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Linear Algebra II (MCS), SS 2006, Exercise 9

Mini-Quiz

(1) A bilinear form on a real vector space V is...?
- ☐ a linear map $\Phi : V \to V \times V$.
- √ a map $\Phi : V \times V \to V$ which is linear in each argument.
- ☐ a map $\Phi : V \times V \to V$ which is linear.

(2) A $*$-sesquilinear map on a vector space V over a field K with involution $*$ is a map $\Phi : V \times V \to V$...?
- ☐ which is linear in each argument.
- ☐ which satisfies $\Phi(u + ru', v + sv') = \Phi(u, v) + r^*\Phi(u', v) + s^*\Phi(u, v') + r^*s^*\Phi(u', v')$.
- √ which is linear in the second argument and satisfies $\Phi(u + ru', v) = \Phi(u, v) + r^*\Phi(u', v)$.

(3) For a $*$-hermitean matrix A, a diagonal matrix S^*AS together with the transformation matrix S can be computed stepwise using...?
- ☐ only column operations on the pair (A, I).
- ☐ successively a column operation and the conjugate row operation on both sides of the pair (A, I).
- √ successively a column operation on both sides of the pair (A, I) and then the conjugate row operation on the left side only.

Groupwork

G 38 Determine a best approximate 'solution' for the following system of linear equations: $Ax = b$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

G 39 Which of the following matrices are positive semi-definite?

$$(i) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (ii) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (iii) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (iv) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (v) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad (vi) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

G 40 Let $\mathcal{Q}$ be the real quadric in affine 2-space V defined by $\mathcal{Q} = \{P \in \mathbb{P} \mid (P^\alpha)^t A P^\alpha = 0\}$ w.r.t. some basis $\alpha = \{O, e_1, e_2\}$ and $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$. Find an affine basis transformation T, such that $\mathcal{Q} = \{x + O_\alpha \in \mathbb{P} \mid \lambda_1 z_1^2 + \lambda_2 z_2^2 = 0\}$, where $x^\beta = z$ and $\beta = T\alpha$.

G 41 Consider the following elementary row operations:

$$(a_1) \quad [Z_k \leftrightarrow Z_l], \qquad (a_2) \quad [Z_k := r \cdot Z_k], r \neq 0, \qquad (a_3) \quad [Z_k := Z_k + r \cdot Z_l].$$

The corresponding column operations are

$$(b_1) \quad [S_k \leftrightarrow S_l], \qquad (b_2) \quad [S_k := r \cdot S_k], r \neq 0, \qquad (b_3) \quad [S_k := S_k + r \cdot S_l].$$

If K is a field with an involution $*$, then the corresponding adjoint column operations are

$$(c_1) \quad [S_k \leftrightarrow S_l], \qquad (c_2) \quad [S_k := r^* \cdot S_k], r \neq 0, \qquad (c_3) \quad [S_k := S_k + r^* \cdot S_l].$$

(i) Show that the elementary matrix corresponding to (b_i) is the transpose of of the elementary matrix corresponding to (a_i).

(ii) Show that the elementary matrix corresponding to (c_i) is the $*$-transpose of of the elementary matrix corresponding to (a_i).

G 42 Is it possible to diagonalize a $*$-hermitean matrix A by using successively a row operation first and then the conjugate column operation? What matrix do you get if, starting from the identity matrix I, you record in each step the row operation you apply to A?

Homework

H 30 For each of the following hermitean matrices $H_i, i = 1,2,3$ find an invertible matrix S_i such that $S_i^* H_i S_i$ is diagonal:

$$(i)\quad H_1 = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}, \qquad (ii)\quad H_2 = \begin{pmatrix} 1 & 2+3i \\ 2-3i & -1 \end{pmatrix}, \qquad (iii)\quad H_3 = \begin{pmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{pmatrix}.$$

Determine the rank and signature in each case.

H 31 (i) We say that two hermitean matrices A and B are congruent to each other, if there is a non-singular matrix S with $B = S^*AS$. Show that congruence is in fact an equivalence relation. I.e. 'A is congruent to B' has the following properties:

(a) A is congruent to A (reflexivity).

(b) If A is congruent to B, then B is congruent to A (symmetry).

(c) If A is congruent to B and B is congruent to C, then A is congruent to C (transitivity).

(ii) Show that two hermitean matrices are congruent to each other if and only if they have the same rank and signature.

H 32 Let Φ be the symmetric bilinear form associated with the quadratic form $q(x, y) := ax^2 + bxy + cy^2$. Show that:

(i) Φ has rank equal to two, if and only if $b^2 - 4ac \neq 0$.

(ii) Φ is positive definite, if and only if $a > 0$ and $b^2 - 4ac < 0$.

H 33 Show that the sum of two positive definite matrices is positive definite. Is the same true for the product of two positive definit matrices?

H 34 (Optional task)
Let V be a finite dimensional vector space over a field K with involution $*$ and let Φ be a $*$-sesquilinear form on V. For a linear subspace $W \subset V$ we define

$$W^\perp = \{v \in V \mid \Psi(v, w) = 0 \text{ for all } w \in W\},$$
$$^\perp W = \{v \in V \mid \Psi(w, v) = 0 \text{ for all } w \in W\}.$$

Show that

(i) (**Corr.:**) $W^\perp$ and $^\perp W$ are linear subspaces of V.

(ii) $W^\perp = {}^\perp W$ if Φ is $*$-hermitean or $*$-skew hermitean.

(iii) $V = \{0\}^\perp = {}^\perp\{0\}$, and $V^\perp = \{0\} = {}^\perp V$ if and only if Φ is non-degenerate.

(iv) (**Corr.:**) $\operatorname{rank}\Phi = \dim V - \dim V^\perp = \dim V - \dim {}^\perp V$, $\dim W^\perp \geq \dim V - \dim W$ and $\dim {}^\perp W \geq \dim V - \dim W$.

(v) The restriction $\Phi|_{W\times W}$ is non-degenerate if and only if $W \cap W^\perp = \{0\} = W \cap {}^\perp W$.

(vi) (**Corr.:**) $V = W \oplus W^\perp = W \oplus {}^\perp W$ if and only if $\Phi|_{W\times W}$ is non-degenerate.

(vii) Let $u = (x_1, x_2)^t, v = (y_1, y_2)^t$ and $\Phi(u, v) = x_1y_1 + 2x_1y_2 + 3x_2y_1$. For $W = \{(t, 0)^t \mid t \in \mathbb{R}\}$ compute $W^\perp$ and $^\perp W$ and verify that $W^\perp \neq {}^\perp W$.

Linear Algebra II (MCS), SS 2006, Exercise 9, Solution

Mini-Quiz

(1) A bilinear form on a real vector space V is...?
- ☐ a linear map $\Phi : V \to V \times V$.
- √ a map $\Phi : V \times V \to V$ which is linear in each argument.
- ☐ a map $\Phi : V \times V \to V$ which is linear.

(2) A $*$-sesquilinear map on a vector space V over a field K with involution $*$ is a map $\Phi : V \times V \to V$...?
- ☐ which is linear in each argument.
- ☐ which satisfies $\Phi(u + ru', v + sv') = \Phi(u, v) + r^*\Phi(u', v) + s^*\Phi(u, v') + r^*s^*\Phi(u', v')$.
- √ which is linear in the second argument and satisfies $\Phi(u + ru', v) = \Phi(u, v) + r^*\Phi(u', v)$.

(3) For a $*$-hermitean matrix A, a diagonal matrix S^*AS together with the transformation matrix S can be computed stepwise using...?
- ☐ only column operations on the pair (A, I).
- ☐ successively a column operation and the conjugate row operation on both sides of the pair (A, I).
- √ successively a column operation on both sides of the pair (A, I) and then the conjugate row operation on the left side only.

Groupwork

G 38 Determine a best approximate 'solution' for the following system of linear equations: $Ax = b$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

A best approximate solution is given by a solution of the following system of linear equations: $A^tAx = A^tb$. *Now* $A^t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, *hence we have the system* $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. *The unique solution to this is* $\tilde{x} = \frac{1}{3}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

G 39 Which of the following matrices are positive semi-definite?

$(i)\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $(ii)\ \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $(iii)\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $(iv)\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $(v)\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $(vi)\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

The only positive semi-definite matrices are the ones in (i) and (v). The one in (v) is even positive definite.

G 40 Let $\mathcal{Q}$ be the real quadric in affine 2-space V defined by $\mathcal{Q} = \{P \in \mathbb{P} \mid (P^\alpha)^t A P^\alpha = 0\}$ w.r.t. some basis $\alpha = \{O, e_1, e_2\}$ and $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$. Find an affine basis transformation T, such that $\mathcal{Q} = \{x + O_\alpha \in \mathbb{P} \mid \lambda_1 z_1^2 + \lambda_2 z_2^2 = 0\}$, where $x^\beta = z$ and $\beta = T\alpha$.

We first diagonalize the block $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ *via* $S = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. *Applying this to the matrix A yields* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$. *Finally, applying a translation, our transition matrix is* $T^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

G 41 Consider the following elementary row operations:

$$(a_1)\ \ [Z_k \leftrightarrow Z_l], \qquad (a_2)\ \ [Z_k := r \cdot Z_k], r \neq 0, \qquad (a_3)\ \ [Z_k := Z_k + r \cdot Z_l].$$

The corresponding column operations are

$$(b_1)\ \ [S_k \leftrightarrow S_l], \qquad (b_2)\ \ [S_k := r \cdot S_k], r \neq 0, \qquad (b_3)\ \ [S_k := S_k + r \cdot S_l].$$

If K is a field with an involution $*$, then the corresponding adjoint column operations are

$$(c_1)\ \ [S_k \leftrightarrow S_l], \qquad (c_2)\ \ [S_k := r^* \cdot S_k], r \neq 0, \qquad (c_3)\ \ [S_k := S_k + r^* \cdot S_l].$$

(i) Show that the elementary matrix corresponding to (b_i) is the transpose of of the elementary matrix corresponding to (a_i).

(ii) Show that the elementary matrix corresponding to (c_i) is the $*$-transpose of of the elementary matrix corresponding to (a_i).

The elementary matrix corresponding to (a_1) is (a^1_{ij}) with

$$a^1_{ij} := \begin{cases} 1 & \text{if } i = j \text{ and } i \neq k \text{ and } i \neq l \\ 0 & \text{if } i = j \text{ and } i = k \text{ or } i = l \\ 1 & \text{if } i \neq j \text{ and } (i,j) = (k,l) \text{ or } (i,j) = (l,k) \\ 0 & \text{else} \end{cases}$$

The elementary matrix corresponding to (a_2) is (a^2_{ij}) with

$$a^2_{ij} := \begin{cases} 1 & \text{if } i = j \text{ and } i \neq k \\ r & \text{if } i = j = k \\ 0 & \text{else} \end{cases}$$

The elementary matrix corresponding to (a_3) is (a^3_{ij}) with

$$a^3_{ij} := \begin{cases} 1 & \text{if } i = j \\ r & \text{if } i = k \text{ and } j = l \\ 0 & \text{else} \end{cases}$$

In each case this is verified by forming $(a^p_i j) \cdot X, p = 1,2,3$. Correspondingly, we get the matrices corresponding to (b_p) and (c_p) as

$$b^1_{ij} := \begin{cases} 1 & \text{if } i = j \text{ and } i \neq k \text{ and } i \neq l \\ 0 & \text{if } i = j \text{ and } i = k \text{ or } i = l \\ 1 & \text{if } i \neq j \text{ and } (i,j) = (k,l) \text{ or } (i,j) = (l,k) \\ 0 & \text{else} \end{cases}$$

$$b^2_{ij} := \begin{cases} 1 & \text{if } i = j \text{ and } i \neq k \\ r & \text{if } i = j = k \\ 0 & \text{else} \end{cases}$$

and

$$b^3_{ij} := \begin{cases} 1 & \text{if } i = j \\ r & \text{if } i = l \text{ and } j = k \\ 0 & \text{else} \end{cases}$$

again, in each case this is verified by forming $X \cdot (b^p_i j), p = 1,2,3$. Finally, we have for the elementary matrices corresponding to $(c^p_{ij}) \cdot X, p = 1,2,3$:

$$c^1_{ij} := \begin{cases} 1 & \text{if } i = j \text{ and } i \neq k \text{ and } i \neq l \\ 0 & \text{if } i = j \text{ and } i = k \text{ or } i = l \\ 1 & \text{if } i \neq j \text{ and } (i,j) = (k,l) \text{ or } (i,j) = (l,k) \\ 0 & \text{else} \end{cases}$$

$$c^2_{ij} := \begin{cases} 1 & \text{if } i = j \text{ and } i \neq k \\ r^* & \text{if } i = j = k \\ 0 & \text{else} \end{cases}$$

and

$$c^3_{ij} := \begin{cases} 1 & \text{if } i = j \\ r^* & \text{if } i = l \text{ and } j = k \\ 0 & \text{else} \end{cases}$$

which is again verified by forming the product $X \cdot (b^p_i j), p = 1,2,3$. From this description it is easy to verify (i) and (ii).

G 42 Is it possible to diagonalize a $*$-hermitean matrix A by using successively a row operation first and then the conjugate column operation? What matrix do you get if, starting from the identity matrix I, you record in each step the row operation you apply to A?

Yes, due to the fact that in the product S^*AS it does not matter if we first compute the product AS and then $S^*(AS)$ or first (S^*A) and then $(S^*A)S$. The matrix you obtain out of the identity after applying the row transformations is S^*, where S^*AS is diagonal.

Homework

H 30 For each of the following hermitean matrices $H_i, i = 1, 2, 3$ find an invertible matrix S_i such that $S_i^* H_i S_i$ is diagonal:

$$(i) \quad H_1 = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}, \qquad (ii) \quad H_2 = \begin{pmatrix} 1 & 2+3i \\ 2-3i & -1 \end{pmatrix}, \qquad (iii) \quad H_3 = \begin{pmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{pmatrix}.$$

Determine the rank and signature in each case.

To (i): $S_1 = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}$, $S_1^* H_1 S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *rank* $= 2$, *signature* $= (2, 0)$.

To (ii): $S_2 = \begin{pmatrix} 1 & -2-3i \\ 0 & 1 \end{pmatrix}$, $S_2^* H_2 S_2 = \begin{pmatrix} 1 & 0 \\ 0 & -14 \end{pmatrix}$, *rank* $= 2$, *signature* $= (1, 1)$.

To (iii): $S_3 = \begin{pmatrix} 1 & -i & -3-i \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{pmatrix}$, $S_3^* H_3 S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$, *rank* $= 3$, *signature* $= (2, 1)$.

H 31 (i) We say that two hermitean matrices A and B are congruent to each other, if there is a non-singular matrix S with $B = S^* AS$. Show that congruence is in fact an equivalence relation. I.e. 'A is congruent to B' has the following properties:

(a) A is congruent to A (reflexivity).

(b) If A is congruent to B, then B is congruent to A (symmetry).

(c) If A is congruent to B and B is congruent to C, then A is congruent to C (transitivity).

(ii) Show that two hermitean matrices are congruent to each other if and only if they have the same rank and signature.

To (i): *Taking for S the identity matrix, then clearly $A = S^* AS$ so the relation is reflexive.*
If A is congruent to B, then there is a non-singular matrix S with $B = S^* AS$. Putting $T = S^-1$ and noting that $T^* = (S^{-1})^* = (S^*)^{-1}$ we see that $A = T^* BT$ and T is non-singular. Hence, the relation is symmetric.
If A is congruent to B and B is congruent to C, then there exist non-singular matrices S and T with $B = S^* AS$ and $C = T^* BT$. Putting $U = ST$ and noting that $U^* = T^* S^*$, we obtain $U^* AU = T^* S^* AST = T^* BT = C$. Hence, A is congruent to C.

To (ii): *If a matrix A has signature (p, q) and rank r, this means that A is congruent to the matrix $\begin{pmatrix} E_p & & \\ & -E_q & \\ & & 0_{r-p-q} \end{pmatrix}$, where 0_n denotes the $n \times n$ matrix with all entries equal to zero. Hence, using (i), by transitivity two matrices are congruent to each other if and only if they have the same rank and signature.*

H 32 Let Φ be the symmetric bilinear form associated with the quadratic form $q(x, y) := ax^2 + bxy + cy^2$. Show that:

(i) Φ has rank equal to two, if and only if $b^2 - 4ac \neq 0$.

(ii) Φ is positive definite, if and only if $a > 0$ and $b^2 - 4ac < 0$.

To (i): *The symmetric matrix associated with Φ is $A := \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$. The rank of Φ is equal to two if and only if the determinant of A is $\neq 0$. We have $\det A = ac - \frac{1}{4}b^2$ and this is $\neq 0$ if and only if $b^2 - 4ac \neq 0$.*

(ii) *By the principal minor criterium, it is easily verified that Φ is positive definite, if and only if $a > 0$ and $b^2 - 4ac < 0$.*

H 33 Show that the sum of two positive definite matrices is positive definite. Is the same true for the product of two positive definit matrices?

It is easy to see that a matrix is positive definite if and only if it is the Gram-matrix associated with a positive definite $*$-hermitian sesquilinear form. Therefore, let Φ_A denote the standard $*$-hermitian sesquilinear form on K^n associated with A. I.e. $\Phi_A(u, v) = (x_1^*, \ldots, x_n^*)A(y_1, \ldots, y_n)^t$, where $u = (x_1, \ldots, x_n)$ and $v = (y_1, \ldots, y_n)$. If A and B are positive definite, then Φ_A and Φ_B are positive definite. this means that $\Phi_A(u, u) > 0$ and $\Phi_B(u, u) > 0$ for all $u \in K^n \setminus \{0\}$. Now $\Phi_A + \Phi_B = \Phi_{A+B}$ and thus $\Phi_{A+B}(u, u) = \Phi_A(u, u) + \Phi_B(u, u) > 0$ for all $u \in K^n \setminus \{0\}$. By the *correspondence of $*$-hermitian sesquilinear forms and $*$-hermitian matrices noted above, we conclude that $A+B$ is positive definite if A and B are positive definite.*

The according statement for the product is false. In fact, the product of two $*$-hermitian matrices A and B need not be hermitian anymore, since $(AB)^* = B^*A^* = BA \neq AB$ in general. For instance, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ are positive definite, but $AB = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$.

If however A and B commute and are positive definite, then it can be shown that their product is positive definite

H 34 (Optional task)

Let V be a finite dimensional vector space over a field K with involution $*$ and let Φ be a $*$-sesquilinear form on V. For a linear subspace $W \subset V$ we define

$$W^\perp = \{v \in V \mid \Psi(v,w) = 0 \text{ for all } w \in W\},$$

$$^\perp W = \{v \in V \mid \Psi(w,v) = 0 \text{ for all } w \in W\}.$$

Show that

(i) (**Corr.:**) $W^\perp$ and $^\perp W$ are linear subspaces of V.

(ii) $W^\perp = {}^\perp W$ if Φ is $*$-hermitean or $*$-skew hermitean.

(iii) $V = \{0\}^\perp = {}^\perp\{0\}$, and $V^\perp = \{0\} = {}^\perp V$ if and only if Φ is non-degenerate.

(iv) (**Corr.:**) $\operatorname{rank}\Phi = \dim V - \dim V^\perp = \dim V - \dim {}^\perp V$, $\dim W^\perp \geq \dim V - \dim W$ and $\dim {}^\perp W \geq \dim V - \dim W$.

(v) The restriction $\Phi|_{W\times W}$ is non-degenerate if and only if $W \cap W^\perp = \{0\} = W \cap {}^\perp W$.

(vi) (**Corr.:**) $V = W \oplus W^\perp = W \oplus {}^\perp W$ if and only if $\Phi|_{W\times W}$ is non-degenerate.

(vii) Let $u = (x_1,x_2)^t, v = (y_1,y_2)^t$ and $\Phi(u,v) = x_1y_1 + 2x_1y_2 + 3x_2y_1$. For $W = \{(t,0)^t \mid t \in \mathbb{R}\}$ compute $W^\perp$ and $^\perp W$ and verify that $W^\perp \neq {}^\perp W$.

To (i): *For any $u, v \in W^\perp$ and $\lambda \in K$ we have $\Phi(\lambda u + v, w) = \lambda^*\Phi(u,w) + \Phi(v,w) = 0$ for all $w \in W$, since $\Phi(u,w) = 0 = \Phi(v,w)$ for all $w \in W$. This shows that $W^\perp$ is a linear subspace of V. Similarly, one shows that $^\perp W$ is a linear subspace of V.*

To (ii): *We have $\Phi(v,w) = \Phi^*(w,v) = \pm\Phi(w,v)$. From this we conclude that $W^\perp = {}^\perp W$ if Φ is $*$-hermitean or $*$-skew hermitean.*

To (iii): *$V = \{0\}^\perp = {}^\perp\{0\}$ is true because $\Phi(v,0) = \Phi(0,v) = 0$ for all $v \in V$. The other statement is true, more or less by the definition of non-degeneracy in the script.*

To (iv): *The rank of Φ is defined as the rank of any Gram-matrix associated with Φ. Let therefore $\{v_1,\dots,v_k\}$ be a basis of $V^\perp$, which we complete to a basis $\alpha = \{v_1,\dots,v_k,v_{k+1},\dots,v_n\}$ of V. W.r.t. this basis, the Gram-matrix $A = \Phi^\alpha$ is*

$$A = \begin{pmatrix} 0 \\ A' \end{pmatrix},$$

where the 0 denotes a $k \times n$ zero matrix and A' a $n-k \times n$ matrix. This shows that $\operatorname{rank}\Phi \leq n-k = \dim V - \dim V^\perp$. Suppose that $\operatorname{rank}\Phi < n-k$, then one row of A' is a linear combination of the other rows. Thus, there is some nonzero vector $(0,\dots,0,a_{k+1},\dots,a_n) \in K^n$, which annihilates A from the right, which implies that for $v := \sum_{i=k+1}^n a_iv_i$ we have $\Phi(v,u) = 0$ for all $u \in V$. However, this would mean that $v \in V^\perp$, which is a contradiction. Hence, $\operatorname{rank}\Phi = \dim V - \dim V^\perp$ and $\operatorname{rank}\Phi = \dim V - \dim {}^\perp V$ is just shown in the same way.

Let $\{v_1,\dots,v_k\}$ be a basis of W which we complete to a basis $\alpha = \{v_1,\dots,v_k,v_{k+1},\dots,v_n\}$ of V. Now any $v \in W^\perp$ can be written as $v = a_1v_1 + \dots + a_nv_n$. For all $i = 1,\dots,k$ we have $0 = \Phi(v,v_i) = \sum_{j=1}^n a_j\Phi(v_j,v_i)$. Thus, if we put $A = \Phi^\alpha$ then $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$, with some $n \times k$-matrix A_1, and $(a_1,\dots,a_n) \cdot A_1 = 0$. This implies $A_1^t \cdot (a_1,\dots,a_n)^t = 0$, meaning that the coordinate vectors of each $v \in W^\perp$ w.r.t. α lie in the kernel of a $k \times n$-matrix, the dimension of which is at least $n-k$. Hence $\dim W^\perp \geq n-k = \dim V - \dim W$. The same argumentation mutatis mutandis gives the desired inequality in the $^\perp W$-case.

To (v): *Clearly, $\Phi|_{W\times W}$ is a $*$-sesquilinear form and $W \cap W^\perp = W^{\perp'}$, where $\perp'$ denotes the complement w.r.t. $\Phi|_{W\times W}$. By (iii) we now have that $\Phi|_{W\times W}$ is non-degenerate if and only if $\{0\} = W^{\perp'} = W \cap W^\perp$. As before, the same argumentation works in the $^\perp W$-case, as well.*

To (vi): *If $V = W \oplus W^\perp$, then in particular, $W \cap W^\perp = \{0\}$. So by (v), $\Phi|_{W\times W}$ is non-degenerate. Conversely, if $\Phi|_{W\times W}$ is non-degenerate, we can read the preceding argument backwards and obtain that $W \cap W^\perp = \{0\}$. All that remains to be shown is $V = W + W^\perp$. However, in* *(iii) we have shown that* $\dim W^{\perp} \geq \dim V - \dim W$ *and for dimension reasons it follows that* $V = W \oplus W^{\perp}$*. Again, analogously it is shown that* $V = W \oplus {}^{\perp}W$ *if and only if* Φ *is non-degenerate.*

To (vii): *We have* $W^{\perp} = \{s(-3,1)^t \mid s \in \mathbb{R}\}$ *and* ${}^{\perp}W = \{s(-2,1)^t \mid s \in \mathbb{R}\}$.