Introduction to Compact Groups

An Introductory Course
from the Fourth Semester up
Qualification Module
Wahlpflichtbereich und Haupstudium

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Chapter 1
Compact Groups: Basics

In the first chapter we introduce basic concepts, look at elementary examples and constructions, and provide the essential tools of the trade.

Definitions and Examples

Definition 1.1. A *compact groupG* is a compact Hausdorff space whose underlying set has a group structure such that the function

$$(*) \qquad (x, y) \mapsto xy^{-1}: G \times G \to G$$

is continuous. □

For the concept of compactness and Hausdorff separation of a space see the set of Lecture Notes "Introductction to Topology." Definition 1.1 is a special case of the definition of a *topological group* , which is a topologcial space and a group such that $(*)$ is continuous.

Our principal source of reference for compact groups is

[1] Hofmann, K. H., and S. A. Morris, The Structure of Compact Groups, de Gruyter Verlag, Berlin, 1998, xvii + 834pp.
Second Completely Revised, Corrected and Augmented Edition 2006, xviii + 860pp. To appear at de Gruyter Verlag, Berlin.

Exercise E1.1. (i) Let G be a group and a topological space Show that the following conditions are equivalent:

(1) The function $(x, y) \mapsto xy^{-1} : G \times G \to G$ is continuous.

(2) Multiplication and inversion are continuous functions.

Here we recall that multiplication is the function $(x, y) \mapsto xy : G \times G \to G$.

Examples 1.2. (i) All finite groups with the discrete topology are compact groups.

(ii) The multiplicative groups

$$\mathbb{S}^0 = \{r \in \mathbb{R} : |r| = 1\},$$
$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\},$$
$$\mathbb{S}^3 = \{q \in \mathbb{H} : |q| = 1\}$$

are compact groups on the unit spheres of the fields of real and complex numbers and of the skew field of quaternions.

The skew field of *quaternions* is isomorphict to a real 4-dimensional subalgebra of the algebra of 2×2 complex matrices

$$M(u,v) \stackrel{\text{def}}{=} \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \in M_2(\mathbb{C}), \quad u,v \in \mathbb{C}.$$

Using the basis $1, i, j, k$ of $\mathbb{H}$, we find the isomorphism via

$$r{\cdot}1 + x{\cdot}i + y{\cdot}j + z{\cdot}k \mapsto M(r + x{\cdot}i, y + z{\cdot}i).$$

Accordingly, $\mathbb{S}^3 \cong \mathrm{SU}(2) \stackrel{\text{def}}{=} \{M(u,v) : u\overline{u} + v\overline{v} = 1\}$.

If G is a compact group on an n-sphere $\mathbb{S}^n$, then $n = 0, 1, 3$. But his is a nontrivial result: see [1] 9.59(iv).

(iii) Each of the groups $\mathrm{O}(n)$ of $n \times n$-orthogonal matrices forms a closed and bounded subset in the vector space $M_n(\mathbb{R})$ of all $n \times n$ real matrices and therefore is a compact group, since matric multiplication, being polynomial in each coefficient, is continuous and inversion agrees with transposition and is, therefore, continuous. By a similar argument, each of the groups $\mathrm{U}(n)$ of $n \times n$-unitary matrices is a compact group; alternatively, one may identify $\mathrm{U}(n)$ with a closed subgroup of $\mathrm{O}(2n)$.

(iv) Every *closed subgroup* of a compact group is a compact group.

(v) If $\{G_j : j \in J\}$ is an arbitrary family of compact groups, then their cartesian product $G \stackrel{\text{def}}{=} \prod_{j\in J} G_j$ with componentwise multiplication and the product topology is a compact group. The compactness of G is a consequence of the *Tychonoff product theorem*; the continuity of $(x,y) \mapsto xy^{-1} : G \times G \to G$ follows from the natural homeomorphy of $G \times G$ and $\prod_{j\in J} G_j \times G_j$ and the continuity of

$$(x_j, y_j) \mapsto x_j y_j^{-1} : G_j \times G_j \to G_j$$

for all $j \in J$.

(va) Let $p \in \mathbb{N}$. Define $f : \mathbb{Z} \to P \stackrel{\text{def}}{=} \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ by $f(x) = (x + p^n\mathbb{Z})_{n\in\mathbb{N}}$. Then $\mathbb{Z}_p \stackrel{\text{def}}{=} \overline{f(\mathbb{Z})}$ is a compact abelian group. If $\mathrm{pr}_n : P \to \mathbb{Z}/p^n\mathbb{Z}$ denotes the projection given by $\mathrm{pr}_n((x_m)_{m\in\mathbb{N}} = x_n$, then $f_n \stackrel{\text{def}}{=} \mathrm{pr}_n|\mathbb{Z}_p$ is a morphism $\mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$. It is an exercise to show that it is surjective. Thus $\ker f_n$ is a subgroup I_n such that $\mathbb{Z}_p/I_n \cong \mathbb{Z}/p^n\mathbb{Z}$. All morphisms in sight also preserve multiplication, so $\mathbb{Z}_p$ is a ring and the I_n are ideals, and I_n turns out to agree exactly with $p^n\mathbb{Z}_p$. One calls $\mathbb{Z}_p$ the *ring of p-adic integers*, and its elements are called *p-adic integers.*

(vb) Every product $\prod_{j\in J} \mathrm{O}(n_j)$ or any product $\prod_{j\in J} \mathrm{U}(n_j)$ of a family of orthogonal, respectively, unitary groups is a compact group, as is any closed subgroup of these. □

It is remarkable and is a first goal of this course to prove that *every compact group is isomorphic as a topological group to a closed subgroup of one of the groups exhibited in* Example 1.2(vb).

Exercise E1.2. Verify the details of the propositions that a product of an arbitrary family $\{G_j : j \in J\}$ of compact groups is a compact group and that a closed subgroup of a compact group is a compact group.

Exercise E1.3. Prove the following assertion:

If G is a compact group and N a closed normal subgroup, then G/N is a compact group with respect to the quotient topology.

We must know that a subset $V \in G/N$ is open iff $q^{-1}V$ is open in G where $q\colon G \to G/N$ is the quotient map given by $q(g) = gN = Ng$.

Exercise E1.4. (p-adic integers). Let L denote the subset of all sequences $(x_n + p^n\mathbb{Z})_{n\in\mathbb{N}} \in P \overset{\text{def}}{=} \prod_{n\in N} \mathbb{Z}/p^n\mathbb{Z}$, $x_n \in \mathbb{Z}$ such that $x_{n+1} \in x_n + p^n\mathbb{Z}$. Show that L is a compact subring of P and that it contains the subring $\mathbb{Z}' \overset{\text{def}}{=} \{(x+p^n\mathbb{Z})_{n\in\mathbb{N}} \in L : x \in \mathbb{Z}\}$. Prove that $\mathbb{Z}' \cong \mathbb{Z}$ and that every open subset of L contains an element of $\mathbb{Z}'$, that is, $\mathbb{Z}'$ is dense in L and $L = \overline{\mathbb{Z}'}$. Conclude that $L = \mathbb{Z}_p$.

Applications to Abelian Groups

An important example arises out of the preceding proposition. For two sets X and Y the set of all functions $f\colon X \to Y$ will be denoted by Y^X.

Definition 1.3. If A is an abelian group (which we prefer to write additively) then the group

$$\mathrm{Hom}(A, \mathbb{T}) \subseteq \mathbb{T}^A$$

of all morphisms of abelian groups into the underlying abelian group of the circle group (no continuity involved!) given the induced group structure and topology of the product group $\mathbb{T}^A$ (that is, pointwise operations and the topology of pointwise convergence) is called *the character group of* A and is written $\widehat{A}$. Its elements are called *characters* of A. □

Proposition 1.4. *The character group $\widehat{A}$ of any abelian group A is a compact abelian group.*

Proof. By Exercise 1.2(v), the product $\mathbb{T}^A$ is a compact abelian group. For any pair $(a, b) \in A \times A$ the set $M(a, b) = \{\chi \in \mathbb{T}^A \mid \chi(a + b) = \chi(a) + \chi(b)\}$ is closed since $\chi \mapsto \chi(c)\colon \mathbb{T}^A \to \mathbb{T}$ is continuous by the definition of the product topology. But then $\widehat{A} = \bigcap_{(a,b)\in A\times A} M(a, b)$ is closed in $\mathbb{T}^A$ and therefore compact. □

Let us look at a few examples: In order to recognize $\widehat{\mathbb{Z}}$ we note that the function $f \mapsto f(1)\colon \mathrm{Hom}(\mathbb{Z}, \mathbb{T}) \to \mathbb{T}$ is an algebraic isomorphism and is continuous by the definition of the topology of pointwise convergence. Since $\widehat{\mathbb{Z}}$ is compact and $\mathbb{T}$

Hausdorff, it is an isomorphism of compact groups. Hence

$$\widehat{\mathbb{Z}} \cong \mathbb{T}. \tag{1}$$

If $\mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z}$ is the cyclic group of order n, then the function $z + n\mathbb{Z} \mapsto \frac{1}{n}z + \mathbb{Z}$ gives an injection $j\colon \mathbb{Z}(n) \to \mathbb{T}$ which induces an isomorphism $\mathrm{Hom}(\mathbb{Z}(n), j)\colon \mathrm{Hom}\big(\mathbb{Z}(n), \mathbb{Z}(n)\big) \to \mathrm{Hom}(\mathbb{Z}(n), \mathbb{T}) = \widehat{\mathbb{Z}(n)}$. Since the function $f \mapsto f(1 + n\mathbb{Z})\colon \mathrm{Hom}\big(\mathbb{Z}(n), \mathbb{Z}(n)\big) \to \mathbb{Z}(n)$ is an isomorphism, we have

$$\widehat{\mathbb{Z}(n)} \cong \mathbb{Z}(n). \tag{2}$$

If X is a set, and $\{A_x \mid x \in X\}$ a family of abelian groups, let us denote with $\bigoplus_{x\in X} A_x$ the direct sum of the A_x, that is, the subgroup of the cartesian product $\prod_{x\in X} A_x$ consisting of all elements $(a_x)_{x\in X}$ with $a_x = 0$ for all x outside some finite subset of X. A special case is $\mathbb{Z}^{(X)} = \bigoplus_{x\in X} A_x$ with $A_x = \mathbb{Z}$ for all $x \in X$. This is the *free abelian group on X*

Proposition 1.5. *The function*

$$\Phi\colon \prod_{x\in X} \mathrm{Hom}(A_x, \mathbb{T}) \to \mathrm{Hom}(\bigoplus_{x\in X} A_x, \mathbb{T})$$

which associates with a family $(f_x)_{x\in X}$ *of morphisms* $f_x\colon A_x \to \mathbb{T}$ *the morphism*

$$(a_x)_{x\in X} \mapsto \sum_{x\in X} f_x(a_x)\colon \bigoplus_{x\in X} A_x \to \mathbb{T}$$

is an isomorphism of compact groups. Notably,

$$(\bigoplus_{x\in X} A_x)\widehat{\ } \cong \prod_{x\in X} \widehat{A_x}. \tag{3}$$

In particular

$$\mathbb{Z}^{(X)\widehat{\ }} \cong \widehat{\mathbb{Z}}^X \cong \mathbb{T}^X. \tag{4}$$

Proof. Abbreviate $\bigoplus_{x\in X} A_x$ by A. We notice that Φ is well defined, since the $f_x(a_x)$ vanish with only finitely many exceptions for $(a_x)_{x\in X}$. Clearly Φ is a morphism of abelian groups. Further $(f_x)_{x\in X} \in \ker\Phi$ if and only if $\sum_{x\in X} f_x(a_x) = 0$ for all $(a_x)_{x\in X} \in A$. Choosing for a given $y \in X$ the family (a_x) so that $a_x = 0$ for $x \neq y$ and $a_y = a$ we obtain $f_y(a) = 0$ for any $a \in A_y$. Thus $f_y = 0$ for all $y \in X$. Hence Φ is injective. If $f\colon A \to \mathbb{T}$ is a morphism, define $f_y\colon A_y \to \mathbb{T}$ by $f_y = f \circ \mathrm{copr}_y$ where $\mathrm{copr}_y\colon A_y \to A$ is the natural inclusion. Then $\Phi\big((f_x)_{x\in X}\big) = f$ follows readily. Thus Φ is surjective, too, and thus is an isomorphism of abelian groups. Next we show that Φ is continuous. By the definition of the topology on $\mathrm{Hom}(A, \mathbb{T}) \subseteq \mathbb{T}^A$, it suffices to show that for each $(a_x)_{x\in X} \in A$, the function $(f_x)_{x\in X} \mapsto \Phi\big((f_x)_{x\in X}\big)\big((a_x)_{x\in X}\big) = \sum_{x\in X} f_x(a_x) : \mathrm{Hom}(A_x, \mathbb{T})^X \to \mathbb{T}$ is continuous. Since only finitely many a_x are nonzero, this is the case if $(f_x)_{x\in X} \mapsto f_y(a_y)$ is continuous for each fixed y, and this holds if $f_y \mapsto f_y(a_y)\colon \mathrm{Hom}(A_y, \mathbb{T}) \to \mathbb{T}$ is continuous. However, by definition of the topology of pointwise convergence, this is indeed the case. Since the domain of Φ is compact by the theorem of Tychonoff and the range is Hausdorff, this suffices for Φ to be a homeomorphism.

The last assertion of the proposition is a special case. This remark concludes the proof of the proposition. $\square$

The compact abelian groups $\mathbb{T}^X$ are called *torus groups*. The finite dimensional tori $\mathbb{T}^n$ are special cases.

We cite from the basic theory of abelian groups the fact that a finitely generated abelian group is a direct sum of cyclic groups. Thus (1), (2) and (3) imply the following remark:

Remark 1.6. If E is a finite abelian group, then $\widehat{E}$ is isomorphic to E (although not necessarily in any natural fashion!). If F is a finitely generated abelian group of rank n, that is, $F = E \oplus \mathbb{Z}^n$ with a finite abelian group E, then $\widehat{F} \cong \widehat{E} \times \mathbb{T}^n$.$\square$

In particular, the character groups of finitely generated abelian groups are compact manifolds. (We shall not make any use of this fact right now.

There are examples of compact abelian groups whose topological nature is quite different.

Example 1.7. Let $\{G_j \mid j \in J\}$ be any family of finite discrete nonsingleton groups. Then $G = \prod_{j\in J} G_j$ is a compact group. All connected components are singleton, and G is discrete if and only if J is finite. $\square$

A topological space in which all connected components are singletons is called *totally disconnected*. Arbitrary products of totally disconnected spaces are totally disconnected, and all discrete spaces are totally disconnected. The standard Cantor middle third set C is a compact metric totally disconnected space. In fact it may be realized as the set of all real numbers r in the closed unit interval, whose expansion $r = \sum_{n=1}^{\infty} a_n 3^{-n}$ with respect to base 3 has all coefficients a_n in the set $\{0, 2\}$. Then the map $f\colon \{-1, 1\}^{\mathbb{N}} \to C$ given by $f\big((r_n)_{n\in\mathbb{N}}\big) = \sum_{n=1}^{\infty}(r_n + 1)3^{-n}$ is a homeomorphism. The set $\mathbb{S}^0 = \{-1, 1\}$ is a finite group, and thus, by Exercise 1.2(v), the domain of f is a compact group.

Hence the Cantor set can be given the structure of a compact abelian group. In this group, every element has order 2, so that in fact, algebraically, it is a vector space over the field GF(2) of 2 elements, and by (2) and (3) above, it is the character group of $\mathbb{Z}(2)^{(\mathbb{N})}$.

One can show that all compact metric totally disconnected spaces without isolated points are homeomorphic to C. In particular, all metric compact totally disconnected infinite groups are homeomorphic to C.

Definition 1.8. Let X and Y be sets and $F \subseteq Y^X$ a set of functions from X to Y. We say that F *separates the points of* X if for any two different points x_1 and x_2 in X, there is an $f \in F$ such that $f(x_1) \neq f(x_2)$. $\square$

If G and H are groups, then a set F of homomorphisms from G to H is easily seen to separate the points of G if and only if for each $g \neq 1$ in G there is an $f \in F$ with $f(g) \neq 1$.

For any abelian group A there is always a large supply of characters. In fact there are enough of them to separate the points. In order to see this we resort to some basic facts on abelian groups:

An abelian group A is called *divisible* if for each $a \in A$ and each natural number n there is an $x \in A$ such that $n{\cdot}x = a$. Examples of divisible groups are $\mathbb{Q}$ and $\mathbb{R}$. Every homomorphic image of a divisible group is divisible, whence $\mathbb{T}$ is divisible. The crucial property of divisible groups is that for every subgroup S of an abelian group A and a homomorphism $f\colon S \to I$ into a divisible group there is a homomorphic extension $F\colon A \to I$ of f, as we shall argue now.

Definition 1.9. An abelian group I is called *injective* if for every injective morphism $i\colon A \to B$ and every morphism $j\colon A \to I$ there is a morphism $f\colon B \to I$ with $j = f \circ i$.

$$\begin{array}{ccc} I & \xrightarrow{\mathrm{id}_I} & I \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle f} \\ A & \xrightarrow[i]{} & B \end{array}$$

□

One may rephrase injectivity in the following convenient fashion: *An abelian group I is injective if and only if any homomorphism $j\colon A \to I$ of a subgroup A of a group B extends to a homomorphism $f\colon B \to I$ on the whole group.*

Proposition 1.10. *For an abelian group G the following conditions are equivalent:*
(1) *G is divisible.*
(2) *G is injective.*

Proof. (1)⇒(2). (AC) Assume that A is a subgroup of B and that a homomorphism $j\colon A \to G$ is given. We must extend j to a morphism $f\colon B \to G$. We consider the set of all morphisms $\varphi\colon C \to G$ with $A \subseteq C \subseteq B$ and $\varphi|A = j$. This set is partially ordered by inclusion of domains and extension of mappings (i.e. $\varphi \leq \varphi'$ if $C \subseteq C'$ and $\varphi'|C = \varphi$). One verifies quickly that this set is inductive, hence by Zorn's Lemma contains a maximal element $\mu\colon M \to G$. We must show $M = B$. Let $b \in B$. Then $M \cap \mathbb{Z}{\cdot}b$ is a cyclic group, say $n\mathbb{Z}{\cdot}b$. Since G is divisible, there is an element $d \in G$ such that $n{\cdot}d = \mu(n{\cdot}b)$. Assume now that $m_1 + z_1{\cdot}b = m_2 + z_2{\cdot}b$. Then $m_2 - m_1 = (z_1 - z_2){\cdot}b \in M \cap \mathbb{Z}{\cdot}b = n\mathbb{Z}{\cdot}b$. In particular, the kernel of $m \mapsto m{\cdot}b\colon \mathbb{Z} \to G$ is contained in $n\mathbb{Z}$. Thus there is a $z \in \mathbb{Z}$ with $(z_1 - z_2 - zn){\cdot}b = 0$, and thus $z_1 - z_2 - zn = z'n$ for some $z' \in \mathbb{Z}$. Hence $\mu(m_2 - m_1) = \mu\big((z_1 - z_2){\cdot}b\big) = \mu\big((z + z')n{\cdot}b\big) = (z + z')n{\cdot}d = (z_1 - z_2){\cdot}d$ and thus $\mu(m_1) + z_1{\cdot}d = \mu(m_2) + z_1{\cdot}d$.

Therefore we define unambiguously a function $\mu'\colon M' \to I$, $M' = M + \mathbb{Z}{\cdot}b$ by $\mu'(m + z{\cdot}b) = \mu(m) + z{\cdot}d$ satisfying $\mu'|M = \mu$. It is easy to verify that μ is a morphism. Hence $\mu \le \mu'$. By the maximality of μ we have $\mu' = \mu$ and thus $M' = M$. Hence $b \in M$. Thus $M = B$.

(2)⇒(1). There is a set X and a surjective homomorphism $p\colon \mathbb{Z}^{(X)} \to G$. We may assume that $G = \mathbb{Z}^{(X)}/K$ with $K = \ker p$ and that $\operatorname{card} X = \operatorname{card} G$. Now $K \subseteq \mathbb{Z}^{(X)} \subseteq \mathbb{Q}^{(X)}$. Then $G = \mathbb{Z}^{(X)}/K \subseteq \mathbb{Q}^{(X)}/K$, and $D = \mathbb{Q}^{(X)}/K$ is divisible. Hence there is a divisible group D with $G \subseteq D$. Since G is injective there is a morphism $f\colon D \to G$ such that $f|G = \mathrm{id}_G$. Hence G is a homomorphic image of a divisible group and is, therefore, divisible. □

$$\begin{array}{ccc}
\mathbb{T} & \xrightarrow{\quad = \quad} & \mathbb{T} \\
{\scriptstyle f}\big\uparrow & & \big\uparrow{\scriptstyle \chi} \\
S & \xrightarrow[\text{incl}]{\qquad} & A
\end{array}$$

Lemma 1.11. *The characters of an abelian group A separate the points.*

Proof. Assume that $0 \ne a \in A$. We must find a morphism $\chi\colon A \to \mathbb{T}$ such that $\chi(a) \ne 0$. Let S be the cyclic subgroup $\mathbb{Z}{\cdot}a$ of A generated by a. If S is infinite, then S is free and for any nonzero element t in $\mathbb{T}$ (e.g. $t = \frac{1}{2} + \mathbb{Z}$) there is an $f\colon S \to \mathbb{T}$ with $f(a) = t \ne 0$. If S has order n, then S is isomorphic to $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{T}$, and thus there is an injection $f\colon S \to \mathbb{T}$. If we let $\chi\colon A \to \mathbb{T}$ be an extension of f which exists by the divisibility of $\mathbb{T}$, then $\chi(a) = f(a) \ne 0$. □

Definitions 1.12. For a compact abelian group G a morphism of compact groups $\chi\colon G \to \mathbb{T}$ is called a *character of* G. The set $\operatorname{Hom}(G, \mathbb{T})$ of all characters is an abelian group under pointwise addition, called the *character group of* G and written $\widehat{G}$. Notice that we do not consider any topology on $\widehat{G}$. □

Now we can of course iterate the formation of character groups and oscillate between abelian groups and compact abelian groups. This deserves some inspection; the formalism is quite general and is familiar from the duality of finite-dimensional vector spaces.

Lemma 1.13. (i) *If A is an abelian group, then the function*

$$\eta_A\colon A \to \widehat{\widehat{A}}, \quad \eta_A(a)(\chi) = \chi(a)$$

is an injective morphism of abelian groups.

(ii) *If G is a compact abelian group, then the function*

$$\eta_G\colon G \to \widehat{\widehat{G}}, \quad \eta_G(g)(\chi) = \chi(g)$$

is a morphism of compact abelian groups.

Proof. (i) The morphism property follows readily from the definition of pointwise addition in $\widehat{A}$. An element g is in the kernel of η_A if $\chi(g) = 0$ for all characters. Since these separate the points by Lemma 1.11, we conclude $g = 0$. Hence η_A is injective.

(ii) Again it is immediate that η_G is a morphism of abelian groups. We must observe its continuity: The function $g \mapsto \chi(g): G \to \mathbb{T}$ is continuous for every character χ by the continuity of characters. Hence the function $g \mapsto \big(\chi(g)\big)_{\chi\in\widehat{G}}: G \to \mathbb{T}^{\widehat{G}}$ is continuous by the definition of the product topology. Since $\widehat{\widehat{G}} = \mathrm{Hom}(\widehat{G}, \mathbb{T}) \subseteq \mathbb{T}^{\widehat{G}}$ inherits its structure from the product, η_G is continuous. □

Exercise E1.5. For a discrete group A and a compact group G the members of $\widehat{\widehat{A}}$ and $\widehat{\widehat{G}}G$ separate the points of $\widehat{A}$, respectively, $\widehat{G}$. Equivalently, the evaluation morphisms $\eta_{\widehat{A}}: \widehat{A} \to \widehat{\widehat{\widehat{A}}}$ and $\eta_{\widehat{G}}: \widehat{G} \to \widehat{\widehat{\widehat{G}}}$ are injective.
[Hint. Observe that already $\eta_A(A)$ separates the points of $\widehat{A}$.] □

Let us look at our basic examples: If A is a finite abelian group, then $\widehat{A}$ is isomorphic to A by Remark 1.6. Hence $\widehat{\widehat{A}}$ is isomorphic to A and $\eta_A: A \to \widehat{\widehat{A}}$ is injective by Lemma 1.23. Hence η_A is an isomorphism.

Every character $\chi: \mathbb{T} \to \mathbb{T}$ yields a morphism of topological groups $f: \mathbb{R} \to \mathbb{T}$ via $f(r) = \chi(r + \mathbb{Z})$. Let $q: \mathbb{R} \to \mathbb{T}$ be the quotient homomorphism. We set $V =]-\frac{1}{3}, \frac{1}{3}[\subseteq \mathbb{R}$ and $W = q(V)$. Then $q|V: V \to W$ is a homeomorphism. Assume that x and y are elements of W such that $x + y \in W$, too. Then $r = (q|V)^{-1}(x)$, $s = (q|V)^{-1}(y)$ and $t = (q|V)^{-1}(x + y)$ are elements of V such that $q(r + s - t) = q(t) + q(s) - q(t) = x + y - (x + y) = 0$ in $\mathbb{T}$. Hence $r + s - t \in \ker q = \mathbb{Z}$. But also $|r + s - t| \le |r| + |s| + |t| < 3\cdot\frac{1}{3} = 1$. Hence $r + s - t = 0$ and $(q|V)^{-1}(x) + (q|V)^{-1}(y) = r + s = t = (q|V)^{-1}(x + y)$. Now let U denote an open interval around 0 in $\mathbb{R}$ such that $f(U) \subseteq W$. If we set $\varphi = (q|V)^{-1} \circ f|U: U \to \mathbb{R}$ then for all $x, y, x + y \in U$ we have $\varphi(x + y) = \varphi(x) + \varphi(y)$. Under these circumstances φ extends uniquely to a morphism $F: \mathbb{R} \to \mathbb{R}$ of abelian groups (see Exercise E1.6 below). Now $q\circ F = f = \chi\circ q$ since F extends φ and U generates the abelian group $\mathbb{R}$. Then $\mathbb{Z} = \ker q \subseteq \ker(q\circ F)$, that is, $F(\mathbb{Z}) \subseteq \ker q = \mathbb{Z}$. Thus if we set $n = F(1)$, then $n \in \mathbb{Z}$. Since φ is continuous, then F is continuous at 0. As a morphism, F is continuous everywhere (see Exercise E1.7 below). As a morphism of abelian groups, F is quickly seen to be $\mathbb{Q}$-linear, and from its continuity it follows that it is $\mathbb{R}$-linear. Thus $F(t) = nt$ and $\chi(t + \mathbb{Z}) = nt + \mathbb{Z}$ follows. Thus the characters of $\mathbb{T}$ are exactly the endomorphisms $\mu_n = (g \mapsto ng)$ and $n \mapsto \mu_n: \mathbb{Z} \to \widehat{\mathbb{T}}$ is an isomorphism.

Exercise E1.6. Prove the following proposition:

The Extension Lemma. *Let U be an arbitrary interval in $\mathbb{R}$ containing* 0 *and assume that $\varphi: U \to G$ is a function into a group such that x, y, $x + y \in U$ implies*

$\varphi(x+y)=\varphi(x)\varphi(y)$. *Then there is a morphism* $F:\mathbb{R}\to G$ *of groups extending* φ. *If* U *contains more than one point then* F *is unique.*

[Hint. Step 1. Show by induction that for any $u\in U$ such that $u, 2u, \dots nu \in U$ we have $\varphi(ku)=\varphi(u)^k$, $k=1,2,\dots,n$. Step 2. If $u, mu\in U$ for a natural number m, then for any $n\in N$, $\varphi(u)^{mn}=\varphi(mu)^n$. Step 3. For $r\in\mathbb{R}$ and two integers m and n with r/m, $r/n\in U$ show $\varphi(r/m)^m=\varphi(r/n)^n$. Indeed assume first that $m,n\geq 1$; let $u=r/mn$, then $\varphi(u)^{mn}=\varphi(r/m)^m$ by Step 2; likewise $\varphi(u)^{mn}=\varphi(r/n)^n$. Reduce the case $m,n<0$ to this case. Step 4. Define $F(r)$ to be the unique element of G for which there is an integer m such that $r/m\in U$ and $F(r)=\varphi(r/m)^m$ and show that F is a morphism.] □

Exercise E1.7. A homomorphism between topological groups is continuous iff it is continuous at the identity.

Now that we have determined $\widehat{\mathbb{T}}$ we look at $\eta_{\mathbb{Z}}$. We have $\eta_{\mathbb{Z}}(n)(\chi)=\chi(n)=n\chi(1)=\mu_n\big(\chi(1)\big)$ for any character χ of $\mathbb{Z}$. Since $\chi\mapsto\chi(1):\widehat{\mathbb{Z}}\to\mathbb{T}$ is an isomorphism by (1) above and since every character of $\mathbb{T}$ is of the form μ_n, this shows that $\eta_{\mathbb{Z}}$ is an isomorphism.

Now we show that $\eta_{\mathbb{T}}$ is an isomorphism, too. We recall that $\widehat{\mathbb{T}}$ is infinite cyclic and is generated by the identity map $\varepsilon:\mathbb{T}\to\mathbb{T}$. In other words, any character $\chi:\mathbb{T}\to\mathbb{T}$ of $\widehat{\mathbb{T}}$ is of the form $\chi=n{\cdot}\varepsilon=\mu_n(\varepsilon)$. Now we observe $\eta_{\mathbb{T}}(g)(n{\cdot}\varepsilon)=n{\cdot}\varepsilon(g)=n{\cdot}g$ for all $n\in\mathbb{Z}$. Taking $n=1$ we note that the kernel of $\eta_{\mathbb{T}}$ is singleton and thus $\eta_{\mathbb{T}}$ is injective. In order to show surjectivity we assume that $\Omega:\widehat{\mathbb{T}}\to\mathbb{T}$ is a character of $\widehat{\mathbb{T}}\cong\mathbb{Z}$. Then $\Omega(\varepsilon)$ is an element $g\in\mathbb{T}$ and we see $\eta_{\mathbb{T}}(g)(n{\cdot}\varepsilon)=n{\cdot}g=n{\cdot}\Omega(\varepsilon)=\Omega(n{\cdot}\varepsilon)$. Thus $\eta_{\mathbb{T}}(g)=\Omega$. This shows that $\eta_{\mathbb{T}}$ is surjective, too. Thus $\eta_{\mathbb{T}}$ is an isomorphism.

Remark 1.24. (i) Assume that A and B are abelian groups such that η_A and η_B are isomorphisms. Then $\eta_{A\oplus B}$ is an isomorphism.

(ii) If G and H are compact abelian groups and η_G and η_H are isomorphisms, then $\eta_{G\times H}$ is an isomorphism.

(iii) For any finitely generated abelian group A, the map $\eta_A: A\to\widehat{\widehat{A}}$ is an isomorphism.

(iv) If $G\cong\mathbb{T}^n\times E$ for a natural number n and a finite abelian group E then $\eta_G: G\to\widehat{\widehat{G}}$ is an isomorphism.

(v) Every torus group $\mathbb{T}^n$ contains an element such that the subgroup generated by it is dense.

Proof. Exercise E1.8. □

Exercise E1.8. Prove Remarks 1.24(i)–(v).

[Hint. For (iii) and (iv) recall that the evaluation morphism is an isomorphism for cyclic groups, for $\mathbb{Z}$ and for $\mathbb{T}$. Also recall the Fundamental Theorem for Finitely Generated Abelian Groups (cf.[1], Appendix A1.11).

For a proof of (v) set $T = \mathbb{T}^n$. Every quotient group of T modulo some closed subgroup is a compact group which is a quotient group of $\mathbb{R}^n$ and is, therefore, a torus by [1] Appendix 1, Theorem 1.12(ii). Now let $x \in T$; then $T/\overline{\langle \mathbb{Z}\cdot x\rangle}$ is a torus, and by (iv) above, its characters separate the points. Thus, $\mathbb{Z}\cdot x$ is dense in T iff all characters of T vanish on $\mathbb{Z}\cdot x$, i.e. on x, iff

$$(\forall \chi \in \widehat{T}) \quad [\chi(\mathbb{Z}\cdot x) = \{0\}] \Rightarrow [\chi = 0]$$

iff the map $\chi \mapsto \big(n \mapsto \chi(n\cdot x)\big) \colon \widehat{T} \to \widehat{\mathbb{Z}}$ is injective iff the map $\chi \mapsto \chi(x) \colon \widehat{T} \to \mathbb{T}$ is injective (via the natural isomorphism $\widehat{Z} \cong \mathbb{T}$). But since $\eta \colon T \to \widehat{\widehat{T}}$ is an isomorphism by (iv) above, any homomorphism $\alpha \colon \widehat{T} \to \mathbb{T}$ is an evaluation, i.e. there is a unique $x \in T$ such that for any $\chi \in \widehat{T}$ we have $\chi(x) = \alpha(\chi)$. Thus, in conclusion, we have an element $x \in T$ such that $\mathbb{Z}\cdot x$ is dense in T iff we have an injective morphism $\mathbb{Z}^n \cong \widehat{T} \to \mathbb{T}$. But the injective morphisms $\mathbb{Z}^n \to \mathbb{R}/\mathbb{Z}$ abound (cf. Appendix A1.43).

Provide a direct proof of Remark 1.24(v) as follows: *Let $r_j \in \mathbb{R}$, $j = 1, \ldots, n$, be n real numbers such that $\{1, r_1, \ldots, r_n\}$ is a set of linearly independent elements of the $\mathbb{Q}$-vector space $\mathbb{R}$. Then the element $x + \mathbb{Z} \in \mathbb{R}^n/\mathbb{Z}^n$, $x = (r_1, \ldots, r_n)$ has the property that $\mathbb{Z}\cdot(x + \mathbb{Z})$ is dense.*] □

Exercise E1.9. Prove the following universal property of the evaluation morphism:

Lemma A. *For every morphism $f \colon A \to \widehat{G}$ from an abelian group A to the character group of a compact abelian group G there is a unique morphism $f' \colon G \to \widehat{A}$ such that $f = \widehat{f'} \circ \eta_A$.*

Lemma B. *For every morphism $f \colon G \to \widehat{A}$ from a compact abelian group G to the character group of an abelian group A there is a unique morphism $f' \colon A \to \widehat{G}$ such that $f = \widehat{f'} \circ \eta_G$.*

[Hint. Lemma B is proved in the same way as Lemma A. In the case of Lemma A, define f' by $f'(g)(a) = f(a)(g)$ for $a \in A$ und $g \in G$. Verify the asserted property and uniqueness by using the definitions of f' and η_A and $\widehat{f'}$.]

Apply this to show

Lemma C. *For each abelian group A we have $\widehat{\eta_A} \circ \eta_{\widehat{A}} = \mathrm{id}_A$ and for each compact abelian group wie have $\widehat{\eta_G} \circ \eta_{\widehat{G}} = \mathrm{id}_G$.*

Exercise E1.10. Prove the following observation on morephisms of abelian groups:

Lemma. *If $f \colon A \to B$ is a morephism of abelian groups and $\widehat{f} \colon \widehat{B} \to \widehat{A}$ its adjoint, then a character β of B is in $\ker \widehat{f}$ iff it annihilates $f(A)$, that is, $\beta(f(A) = \{0\}$.*

Projective Limits

Definition 1.25. Let J be a directed set, that is, a set with a reflexive, transitive and antisymmetric relation $\leq$ such that every finite nonempty subset has an upper bound. A *projective system of topological groups over* J is a family of morphisms $\{f_{jk}: G_k \to G_j \mid (j,k) \in J \times J,\, j \leq k\}$, where G_j, $j \in J$ are topological groups, satisfying the following conditions:
(i) $f_{jj} = \mathrm{id}_{G_j}$ for all $j \in J$
(ii) $f_{jk} \circ f_{kl} = f_{jl}$ for all $j, k, l \in J$ with $j \leq k \leq l$. □

Lemma 1.26. (i) *For a projective system of topological groups, define the topological group* P *by* $P = \prod_{j \in J} G_j$. *Set*

$$G = \{(g_j)_{j \in J} \in P \mid (\forall j, k \in J)\, j \leq k \Rightarrow f_{jk}(g_k) = g_j\}.$$

Then G *is a closed subgroup of* P. *If* $\mathrm{incl}: G \to P$ *denotes the inclusion and* $\mathrm{pr}_j: P \to G_j$ *the projection, then the function* $f_j = \mathrm{pr}_j \circ \mathrm{incl}: G \to G_j$ *is a morphism of topological groups for all* $j \in J$, *and for* $j \leq k$ *in* J *the relation* $f_j = f_{jk} \circ f_k$ *is satisfied.*

(ii) *If all groups* G_j *in the projective system are compact, then* P *and* G *are compact groups.*

Proof. (i) Assume that $j \leq k$ in J. Define $G_{jk} = \{(g_l)_{l \in J} \in P \mid f_{jk}(g_k) = g_j\}$. Since f_{jk} is a morphism of groups, this set is a subgroup of P, and since f_{jk} is continuous, it is a closed subgroup. But $G = \bigcap_{(j,k) \in J \times J,\, j \leq k} G_{jk}$. Hence G is a closed subgroup. The remainder is straightforward.

(ii) If all G_j are compact, then P is compact by Tychonoff's Theorem, and thus G as a closed subgroup of P is compact, too. □

Definitions 1.27. If $\mathcal{P} = \{f_{jk}: G_k \to G_j \mid (j,k) \in J \times J,\, j \leq k\}$ is a projective system of topological groups, then the group G of Lemma 1.26 is called its *projective limit* and is written $G = \lim \mathcal{P}$. As a rule it suffices to remind oneself of the entire projective system by recording the family of groups G_j involved in it; therefore the notation $G = \lim_{j \in J} G_j$ is also customary. The morphisms $f_j: G \to G_j$ are called *limit maps* and the morphisms $f_{jk}: G_k \to G_j$ are called *bonding maps.* □

Example 1.28. Assume that we have a sequence $\varphi_n: G_{n+1} \to G_n$, $n \in \mathbb{N}$ of morphisms of compact groups:

$$G_1 \overset{\varphi_1}{\leftarrow} G_2 \overset{\varphi_2}{\leftarrow} G_3 \overset{\varphi_3}{\leftarrow} G_4 \overset{\varphi_4}{\leftarrow} \cdots$$

Then we obtain a projective system of compact groups by defining, for natural numbers $j \leq k$, the morphisms

$$f_{jk} = \varphi_j \circ \varphi_{j+1} \circ \cdots \circ \varphi_{k-1}: G_k \to G_j.$$

Then $G = \lim_{n \in \mathbb{N}} G_n$ is simply given by $\{(g_n)_{n \in \mathbb{N}} \mid (\forall n \in \mathbb{N})\, \varphi_n(g_{n+1}) = g_n\}$.

(i) Choose a natural number p and set $G_n = \mathbb{Z}(p^n) = \mathbb{Z}/p^n\mathbb{Z}$. Define $\varphi_n: \mathbb{Z}(p^{n+1}) \to \mathbb{Z}(p^n)$ by $\varphi_n(z + p^{n+1}\mathbb{Z}) = z + p^n\mathbb{Z}$:

$$\mathbb{Z}(p) \overset{\varphi_1}{\leftarrow} \mathbb{Z}(p^2) \overset{\varphi_2}{\leftarrow} \mathbb{Z}(p^3) \overset{\varphi_3}{\leftarrow} \mathbb{Z}(p^4) \overset{\varphi_4}{\leftarrow} \cdots$$

The projective limit of this system is none other than our *group* $\mathbb{Z}_p$ *of p-adic integers.*

(ii) Set $G_n = \mathbb{T}$ for all $n \in \mathbb{N}$ and define $\varphi_n(g) = p{\cdot}g$ for all $n \in \mathbb{N}$ and $g \in \mathbb{T}$. (It is customary, however, to write p in place of φ_p):

$$\mathbb{T} \xleftarrow{p} \mathbb{T} \xleftarrow{p} \mathbb{T} \xleftarrow{p} \mathbb{T} \xleftarrow{p} \cdots$$

The projective limit of this system is called the *p-adic solenoid* $\mathbb{T}_p$. □

Proposition 1.29. *Assume that* $G = \lim_{j\in J} G_j$ *for a projective system* $f_{jk}: G_k \to G_j$ *of compact groups,* $j \le k$ *in* J*, and denote with* $f_j: G \to G_j$ *the limit maps. Then the following statements are equivalent:*

(1) *All bonding maps* f_{jk} *are surjective.*

(2) *All limit maps* f_j *are surjective.*

Proof. (1)⇒(2) Fix $i \in J$. Let $h \in G_i$; we must find an element $g = (g_j)_{j\in J} \in G$ with $g_i = f_i(g) = h$. For all $k \in J$ with $i \le k$ we define $C_k \subseteq \prod_{j\in J} G_j$ by

$$\{(x_j)_{j\in J} \mid (\forall j \le k)\, x_j = f_{jk}(x_k) \text{ and } x_i = h\}.$$

Since f_{ik} is surjective, $C_k \ne \emptyset$. If $i \le k \le k'$ then we claim $C_{k'} \subseteq C_k$. Indeed $(x_j)_{j\in J} \in C_{k'}$ implies $f_{jk}(x_k) = f_{jk}f_{kk'}(x_{k'}) = f_{jk'}(x_{k'}) = x_j$ and $x_i = h$. Thus $(x_j)_{j\in J} \in C_k$ and the claim is established. Now $\{C_k \mid k \in J, i \le k\}$ is a filter basis of compact sets in $\prod_{j\in J} G_j$ and thus has nonempty intersection. Assume that $g = (g_m)_{m\in J}$ is in this intersection. Then, firstly, $g_i = h$. Secondly, let $j \le k$. Since J is directed, there is a k' with $i, k \le k'$. Then $(g_m)_{m\in J} \in C_{k'}$. Hence $g_j = f_{jk'}(g_{k'}) = f_{jk}f_{kk'}(g_{k'}) = f_{jk}(g_k)$ by the definition of $C_{k'}$. Hence $g \in \lim_{j\in J} G_j$. Thus g is one of the elements we looked for.

(2)⇒(1) Let $j \le k$. Then $f_j = f_{jk}f_k$. Thus the surjectivity of f_j implies that of f_{jk}. □

Definition 1.30. A projective system of topological groups in which all bonding maps and all limit maps are surjective is called a *strict projective system* and its limit is called a *strict projective limit.* □

Proposition 1.31. (i) *Let* $G = \lim_{j\in J} G_j$ *be a projective limit of compact groups. Let* $\mathcal{U}_j$ *denote the filter of identity neighborhoods of* G_j*,* $\mathcal{U}$ *the filter of identity neighborhoods of* G*, and* $\mathcal{N}$ *the set* $\{\ker f_j \mid j \in J\}$*. Then*

(a) $\mathcal{U}$ *has a basis of identity neighborhoods* $\{f_k^{-1}(U) \mid k \in J, U \in \mathcal{U}_k\}$.

(b) $\mathcal{N}$ *is a filter basis of compact normal subgroups converging to* $\mathbf{1}$. (*That is, given a neighborhood* U *of* $\mathbf{1}$*, there is an* $N \in \mathcal{N}$ *such that* $N \subseteq U$.)

(ii) *Conversely, assume that* G *is a compact group with a filter basis* $\mathcal{N}$ *of compact normal subgroups with* $\bigcap \mathcal{N} = \{\mathbf{1}\}$*. For* $M \subseteq N$ *in* $\mathcal{N}$ *let* $f_{NM}: G/M \to G/N$ *denote the natural morphism given by* $f_{NM}(gM) = gN$*. Then the* f_{NM} *constitute a strict projective system whose limit is isomorphic to* G *under the map*

$g \mapsto (gN)_{N\in\mathcal{N}}: G \to \lim_{N\in\mathcal{N}} G/N$. With this isomorphism, the limit maps are equivalent to the quotient maps $G \to G/N$.

Proof. (i)(a) Let $V \in \mathcal{U}$. Then by the definition of the projective limit there is an identity neighborhood of $\prod_{j\in J} G_j$ of the form $W = \prod_{j\in J} W_j$ with $W_j \in \mathcal{U}_j$ for which there is a finite subset F of J such that $j \in J \setminus F$ implies $W_j = G_j$ such that $W \cap \lim_{j\in J} G_j \subseteq V$. Since J is directed, there is an upper bound $k \in J$ of F. There is a $U \in \mathcal{U}_k$ such that $f_{jk}(U) \subseteq W_j$ for all $j \in J$. Then $f_k^{-1}(U) \subseteq W \cap \lim_{j\in J} G_j \subseteq V$.

(i)(b) Evidently, each $\ker f_j$ is a compact normal subgroup. Since $i, j \le k$ implies $\ker f_k \subseteq \ker f_i \cap \ker f_j$ and J is directed, $\mathcal{N}$ is a filter basis. For each $j \in J$ we have $\ker f_j = f_j^{-1}(1) \subseteq f_j^{-1}(U)$ for any $U \in \mathcal{U}_j$. Since $f_j^{-1}(U)$ is a basic neighborhood of the identity by (a), we are done.

(ii) It is readily verified that the family of all morphisms $f_{NM}: G/M \to G/N$ for $M \subseteq N$ in $\mathcal{N}$ constitutes a strict projective system of compact groups. An element $(g_N N)_{N\in\mathcal{N}} \in \prod_{N\in\mathcal{N}} G/N$ with $g_N \in G$ is in its limit L if and only if for each pair $M \supseteq N$ in $\mathcal{N}$ we have $f_{MN}(g_N N) = g_M M$, that is, $g_M^{-1} g_N \in M$. Thus for each $g \in G$ certainly $(gN)_{N\in\mathcal{N}} \in L$. The kernel of the morphism $\varphi = (g \mapsto (gN)_{N\in\mathcal{N}}): G \to L$ is $\bigcap \mathcal{N} = \{\mathbf{1}\}$. Hence φ is injective. Assume $\gamma = (g_N N)_{N\in\mathcal{N}} \in L$. Then $\{g_N N \mid N \in \mathcal{N}\}$ is a filter basis of compact sets in G, for if $M \supseteq N$ then $g_M^{-1} g_N \in M$, and thus $g_N \in g_M M \cap g_N N$. Hence its intersection contains an element g and then $g \in g_N N$ is equivalent to $gN = g_N N$. Thus $\varphi(g) = \gamma$. We have shown that φ is also surjective and thus is an isomorphism of compact groups (see Remark 1.8). If $q_N: G \to G/N$ is the quotient map, and if $f_N: L \to G/N$ is the limit map defined by $f_N\big((g_N N)_{N\in\mathcal{N}}\big) = g_N N$, then clearly $q_N = f_N \circ \varphi$. The proof of the proposition is now complete. □

The significance of the preceding proposition is that we can think of a strict projective limit G as a compact group which is approximated by factor groups G/N modulo smaller and smaller normal subgroups N. This is not a bad image. The group G is decomposed into cosets gN whose size can be made as small as we wish using the normal subgroups in the filter basis $\mathcal{N}$.

More Duality Theory

Let A be an arbitrary abelian group. Let $\mathcal{F}$ denote the family of all finitely generated subgroups. This family is directed, for if $F, E \in \mathcal{F}$ then $F + E \in \mathcal{F}$. Also, $A = \bigcup_{F\in\mathcal{F}} F$. If $E, F \in \mathcal{F}$ and $E \subseteq F$ then the inclusion $E \to F$ induces a morphism $f_{EF}: \widehat{F} \to \widehat{E}$ via $f_{EF}(\chi) = \chi|E$ for $\chi: F \to \mathbb{T}$. The family $\{f_{EF}: \widehat{F} \to \widehat{E} \mid E, F \in \mathcal{F}, E \subseteq F\}$ is a projective system of compact abelian groups. By the divisibility of $\mathbb{T}$, each character on $E \subseteq F$ extends to one on F and so this system is strict. The inclusion $F \to A$ induces a morphism $f_F: \widehat{A} \to \widehat{F}$ by $f_F(\chi) = \chi|F$ for each character $\chi: A \to \mathbb{T}$.

Proposition 1.32. *The map* $\chi \mapsto (\chi|F)_{F\in\mathcal{F}}: \widehat{A} \to \lim_{F\in\mathcal{F}} \widehat{F}$ *is an isomorphism of compact abelian groups.*

Proof. Define $\varphi: \mathrm{Hom}(A,\mathbb{T}) \to \lim_{F\in\mathcal{F}} \mathrm{Hom}(F,\mathbb{T})$ by $\varphi(\chi) = (\chi|F)_{F\in\mathcal{F}}$. This definition yields a morphism of compact groups. A character χ of A is in its kernel if and only if $\chi|F = 0$ for all $F \in \mathcal{F}$. But since $A = \bigcup_{F\in\mathcal{F}} F$ this is the case if and only if $\chi = 0$. Thus φ is injective. Now let $\gamma = (\chi_F)_{F\in\mathcal{F}} \in \lim_{F\in\mathcal{F}} \widehat{F}$. By the definition of the bonding maps, this means that for every pair of finitely generated subgroups $E \subseteq F$ in A we have $\chi_F|E = \chi_E$. Now we can unambiguously define a function $\chi: A \to \mathbb{T}$ as follows. We pick for each $a \in A$ an $F \in \mathcal{F}$ with $a \in F$ (for instance, $F = \mathbb{Z}{\cdot}a$). By the preceding, the element $\chi_F(a)$ in $\mathbb{T}$ does not depend on the choice of F. Hence we define a function $\chi: A \to \mathbb{T}$ by $\chi(a) = \chi_F(a)$. If a, $b \in A$, take $F = \mathbb{Z}{\cdot}a + \mathbb{Z}{\cdot}b$ and observe $\chi(a+b) = \chi_F(a+b) = \chi_F(a)+\chi_F(b) = \chi(a)+\chi(b)$. Thus $\chi \in \mathrm{Hom}(A,\mathbb{T})$ and $\chi|F = \chi_F$. Hence $\varphi(\chi) = \gamma$. Thus φ is bijective and hence an isomorphism of compact groups (see Remark 1.8). □

In short: *The character group* $\widehat{A}$ *of any abelian group* A *is the strict projective limit of the character groups* $\widehat{F}$ *of its finitely generated subgroups* F. We know that $\widehat{F}$ is a direct product of a finite group and a finite-dimensional torus group (see Remark 1.18). In particular, every character group of an abelian group is approximated by compact abelian groups on manifolds.

Assume that $G = \lim_{j\in J} G_j$ is a strict projective limit of compact abelian groups with limit maps $f_j: G \to G_j$. Every character $\chi: G_j \to \mathbb{T}$ gives a character $\chi \circ f_j: G \to \mathbb{T}$ of G. Since f_j is surjective, $\chi \mapsto \chi \circ f_j: \widehat{G_j} \to \widehat{G}$ is injective. Under this map, we identify $\widehat{G_j}$ with a subgroup of $\widehat{G}$.

Proposition 1.33. *If* G *is a strict projective limit* $\lim_{j\in J} G_j$ *then* $\widehat{G} = \bigcup_{j\in J} \widehat{G_j}$.

Proof. With our identification of $\widehat{G}_j$ as a subgroup of $\widehat{G}$, the right side is contained in the left one. Now assume that $\chi: G \to \mathbb{T}$ is a character of G. If we denote with V the image of $]-\frac{1}{3}, \frac{1}{3}[$ in $\mathbb{T}$, then $\{\mathbf{0}\}$ is the only subgroup of $\mathbb{T}$ which is contained in V. Now $U = \chi^{-1}(V)$ is an open neighborhood of 0 in G. Hence by Proposition 1.31(i) there is a $j \in J$ such that $\ker f_j \subseteq U$. Hence $\chi(\ker f_j)$ is a subgroup of $\mathbb{T}$ contained in V and therefore is $\{\mathbf{0}\}$. Thus $\ker f_j \subseteq \ker \chi$ and there is a unique morphism $\chi_j: G_j \to \mathbb{T}$ such that $\chi = \chi_j \circ f_j$. With our convention, this means exactly $\chi \in \widehat{G_j}$. Thus $\widehat{G} \subseteq \bigcup_{j\in J} \widehat{G_j}$. □

The next theorem is one half of the famous Pontryagin Duality Theorem for compact abelian groups.

Theorem 1.34. *For any abelian group* A *the morphism* $\eta_A: A \to \widehat{\widehat{A}}$ *is an isomorphism.*

Proof. We know that $\widehat{A}$ is the strict projective limit $\lim_{F\in\mathcal{F}}\widehat{F}$ with the directed family $\mathcal{F}$ of finitely generated subgroups of A. (See Proposition 1.32.) The limit maps $f_F\colon \widehat{A}\to\widehat{F}$ are given by $f_F(\chi)=\chi|F$, and these surjective maps induce injective morphisms $\mathrm{Hom}(f_F,\mathbb{T})\colon\mathrm{Hom}(\widehat{F},\mathbb{T})\to\mathrm{Hom}(\widehat{A},\mathbb{T})$ with $\mathrm{Hom}(f_F,\mathbb{T})(\Sigma)=\Sigma\circ f_F$. By Proposition 1.33, $\mathrm{Hom}(\widehat{A},\mathbb{T})$ is the union of the images of the injective morphisms $\mathrm{Hom}(f_F,\mathbb{T})$. Thus for any $\Omega\in\mathrm{Hom}(\widehat{A},\mathbb{T})$ there is an $F\in\mathcal{F}$ such that Ω is in the image of $\mathrm{Hom}(f_F,\mathbb{T})$. Hence there is a $\Sigma\in\mathrm{Hom}(\widehat{F},\mathbb{T})$ such that $\Omega=\mathrm{Hom}(f_F,\mathbb{T})(\Sigma)=\Sigma\circ f_F$. But $\eta_F\colon F\to\mathrm{Hom}(\widehat{F},\mathbb{T})$ is an isomorphism by Remark 1.24(i). Hence there is an $a\in F$ such that $\Sigma=\eta_F(a)$. Thus $\Omega=\eta_f(a)\circ f_F$. Therefore, for any character $\chi\colon A\to\mathbb{T}$ of A we have $\Omega(\chi)=\eta_F(a)\big(f_F(\chi)\big)=\eta_F(a)(\chi|F)=(\chi|F)(a)=\chi(a)=\eta_A(a)(\chi)$. Thus η_A is surjective. The injectivity was established in Lemma 1.23. $\square$

It is helpful to visualize our argument by diagram chasing:

$$\begin{array}{ccc}
F & \xrightarrow{\ \eta_F\ } & \mathrm{Hom}(\widehat{F},\mathbb{T})\\
{\scriptstyle\mathrm{inc}}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathrm{Hom}(\widehat{\mathrm{inc}},\mathbb{T})}\\
A & \xrightarrow[\ \eta_A\ ]{} & \mathrm{Hom}(\widehat{A},\mathbb{T}).
\end{array}$$

The other half of the Pontryagin Duality Theorem claims that $\eta_G\colon G\to\widehat{\widehat{G}}$ is an isomorphism for any compact abelian group G, too. We cannot prove this at the present level of information. However, in practicing the concept of a projective limit we can take one additional step.

Let us, at least temporarily, use the parlance that a compact abelian group G is said *to have duality* if $\eta_G\colon G\to\widehat{\widehat{G}}$ is an isomorphism. We propose the following exercise whose proof we indicate rather completely since it is of independent interest.

Lemma 1.35. *If a compact abelian group G is the limit $\lim_{j\in J}G_j$ of a strict projective system of compact abelian groups G_j which have duality, then G has duality.*

Proof. After Lemma 1.23, we have to show that $\eta_G\colon G\to\widehat{\widehat{G}}$ is bijective. We attack the harder part first and show that η_G is surjective. Assume that $\Omega\in\widehat{\widehat{G}}$; that is, Ω is a morphism of abelian groups $\widehat{G}\to\mathbb{T}$. By Proposition 1.36, $\widehat{G}=\bigcup_{j\in J}\widehat{G_j}$. If we denote with Ω_j the restriction $\Omega|\widehat{G_j}$, then $\Omega_j\colon\widehat{G_j}\to\mathbb{T}$ is an element of $\widehat{\widehat{G_j}}$. Since G_j has duality by hypothesis, η_{G_j} is *surjective* and thus there is a $g_j\in G_j$ such that $\eta_{G_j}(g_j)=\Omega_j$. We claim that $g\overset{\mathrm{def}}{=}(g_j)_{j\in J}\in\prod_{j\in J}G_j$ is an element of $\lim_{j\in J}G_j=G$. For this purpose assume that $j\le k$ in J. We have a commutative diagram

$$\begin{array}{ccc} G_k & \xrightarrow{\eta_{G_k}} & \widehat{\widehat{G_k}} \\ {\scriptstyle f_{jk}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\widehat{f_{jk}}}} \\ G_j & \xrightarrow[\eta_{G_j}]{} & \widehat{\widehat{G_j}}. \end{array}$$

(We shall consider this claim in a separate exercise below.) We notice that

$$\widehat{\widehat{f_{jk}}}: \widehat{\widehat{G_k}} \to \widehat{\widehat{G_j}}$$

is the restriction map sending Ω_k to $\Omega_k|\widehat{G_j} = \Omega_j$. Thus

$$\eta_{G_j}\big(f_{jk}(g_k)\big) = \widehat{\widehat{f_{jk}}}\big(\eta_{G_k}(g_k)\big) = \widehat{\widehat{f_{jk}}}(\Omega_k) = \Omega_j = \eta_{G_j}(g_j).$$

But since G_j has duality, η_{G_j} is *injective*, and thus

$$f_{jk}(g_k) = g_j,$$

which establishes the claim $g \in \lim_{j\in J} G_j$. For each limit map $f_j: G \to G_j$, as before, we have a commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\eta_{G}} & \widehat{\widehat{G}} \\ {\scriptstyle f_{j}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\widehat{f_{j}}}} \\ G_j & \xrightarrow[\eta_{G_j}]{} & \widehat{\widehat{G_j}}. \end{array}$$

Thus $\widehat{\widehat{f}}_j\big(\eta_G(g)\big) = \eta_{G_j}\big(f_j(g)\big) = \eta_{G_j}(g_j) = \Omega_j$ for all $j \in J$. Now we observe that $\widehat{\widehat{f}}_j: \widehat{\widehat{G}} \to \widehat{\widehat{G_j}}$ is the restriction $\Sigma \to \Sigma|\widehat{G_j}$. Thus the restriction of the morphism $\eta_G(g): \widehat{G} \to \mathbb{T}$ to each $\widehat{G_j}$ is Ω_j, and therefore this morphism is none other than the given map Ω. Hence $\eta_G(g) = \Omega$ and the claim of the surjectivity of η_G is proved.

As a second step we show that η_G is injective. We have observed before that this statement is equivalent to the assertion that the characters of G separate the points. Hence we assume that $0 \neq g \in G$. Set $\mathcal{N} = \{\ker f_j \mid j \in J\}$. From Proposition 1.33(i) we know that $\bigcap \mathcal{N} = \{0\}$. Hence there is a $j \in J$ such that $g \notin \ker f_j$, that is, $f_j(g) \neq 0$. Since the group G_j has duality, its characters separate its points. Hence there is a $\chi \in \widehat{G_j}$ such that $\chi\big(f_j(g)\big) \neq 0$. Hence $\chi \circ f_j \in \widehat{G}$ is a character of G which does not annihilate g. The assertion is now proved. □

Let us now assume the following result

Proposition 1.36. *The characters of a compact abelian groups separate the points.* □

The proof is a consequence of the basic theorem for compact groups saying that every compact group has enough finite dimensional continuous lienar representations to separate the points. [See e.g. Lecture Notes "Introduction to Topological Groups", WS 2005-06, `topgr.pdf`.]

Now we can prove the following result.

Theorem 1.37. *For any compact abelian group G the morphism $\eta_G: G \to \widehat{\widehat{G}}$ is an isomorphism.*

Proof. From Proposition 1.36, it follows at once that $\eta_G: G \to \widehat{\widehat{G}}$ is injective. Hence the corestriction $g \mapsto \eta_G \to \Gamma \stackrel{\text{def}}{=} \eta_G(G)$ is an isomorphism onto the subgroup $\Gamma \subseteq \widehat{\widehat{G}}$. We claim that $\Gamma = \widehat{\widehat{G}}$; a proof of this claim will finish the proof. By Proposition 1.36 once again, the claim is proved if every character of $\widehat{\widehat{G}}/\Gamma$ is zero, that is, if every character of $\widehat{\widehat{G}}$ which vanishes on Γ is zero. By Theorem 1.34 we may identify $\widehat{G}$ with the character group of $\widehat{\widehat{G}}$ under the evaluation isomorphism. Thus a character f of $\widehat{\widehat{G}}$ vanishing on Γ is given by an element $\chi \in \widehat{G}$ such that $f(\Omega) = \Omega(\chi)$. But we have $0 = f\big(\eta_G(g)\big) = \eta_G(g)(\chi)$ for all $g \in G$ since f annihilates Γ. By the definition of η_G we then note $\chi(g) = \eta_G(g)(\chi) = 0$ for all $g \in G$, that is, $\chi = 0$ and thus $f = 0$. □

Theorems 1.34 and 1.37 constitute the object portion of the *Pontryagin Duality Theorem for discrete and compact abelian groups*. Up to natural isomorphism it sets up a bijection between the class of discrete and that of compact abelian groups. It shall reveal its true power when it is complemented by the morphism part which sets up a similar bijection between morphisms. However, this belongs to the domain of generalities and does, in fact, not require more work in depth. The nontrivial portion of the duality is accomplished.

The following consequence of the duality theorem turns out to be very useful.

Corollary 1.38. (i) *Let G be a compact abelian group and A a subgroup of the character group $\widehat{G}$. The following two conditions are equivalent:*

(1) *A separates the points of G.*

(2) *$A = \widehat{G}$.*

(ii) (The Extension Theorem for Characters) *If H is a closed subgroup of G, then every character of H extends to a character of G.*

Proof. (i) Proposition 1.36 says that (2) implies (1), and so we have to prove that (1) implies (2). Since the characters of the discrete group $\widehat{G}/A$ separate the points by Lemma 1.11, in order to prove (2) it suffices to show that every character of $\widehat{G}$ vanishing on A must be zero. Thus let Ω be a character of $\widehat{G}$ vanishing on A. By Theorem 1.37, there is a $g \in G$ with $\eta_G(g) = \Omega$. Thus $\chi \in A$ implies $0 = \Omega(\chi) = \eta_G(g)(\chi) = \chi(g)$. From (1) we now conclude $g = 0$. Hence $\Omega = \eta_G(g) = 0$.

(ii) The collection of all restrictions $\chi|H$ of characters of G to H separates the points of H since the characters of G separate the points of G by Proposition 1.36. Then (i) above shows that the function $\chi \mapsto \chi|H: \widehat{G} \to \widehat{H}$ is surjective, and this proves the assertion. □

Corollary 1.39. *For every compact abelian group G there is a filter basis $\mathcal{N}$ of compact subgroups such that G is the strict projective limit $\lim_{N\in\mathcal{N}} G/N$ of factor groups each of which is a character group of a finitely generated abelian group.*

Proof. Let $A = \widehat{G}$ denote the character group of G and $\mathcal{F}$ the family of finitely generated subgroups. If $F \in \mathcal{F}$, let $N_F = F^\perp$ denote the annihilator $\{g \in G \mid \chi(g) = 0 \text{ for all } \chi \in F\}$. Since $F \subseteq F'$ in $\mathcal{F}$ implies $N_{F'} \subseteq N_F$, the family $\mathcal{N} = \{N_F \mid F \in \mathcal{F}\}$ is a filter basis of closed subgroups. An element g is in $\bigcap \mathcal{N}$ if and only if it is in the annihilator of every finitely generated subgroup of A, hence if and only if it is annihilated by all of A, since A is the union of all of its finitely generated subgroups. Thus $g = 0$ by Proposition 1.36. By Proposition 1.31(ii), therefore, G is the strict projective limit $G = \lim_{F\in\mathcal{F}} G/N_F$.

Now we claim that the character group of G/N_F may be identified with F. This will finish the proof of the corollary. If $q_F: G \to G/N_F$ denotes the quotient map, then the function $\varphi \mapsto \varphi \circ q_F: (G/N_F)\widehat{\ } \to \widehat{G}$ is injective as q_F is surjective. Its image is precisely the group $F^{\perp\perp}$ of all characters vanishing on N_F. Since every character $\chi \in F$ vanishes on N_F, we have $F \subseteq F^{\perp\perp}$. We shall now show equality and thereby prove the claim. But when $F^{\perp\perp}$ is identified with the Extension character group of G/N_F then the subgroup F separates the points of G/N_F since the only coset $g+N_F \in G/N_F$ annihilated by all of F is N_F by the definition of N_F. Now the Extension Theorem for Characters, Corollary 1.38(ii) shows $F = F^{\perp\perp}$. □

Corollary 1.39 yields the following remark:

Corollary 1.40. *Every compact abelian group is the strict projective limit of a projective system of groups G/N isomorphic to $\mathbb{T}^{n(N)} \times E_N$ with suitable numbers $n(N) = 0, 1, \ldots,$ and finite abelian groups E_N.* □

Thus every compact abelian groups is the strict projective limit of compact abelian groups defined on compact manifolds. Such groups are called compact abelian *Lie groups*

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