Introduction to Compact Groups
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Exercise Sheet 2
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TECHNISCHE UNIVERSITÄT DARMSTADT

Morphisms between topological abelian groups

1. Exercise. Let U b an arbitrary interval on $\mathbb{R}$ containing 0 and assume that $\phi: U \to G$ is a function into a group such that $x, y, x+y \in U$ implies $\phi(x+y) = \phi(x)\phi(y)$. Then there is a morphism of groups $F: \mathbb{R} \to G$ extending f. If U contains more than one point, then F is unique.

[Hint. (a) For $r \in \mathbb{R}$ and two integers m and n such that $r/m, r/n \in U$ prove that $m{\cdot}\phi(r/m) = n{\cdot}\phi(r/n)$.
(b) Assume that U is not singleton. (How do you define F if U is singleton?) Let $r \in \mathbb{R}$. Show that there is at least one integer n such that $r/n \in U$.
(c) Define $F: \mathbb{R} \to G$ by setting $F(r) = n{\cdot}\phi(r/n)$ where $n/r \in U$. Why is this possible?
(d) Show that F is a homomorphism.]

2. Exercise. Let $f: G \to H$ be a homomorphism between two topological groups. Show that the following conditions are equivalent:
(1) f is continuous.
(2) f is continuous at every piont $g \in G$.
(3) There is a point $g \in G$ such that f is continuous at g.
(4) f is continuous at 1.

[Hint. (1) $\iff$ (2) has nothing to do with groups: This is true for any function between topological spaces.
What about (2) $\Rightarrow$(3) $\Rightarrow$(4)?
Prove (4) $\Rightarrow$(2) using the fact that for each $g \in G$, the function $x \mapsto gx$ is a homeomorphism of G. (Is that ok?)]

3. Exercise. Let G and H be compact abelian groups and let K be a topological abelian group. Show that $\mathrm{Hom}(G \times H, K)$ and $\mathrm{Hom}(G, K) \oplus \mathrm{Hom}(H, K)$ are isomorphic.
Conclude that the character group of a product of compact abelian groups is the direct sum of the character groups.
Is this true for families of compact groups with more than two members? Proof?