Introduction to Compact Groups
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Exercise Sheet 3
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Morphisms between abelian groups, continued

1. Exercise. (Injectivity-Projectivity) A n abelian group P is called *projective* if for every surjective morphism $e: A \to B$ and every morphism $f: P \to B$ there is a morphism $F: P \to A$ such that $e \circ F = f$.
Draw a diagram. Compare it with the diagram drawn for the definition of an injective group.
Prove the following Theorem:
For an abelian group G, the following conditions are equivalent:
(1) G *is projective.*
(2) G *is free.*

[Hint. (1) $\Rightarrow$(2): G is a quotient of a free group, say via $e: F \to G$. Let $f: G \to G$ be the identity morphism. Since G is projective, find $F: G \to F$ such that $e \circ F = f$. conclude that G is isomorphic to a subgroup of a free group. You are no allowed to use a Theorem of Schreier: A subgroup of a free abelian group is free.
(2)$\Rightarrow$(1): Let X be a free generating set of G, i.e, $G = \mathbb{Z}^{(X)}$ and let $f: G \to B$ be given. Let $e: A \to B$ be a surjective morphism. Use the Axiom of Choice to find a function $s: B \to A$ such that $e \circ s$ is the identity function of B. The function $s: f|X: X \to A$ extends to a morphism $F: G \to A$. Show that F satisfies the requirement.]

2. Exercise. Let G be a cyclic group (finite or infinite). Show that $\eta_G: G \to \widehat{\widehat{G}}$ is an isomorphism.

[Hint. Use your knowledge on the character groups of finite cyclic groups and the character groups of $\mathbb{Z}$ and $\mathbb{T}$.]

3. Exercise. If A is a direct sum of groups A_1 and A_2 with the property that η_{A_n} is an isomorphhism for $n = 1, 2$, then η_A is an isomorphism.

4. Exercise. Let G be a finitely generated abelian group.
Show that $\eta_G: G \to \widehat{\widehat{G}}$ is an isomorphism.
[Hint. Use the following facts: (a) A finitely generated abelian group is a finite direct sum of cuclic groups. (b) Exercise 3.]