Introduction to Compact Groups
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Exercise Sheet 4
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**Projective Limits**

**1. Exercise.** (Nonstrict projective limits) Let $G$ be a compact group and $\mathcal{F}$ a filter basis of closed subgroups. Then

$$\{H, H \in \mathcal{F}; f_{HK}: K \to H,\ H, K \in \mathcal{N}, K \subseteq H,\}$$

$f_{HK}(k) = k$, is a projective system. Show that $\lim_{H \in \mathcal{F}} H = \bigcap_{H \in \mathcal{F}} H = \bigcap \mathcal{F}$

[Hint. Define $f: \bigcap \mathcal{F} \to \lim_{H \in \mathcal{F}} H$, $f(g) = (g)_{H \in \mathcal{F}} \in \prod_{H \in \mathcal{F}} H$. Prove that $f$ is bijective. For this purpose first observe that $f$ is injective. Easy. Indeed very easy!. Next $f$ is surjective. So let $(g_H)_{H \in \mathcal{F}}$. What does that mean for the $g_H$? Find $g \in \bigcap \mathcal{F}$ so that $g = g_H$ for all $H$.]

**2. Exercise.** (Strict projective limits) Let $\phi_n: G_{n+1} \to G_n$, $n = 1, 2, \ldots$ be a sequence of morphisms of compact groups. For $m \leq n$ in $\mathbb{N}$ define $\phi_{mn}: G_n \to G_m$ by $\phi_{mn} = \phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_n$. Show that $\{G_n, n \in \mathbb{N}; \phi_{mn}: G_n \to G_m, m \leq n\}$ is a projective system.

Give an elementary proof that the limit maps $f_m: \lim_{n \in \mathbb{N}} G_n \to G_m$ are surjective if all $\phi_n$ are surjective.