot f(x) + r\cdot c_0$.

(ii) Show that in all spheres of dimension 2 or more each loop is contractible. Observe that $\mathbb{S}^0$ fails to be connected. Show that for the covering $p: \mathbb{R} \to \mathbb{S}^1$, $p(t) = e^{it}$ the identity map $f: \mathbb{S}^1 \to \mathbb{S}^1$ does not lift to a continuous function $\widetilde{f}: \mathbb{S}^1 \to \mathbb{R}$; if it did, the map $\widetilde{f}$ and then f would be contractible, but the coextension $f: \mathbb{S}^1 \to \mathbb{C} \setminus \{0\}$ has winding number one, and contractible loops in $\mathbb{C} \setminus \{0\}$ would have winding number 0. (A little elementary complex analysis is used here!)

(iii) The product $S = \prod_{j \in J} S_j$ is arcwise and locally arcwise connected by the definition of the product topology. If $\alpha: \mathbb{S}^1 \to S$ is a loop, then $\alpha(t) = \big(\alpha_j(t)\big)_{j \in J}$ and $\alpha_j: \mathbb{S}^1 \to S_j$ is a loop in S_j. Then by 4.10, since A_j is simply connected, there is continuous extension $A_j: \mathbb{D} \to S_j$ to the complex unit disc $\mathbb{D}$. Then $A: \mathbb{D} \to S$, $A(t) = \big(A_j(t)\big)_{j \in J}$ is a continuous extension of α. Hence every loop in S is contractible and thus S is simply connected by 4.10.] □

In the Definition 4.1 of a covering, the open cover $\{U_j : j \in J\}$ plays a somewhat volatile role; there is however, a class of spaces in which such a cover may be chosen in a canonical fashion. Indeed, a space X will be called *locally simply connected* if the set $\mathcal{S}(X)$ of simply connected open subsets of X covers X. (In the constructions of 4.10 we have used a similar hypothesis.)

Lemma 4.12. *Let X be a locally simply connected space. Then for each covering $p: E \to X$ there is a family $(F_S)_{S\in\mathcal{S}(X)}$ of discrete spaces and a family of homeomorphisms $(h_S)_{S\in\mathcal{S}(X)}$, $h_S: F_S \times S \to f^{-1}(S)$ such that $f\big(h_S(y,s)\big) = s$ for all $s \in S$.*

Proof . Let $S \in \mathcal{S}(X)$. The restriction $f|f^{-1}(S) : f^{-1}(S) \to S$ is a covering by 4.4(iii). Let F_S denote the set of connected components of $f^{-1}(S)$. By 4.9 each restriction $f|T: T \to S$ for $T \in F_S$ is a homeomorphism. Define $h_S: F_S \times S \to f^{-1}(S)$ by $h_S(T,s) = (f|T)^{-1}(s)$. Then $f\big(h_S(T,s)\big) = f\big((f|T)^{-1}(s)\big) = s$. □

We fix a connected and locally simply connected space X and consider the class $\mathcal{C}(X)$ of all coverings $p: E \to X$, denoted (E,p), together with the maps $f: E_1 \to E_2$ for objects (E_j, p_j), $j = 1, 2$ satisfying $p_2 \circ f = p_1$. This class forms a category with these maps as morphisms $f: (E_1, p_1) \to (E_2, p_2)$.

We assume that X is connected and consider the subclass $\mathcal{C}_0(X)$ of *connected* coverings (meaning, of course those coverings (E,p) for which E is connected. We claim that there is an upper bound to the cardinality of E depending on X only. We define an equivalence relation R on E consisting of all pairs $(x,y) \in E \times E$ such that there is a finite sequence of open subsets $U_1, \dots, U_n$ in E such that

(i) $p(U_j) \in \mathcal{S}(X)$ for $j = 1, \dots, n$,
(ii) $U_{j-1} \cap U_j \neq \emptyset$, $j = 2, \dots, n$,
(iii) $x \in U_1$ and $y \in U_n$.

Undoubtedly R is an equivalence relation, and obviously its cosets are all open. But each coset of an equivalence relation whose cosets are open is closed (as the complement of the union of all the other cosets). Since E is connected, there is only one equivalence class. At this point we pass to pointed spaces and fix an $x_0 \in X$ and consider each covering (E,p) of X to be equipped with a base point $e_0 \in E$ such that $p(e_0) = x_0$. For each x we find a chain $U_1, \dots, U_n$ satisfying (i), (ii) and

(iii)$_0$ $e_0 \in U_1$ and $x \in U_n$.

Then we set $V_j \stackrel{\text{def}}{=} p(U_j)$, $j = 1, \dots, n$ and notice

(a) $V_j \in \mathcal{S}(X)$ for $j = 1, \dots, n$,
(b) $V_{j-1} \cap V_j \neq \emptyset$, $j = 2, \dots, n$,
(c) $x_0 \in V_1$ and $p(x) \in V_n$.

We observe that every such chain V_j and every choice of an element $y \in V_n$ gives rise to only one lifting to a chain of sets U_j satisfying (i), (ii), and (iii) and the selection of exactly one $x \in U_n$ such that $p(x) = y$. The cardinality of the set of all finite chains Vj is not bigger than the cardinality of finite sequences of the infinite set of all subsets of X and is therefore not bigger than $2^{\operatorname{card}(X)}$. The function assigning to $(V_1, \dots, V_n; y)$ where $(V_1, \dots, V_n)$ satisfies (a), (b), and (c) and $y \in V_n$ the unique $x \in U_n$ with the unique lifting $(U_1, \dots, U_n)$ and $f(x) = y$ is surjective. Hence

$$\operatorname{card} E \le \operatorname{card}(X) \cdot 2^{\operatorname{card}(X)} = 2^{\operatorname{card}(X)}.$$

It follows that there is a *set* J of coverings $\big((E,e_0),p)\big)$ of (X,x_0) with a connected covering space E such that every isomorphy class of $\mathcal{C}(X)$ contains exactly one member of J, and we may assume that (X,id_X) is one of them. We say that a covering $j_1 = \big((E_1,e_{10}),p_1\big)$ is *above* a covering $j_2 = \big((E_2,e_{20},p_2\big)$ iff there is a morphism of coverings of pointed spaces $f\colon j_1 \to j_2$ and write $j_2 \le j_1$. Since a base point preserving morphism is a lifting of the base point preserving covering $p_1\colon E_1 \to X$, it is unique by Proposition 4.5(ii). Hence J is a partially ordered set with respect to the "above" relation. Due to the pullback construction in 2.4(ii) this partially ordered set is directed, since for any two coverings there will be one which is above the two.

We propose to show that J contains a maximal element $(\widetilde{X},\widetilde{p})$ which is above all others. If there is such an element then $\widetilde{X}$ will be simply connected by Proposition 4.9, and up to isomorphisms of coverings, it will be unique.

Definition 4.13. A covering $(\widetilde{X},\widetilde{p})$, $\widetilde{p}\colon \widetilde{X} \to X$ is called a *universal covering* if $\widetilde{X}$ is simply connected. □

As an example consider the one-sphere $\mathbb{S}^1$. We take $x_0 = 1$ as base point. Among the coverings we have the following morphisms

1) All maps $\mu_n\colon \mathbb{S}^1 \to \mathbb{S}^1$, $n \in \mathbb{Z}$. The fiber over any simply connected open set in $\mathbb{S}^1$ (here being homeomorphic to an interval) is isomorphic to $\ker \mu_n = \{e^{2\pi i m/n} \mid m = 0,\dots,n-1\}$,
2) The map $\exp\colon \mathbb{R} \to \mathbb{S}^1$, $\exp r = e^{2\pi i r}$. The fiber over any simply connected open set in $\mathbb{S}^1$ is $\ker \exp \mathbb{Z}$.

Since $\mathbb{R}$ is simply connected by Proposition 4.7 the covering in 2) is universal, and indeed given any other one in 1) there is a covering from the universal one to it. The issue is now: Do we always find a universal covering?

The construction of an inverse limit in Chapter 1, notably 1.25ff. (for which category theory provides sweeping generalisations) suggests that we construct a limit. For this purpose, we let M denote the set of all morphisms $f\colon (E^f,p^f) \to (E_f,p_f)$ between the coverings of X in J and consider, in the category of pointed spaces , the projective limit

$$L(X) \overset{\text{def}}{=} \{(x_E)_{(E,p)\in J} \in \textstyle\prod_{(E,p)\in J} E \mid (\forall f \in M)\ f(x_{E^f}) = x_{E_f}\}.$$

We define the map $\overline{p}\colon L(X) \to X$, $\overline{p}\big((x_E)_{(E,p)\in J}\big) = x_X$ recalling (X,id_X) to be the minimal element of J. Set $e = (e_j)_{j\in J}$ where e_j is the base point of E where $j = (E,p) \in J$. Then $\overline{p}(e) = x_0$. Now let $\xi = (x_j)_{j\in J} \in L(X)$ and set $x = \overline{p}(\xi)$. Let $S \in \mathcal{S}(X)$ be a simply connected open neighborhood of x in X. Then for each $j = (E,p) \in J$ there is a unique cross section $\sigma_j\colon S \to E$ of pointed spaces such that $p \circ \sigma_j = \mathrm{id}_S$ and $\sigma_j(x) = x_j$ see 4.9). Then $\sigma(x') \overset{\text{def}}{=} (\sigma_j(x'))_{j\in J} \in \prod_{(E,p)\in J} E$ is seen to be in $L(X)$ for all $x' \in S$ by the uniqueness of liftings (4.5(ii)). Hence $\sigma_{(\xi,S)}\colon X \to L(X)$, is a cross section satisfying $\sigma_{(\xi,S)}(\overline{p}|\overline{p}^{-1}(S)) = \mathrm{id}_S$ and $\sigma_{(\xi,S)}(x) = \xi$. In particular, if x is in the image of $\overline{p}$ then every simply connected neighborhood S of X is in the image I of $\overline{p}$. This shows that I is open. If $x \in \overline{I}$

then some simply connected neighborhood S of x meets I. Hence $x \in S \subseteq I$. Thus $x \in I$ and I is also closed. Since X is connected, $I = X$ and thus $\overline{L}(X) \to X$ is surjective. The set $\mathcal{B}$ of all open subsets U of X for which there is an $S \in \mathcal{S}(X)$ with $U \subseteq S$ form a basis for the topology of X. The set of all $V_{\xi,U} \stackrel{\text{def}}{=} \sigma_{\xi,S}(U)$ for any $S \in \mathcal{S}(X)$, $U \subseteq S$ is a basis for a topology $\mathcal{O}$ on $L(X)$ such that the components of $\overline{p}^{-1}(S)$ are exactly the sets $\sigma_{\xi,S}(S)$. For $S \in \mathcal{S}$ we let F_S denote the set of these components and define $h_S\colon F_S \times S \to \overline{p}^{-1}(S)$ by $h_S(\sigma_{\xi,S}(S), y) = \sigma_{\xi,S}(y)$. Then $\overline{p}\big(h_S(C,y)\big) = y$. This shows that $\overline{p}\colon \big(L(X), \mathcal{O}\big) \to X$ is a covering of pointed spaces. Let $\widetilde{X}$ be the connected component of e in $\big(L(X), \mathcal{O}\big)$, and let $\widetilde{p}$ be the restriction $\overline{p}|\widetilde{X}$. Then $\widetilde{p}\colon (\widetilde{X}, e) \to (X, x_0)$ is a covering by 4.4(iv). Thus $\big((\widetilde{X}, e), \widetilde{p}\big) \in J$.

For each $k \in J$ we have a limit map $\overline{p}_k\colon L(X) \to E$ for $k = (E, p)$ given by $\overline{p}\big((x_j)_{j\in J}\big) = x_k$. The space E is locally simply connected; e.g. the connected components of $p^{-1}(S)$, $S \in \mathcal{S}(X)$ are homeomorphic to S by 4.4(iv) and 4.9. Then just as in the case of the minimal $k = (X, \mathrm{id}_X)$ we see that $\overline{p}_k\colon \big(L(X), \mathcal{O}\big) \to (E, p) = k$ is a covering. By 4.4(iv), accordingly, write $\widetilde{x} = (x_j)_{j\in J}$ for an element in $\widetilde{X}$ and note that $p\big(\overline{p}_k(\widetilde{x})\big) = p(x_k) = x_{(X,\mathrm{id}_X)} = \widetilde{p}\big((x_j)_{j\in J}\big) = \widetilde{p}(\widetilde{x})$. Thus the map $\overline{p}_k|\widetilde{X}\colon (\widetilde{X}, e) \to (E, e_j)$ is a morphism of pointed coverings of (X, x_0). Thus $(\widetilde{X}, \widetilde{p})$ is maximal in J and $\widetilde{p}\colon (\widetilde{X}, e) \to (X, x_0)$ is a universal covering.

We have now proved the following existence theorem:

Theorem 4.14. (Existence of Universal Coverings) *Every connected locally simply connected Hausdorff space has a universal covering.* □

Since each open n-ball in $\mathbb{R}^n$ is simply connected (see E4.8(iii)) every locally euclidean space (i.e. every space having an open cover consisting of sets homeomorphic to an open ball of $\mathbb{R}^n$) is locally simply connected. By 4.10 and 4.11(i) all open balls in a Banach space are simply connected. Thus every space covered by a family of open sets each homeomorphic to an open ball in some Banach space is locally simply connected. Let us call such spaces *topological manifolds.*

Corollary 4.15. (Universal Coverings of Manifolds) *Every connected topological manifold has a universal covering.* □

From hindsight the somewhat lengthy proof of Theorem 4.14 exhibits a curiosity as far as limit constructions go: After we were all through we discovered that the limit was none other than a member of the inverse system itself because the index set turned out to have a maximal element . The example of the one-sphere mentioned above illustrates this fact: The limit of the coverings listed under 1) alone is a genuine solenoid; if we include the covering under 2), the limit degenerates to the universal covering itself.