Introduction to Compact Groups
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Exercise Sheet 1
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Definition of Compact Groups and Examples

1. Exercise. (i) Let G be a group and a topological space Show that the following conditions are equivalent:
(1) The function $(x, y) \mapsto xy^{-1} : G \times G \to G$ is continuous.
(2) Multiplication and inversion are continuous functions.
Here we recall that multiplication is the function $(x, y) \mapsto xy : G \times G \to G$.

2. Exercise. Verify the details of the propositions that a product of an arbitrary family $\{G_j : j \in J\}$ of compact groups is a compact group and that a closed subgroup of a compact group is a compact group.

3. Exercise. Prove the following assertion:
If G is a compact group and N a closed normal subgroup, then G/N is a compact group with respect to the quotient topology.
We must know that a subset $V \in G/N$ is open iff $q^{-1}V$ is open in G where $q: G \to G/N$ is the quotient map given by $q(g) = gN = Ng$.

4. Exercise. (p-adic integers). Let L denote the subset of all sequences $(x_n+p^n\mathbb{Z})_{n\in\mathbb{N}} \in P \stackrel{\text{def}}{=} \prod_{n\in N} \mathbb{Z}/p^n\mathbb{Z}$, $x_n \in \mathbb{Z}$ such that $x_{n+1} \in x_n+p^n\mathbb{Z}$. Show that L is a compact subring of P and that it contains the subring $\mathbb{Z}' \stackrel{\text{def}}{=} \{(x+p^n\mathbb{Z})_{n\in\mathbb{N}} \in L : x \in \mathbb{Z}\}$. Prove that $\mathbb{Z}' \cong \mathbb{Z}$ and that every open subset of L contains an element of $\mathbb{Z}'$, that is, $\mathbb{Z}'$ is dense in L and $L = \overline{\mathbb{Z}'}$. Conclude that $L = \mathbb{Z}_p$.