Department of Mathematics
Prof. Dr. Christian Herrmann
Dipl.-Math. Frederick Magata
Dr. Abdelhadi Es-Sarhir

TECHNISCHE
UNIVERSITÄT
DARMSTADT

8. Juni 2006

Linear Algebra II (MCS), SS 2006, Exercise 8

Groupwork

G 33 We put $u = (x_1, x_2)^t$ and $v = (y_1, y_2)^t$. Let f be the bilinear form on $\mathbb{R}^2$ defined by

$$f(u, v) = 2x_1y_1 - 3x_1y_2 + x_2y_2.$$

(i) Determine the Gram-matrix A of f w.r.t. the basis $\{e_1 = (1, 0)^t, e_2 = (1, 1)^t\}$.
(ii) Determine the Gram-matrix B of f w.r.t. the basis $\{e'_1 = (2, 1)^t, e'_2 = (1, -1)^t\}$.
(iii) Determine the transition matrix P from $\{e_1, e_2\}$ to $\{e'_1, e'_2\}$ and verify that $B = P^tAP$.

G 34 For each of the following matrices $A_i, i = 1, 2$ find matrices P_i, such that $P_i^tA_iP_i$ is diagonal. What is the signature of A_i?

$$(i) \quad A_1 = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix}, \qquad (ii) \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix}.$$

G 35 Let V be a n-dimensional vector space and let $\beta = \{e_1, \dots, e_n\}$ be a basis of V. Show that a $*$-sesquilinear form Φ on V is $*$-hermitean, resp. $*$-skew hermitean, if and only if its associated Gram-Matrix A w.r.t. β is $*$-hermitean, resp. $*$-skew hermitean. I.e. A satisfies $A^* = A$, resp. $A^* = -A$.

G 36 Let $P = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Compute

$$(i) \quad P^{-1}AP, \qquad (ii) \quad P^*AP, \qquad (iii) \quad P^tAP.$$

Which transformation is the correct one if we view A as the matrix associated with
(a) a linear endomorphism of $\mathbb{C}^2$
(b) a complex bilinear form on $\mathbb{C}^2$ (i.e. $*$-sesquilinear w.r.t. the trivial involution)
(c) a sesquilinear form on $\mathbb{C}^2$ (i.e. $*$-sequilinear w.r.t. complex conjugation)
and perform a change of basis from $\{(1, 0)^t, (0, 1)^t\}$ to $\{(i, -1)^t, (i, 1)^t\}$ in $\mathbb{C}^2$? If $a = 1, b = 0$, which transformations preserve the eigenvalues of A, which the signs of the eigenvalues and which the invertability of A?

G 37 Let V be a n-dimensional vector space over a field K with involution $*$ and let $\beta = \{e_1, \dots, e_n\}$ be a basis of V.

(i) Show that the set of $*$-sesquilinear forms $\mathrm{Sesq}(V)$ on V is a vector space over K with respect to the following definitions of scalar multiplication and addition for every $\lambda \in K$ and $\Phi, \Psi \in \mathrm{Sesq}(V)$:

$$\lambda \cdot \Phi : V \times V \to K, (u, v) \mapsto \lambda \cdot \Phi(u, v)$$

$$\Phi + \Psi : V \times V \to K, (u, v) \mapsto \Phi(u, v) + \Psi(u, v).$$

(ii) To a bilinear form Φ on V, let A_Φ denote its Gram-matrix w.r.t. β. Show that the map $\Phi \mapsto A_\Phi$ is a linear isomorphism from $\mathrm{Sesq}(V)$ onto the space $M_n(K)$ of $n \times n$-square matrices over K.
(iii) What is the dimension of $\mathrm{Sesq}(V)$?

Homework

H 27 Determine the rank and signature of the quadratic form on $\mathbb{R}^3$ defined by

$$q(x_1, x_2, x_3) = x_1x_2 + 2x_1x_3 + x_3^2.$$

H 28 Let V be the *real* vector space of complex 2×2 hermitean matrices and let $W \subset V$ be the subspace of matrices with trace (i.e. sum of the diagonal entries) equal to 0. Show:

(i) $q = \det$ is a quadratic form on V, i.e. there is a symmetric bilinear form Φ with $\Phi(v, v) = q(v)$.

(ii) The symmetric bilinear form Φ is negative definite on W. I.e. $\Phi(w, w) < 0$ for all $w \in W \setminus \{0\}$.

H 29 Let V be a vector space over some field K with involution $*$ in which $2 = 1 + 1 \neq 0$. Recall that a $*$-sesquilinear form Ψ on V is called $*$-*hermitean* (resp. $*$-*skew hermitean*), if $\Psi = \Psi^*$ (resp. $\Psi = -\Psi^*$). Show that every $*$-sesquilinear form Φ on V is the unique sum of a $*$-hermitean and a $*$-skew hermitean sesquilinear form.

Please note that the next exercise groups will, instead of Thursday, take place on the coming Monday 12.6.2006 3:20 pm-5:00 pm in room S2 15/201. Homework solutions may also be submitted on Thursday, 22.6.2006.

Linear Algebra II (MCS), SS 2006, Exercise 8, Solution

Groupwork

G 33 We put $u = (x_1, x_2)^t$ and $v = (y_1, y_2)^t$. Let f be the bilinear form on $\mathbb{R}^2$ defined by

$$f(u, v) = 2x_1y_1 - 3x_1y_2 + x_2y_2.$$

(i) Determine the Gram-matrix A of f w.r.t. the basis $\{e_1 = (1,0)^t, e_2 = (1,1)^t\}$.
(ii) Determine the Gram-matrix B of f w.r.t. the basis $\{e_1' = (2,1)^t, e_2' = (1,-1)^t\}$.
(iii) Determine the transition matrix P from $\{e_1, e_2\}$ to $\{e_1', e_2'\}$ and verify that $B = P^tAP$.

To (i): *$A = (a_{ij})$ with $a_{ij} = f(e_i, e_j)$. E.g. $a_{12} = f(e_1, e_2) = 2 - 3 + 0 = -1$. This gives $A = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$.*

To (ii): *As above, $B = (b_{ij})$ with $b_{ij} = f(e_i', e_j')$. Thus $B = \begin{pmatrix} 3 & 9 \\ 0 & 6 \end{pmatrix}$.*

To (iii): *$P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ thus $P^t = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $P^tAP = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 0 & 6 \end{pmatrix} = B$.*

G 34 For each of the following matrices $A_i, i = 1, 2$ find matrices P_i, such that $P_i^tA_iP_i$ is diagonal. What is the signature of A_i?

$$(i) \quad A_1 = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix}, \qquad (ii) \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix}.$$

To (i): *First, we form the block matrix $(A_1, I) = \left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ -3 & 7 & -5 & 0 & 1 & 0 \\ 2 & -5 & 8 & 0 & 0 & 1 \end{array}\right)$. Then we proceed by the 'symmetric Gauss algorithm' given in the script. I.e. we transform the left hand block in several steps to diagonal form, where in each step an elementary column operation is followed by the corresponding row operation, whereas the right hand block only records the column operation we have made. For instance, in the left hand block add three times the first column to the second column and then three times the (new) firs row to the (new) second row. In the right hand block just add three times the first column to the second column. This gives $\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 3 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 8 & 0 & 0 & 1 \end{array}\right)$.*

After several such steps we end up with $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 18 & 0 & 0 & 2 \end{array}\right)$. Thus $P_1 = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ and $P_1^tA_1P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 18 \end{pmatrix}$. The signature of A_1 is easily read off as $(2, 1)$.

To (ii): *Just as in (i), we form $(A_2, I) = \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$. As above, we transform this pair and obtain $P_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & 2 & -4 \end{pmatrix}$ and $P_2^tA_2P_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -14 \end{pmatrix}$. The signature of A_2 is $(1, 2)$.*

G 35 Let V be a n-dimensional vector space and let $\beta = \{e_1, \ldots, e_n\}$ be a basis of V. Show that a $*$-sesquilinear form Φ on V is $*$-hermitean, resp. $*$-skew hermitean, if and only if its associated Gram-Matrix A w.r.t. β is $*$-hermitean, resp. $*$-skew hermitean. I.e. A satisfies $A^* = A$, resp. $A^* = -A$.

Suppose that Φ is $*$-hermitean. Since the entries of the Gram-matrix $A = (a_{ij})$ are given by $a_{ij} = \Phi(e_i, e_j)$, we have by symmetry of Φ that $a_{ij} = \Phi(e_i, e_j) = \Phi^*(e_j, e_i) = a_{ji}^*$, hence $A = A^*$. Similarly, if Φ is $*$-skew hermitean one shows that $A = -A^*$. Conversely, suppose that $A = A^*$. Then $\Phi(e_i, e_j) = a_{ij} = a_{ji}^* = \Phi^*(e_j, e_i)$. If now $u, v \in V$ are arbitrary elements, we can write them w.r.t. β as

$u = \sum_{i=1}^n u_i e_i$ and $v = \sum_{i=1}^n v_i e_i$. We therefore have

$$\begin{aligned}\Phi(u,v) &= \Phi(\sum_{i=1}^n u_i e_i, \sum_{j=1}^n v_j e_j) = \sum_{i,j=1}^n u_i^* v_j \Phi(e_i,e_j) = \sum_{i,j=1}^n u_i^* v_j \Phi^*(e_j,e_i) = \sum_{i,j=1}^n (u_i v_j^* \Phi(e_j,e_i))^* \\ &= \left(\sum_{i,j=1}^n v_j^* u_i \Phi(e_j,e_i)\right)^* = \Phi^*(v,u).\end{aligned}$$

Similarly, one shows that $\Phi = -\Phi^*$ if $A = -A^*$.

G 36 Let $P = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Compute

$$(i)\quad P^{-1}AP, \qquad (ii)\quad P^*AP, \qquad (iii)\quad P^tAP.$$

Which transformation is the correct one if we view A as the matrix associated with
(a) a linear endomorphism of $\mathbb{C}^2$
(b) a complex bilinear form on $\mathbb{C}^2$ (i.e. $*$-sesquilinear w.r.t. the trivial involution)
(c) a sesquilinear form on $\mathbb{C}^2$ (i.e. $*$-sequilinear w.r.t. complex conjugation)
and perform a change of basis from $\{(1,0)^t, (0,1)^t\}$ to $\{(i,-1)^t, (i,1)^t\}$ in $\mathbb{C}^2$? If $a = 1, b = 0$, which transformations preserve the eigenvalues of A, which the signs of the eigenvalues and which the invertability of A?

We have $P^{-1} = \begin{pmatrix} -i/2 & -1/2 \\ -i/2 & 1/2 \end{pmatrix}$, $P^* = \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$ and $P^t = \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}$. Then

$$(i)\quad P^{-1}AP = \begin{pmatrix} a & -ib \\ ib & a \end{pmatrix}, \qquad (ii)\quad P^*AP = \begin{pmatrix} 2a & -2ib \\ 2ib & 2a \end{pmatrix}, \qquad (iii)\quad P^tAP = \begin{pmatrix} -2ib & -2a \\ -2a & 2ib \end{pmatrix}.$$

(i) describes the transformation of the linear endomorphism, (ii) describes the transformation of the sequilinear form and (iii) the transformation of the complex bilinear form. Furthermore, (i), (ii) and (iii) preserve the invertability of A, (i) and (ii) preserve the signs of the eigenvalues of A and (i) preserves the eigenvalues of A.

G 37 Let V be a n-dimensional vector space over a field K with involution $*$ and let $\beta = \{e_1, \dots, e_n\}$ be a basis of V.

(i) Show that the set of $*$-sesquilinear forms $\mathrm{Sesq}(V)$ on V is a vector space over K with respect to the following definitions of scalar multiplication and addition for every $\lambda \in K$ and $\Phi, \Psi \in \mathrm{Sesq}(V)$:

$$\lambda \cdot \Phi : V \times V \to K, (u,v) \mapsto \lambda \cdot \Phi(u,v)$$
$$\Phi + \Psi : V \times V \to K, (u,v) \mapsto \Phi(u,v) + \Psi(u,v).$$

(ii) To a bilinear form Φ on V, let A_Φ denote its Gram-matrix w.r.t. β. Show that the map $\Phi \mapsto A_\Phi$ is a linear isomorphism from $\mathrm{Sesq}(V)$ onto the space $M_n(K)$ of $n \times n$-square matrices over K.

(iii) What is the dimension of $\mathrm{Sesq}(V)$?

To (i): *One could either verify all vector space axioms for $\mathrm{Sesq}(V)$, which is a somewhat tedious or instead show that $\mathrm{Sesq}(V)$ is a linear subspace of a suitable vector space. In fact, $\mathrm{Sesq}(V)$ is a subset of the vector space $V^{V\times V}$ of all maps from $V \times V$ to V and the addition and scalar multiplication we have defined, coincides with the ones on $V^{V\times V}$. We just have to show that the zero vector is contained in $\mathrm{Sesq}(V)$ and that for $\Phi, \Psi \in \mathrm{Sesq}(V)$ and $\lambda \in K$ we have $\lambda \cdot \Phi + \Psi \in \mathrm{Sesq}(V)$. In fact, the zero vector in $V^{V\times V}$ is the map which is zero everywhere on $V \times V$. This is clearly a $*$-sesquilinear form. Furthermore,*

$$\begin{aligned}(\lambda\cdot\Phi+\Psi)(u+ru',v) &= \lambda\Phi(u+ru',v) + \Psi(u+ru',v) \\ &= \lambda\Phi(u,v) + \lambda r^*\Phi(u',v) + \Psi(u,v) + r^*\Psi(u',v) \\ &= (\lambda\cdot\Phi+\Psi)(u,v) + r^*(\lambda\cdot\Phi+\Psi)(u',v).\end{aligned}$$

This proves linearity in the first argument. The second argument can be dealt with analogously.

To (ii): *In the script it is already shown that the assignment $\Phi \mapsto A_\Phi$ is a bijection. It remains to show that it is linear. That is, we have to show that $A_{\lambda\cdot\Phi+\Psi} = \lambda \cdot A_\Phi + A_\Psi$. However, since $(\lambda\cdot\Phi+\Psi)(e_i,e_j) = \lambda\Phi(e_i,e_j) + \Psi(e_i,e_j)$, describes on the left hand side the (i,j)-th entry of $A_{\lambda\cdot\Phi+\Psi}$ and the right hand side the (i,j)-th entry of $\lambda \cdot A_\Phi + A_\Psi$, we have proven our claim.*

To (iii): *From (ii) we see that $\dim_K \mathrm{Sesq}(V) = n^2$.*

Homework

H 27 Determine the rank and signature of the quadratic form on $\mathbb{R}^3$ defined by

$$q(x_1, x_2, x_3) = x_1x_2 + 2x_1x_3 + x_3^2.$$

The associated symmetric matrix is $\begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. *Using, for instance, the symmetric Gauss-algorithm we obtain that the signature of q is $(2, 1)$, hence the rank is 3.*

H 28 Let V be the *real* vector space of complex 2×2 hermitean matrices and let $W \subset V$ be the subspace of matrices with trace (i.e. sum of the diagonal entries) equal to 0. Show:

(i) $q = \det$ is a quadratic form on V, i.e. there is a symmetric bilinear form Φ with $\Phi(v, v) = q(v)$.

(ii) The symmetric bilinear form Φ is negative definite on W. I.e. $\Phi(w, w) < 0$ for all $w \in W \setminus \{0\}$.

To (i): *A general 2×2-hermitian matrix has the form $A = \begin{pmatrix} a & b + ic \\ b - ic & d \end{pmatrix}$ with arbitrary $a, b, c, d \in \mathbb{R}$. Hence, $\det A = ad - b^2 - c^2$. The sum of two hermitian matrices as well as a real(!) scalar multiple of a hermitian matrix is again hermitian. Thus V is a subspace of the real vector space of all complex 2×2-matrices. From the above description of a general 2×2-hermitian matrix, we see that V has dimension 4 and a basis is given by*

$$\{e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

W.r.t. this basis, we can write down a Gram-matrix for q, showing that q is indeed a quadratic form: $\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$. *If we put $u = \begin{pmatrix} a_1 & b_1 + ic_1 \\ b_1 - ic_1 & d_1 \end{pmatrix}$ and $v = \begin{pmatrix} a_2 & b_2 + ic_2 \\ b_2 - ic_2 & d_2 \end{pmatrix}$, the associated symmetric bilinear form is $\Phi(u, v) = \frac{1}{2}a_1d_2 + \frac{1}{2}a_2d_1 - b_1b_2 - c_1c_2$.*

To (ii): *A general element of W has the form $B = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix}$ and W is easily recognized as a three-dimensional (real) subspace of V, with a basis given by*

$$\{f_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\}.$$

Now if $w = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix}$, then $\Phi(w, w) = \det \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} = -a^2 - b^2 - c^2 < 0$, if $w \neq 0$ and we have shown that Φ is indeed negative definite on W.

H 29 Let V be a vector space over some field K with involution $*$ in which $2 = 1 + 1 \neq 0$. Recall that a $*$-sesquilinear form Ψ on V is called $*$-*hermitean* (resp. $*$-*skew hermitean*), if $\Psi = \Psi^*$ (resp. $\Psi = -\Psi^*$). Show that every $*$-sesquilinear form Φ on V is the unique sum of a $*$-hermitean and a $*$-skew hermitean sesquilinear form.

Put $\Phi_1 := \frac{1}{2}(\Phi + \Phi^*)$ and $\Phi_2 := \frac{1}{2}(\Phi - \Phi^*)$. Obviously $\Phi = \Phi_1 + \Phi_2$ and $\Phi_1^* = \Phi_1$ as well as $\Phi_2^* = -\Phi_2$. This proves one half of the claim. Suppose now that $\Phi = \Psi_1 + \Psi_2$ for some $*$-hermitian Ψ_1 and some $*$-skew hermitian Ψ_2. From our first desomposition we obtain the equation $\Psi_1 - \Phi_1 = \Phi_2 - \Psi_2$. It is easy to verify that the sets of $*$-hermitian and $*$-skew hermitian sesquilinear forms each form a linear subpace of the space of all $*$-sesquilinear forms $\mathrm{Sesq}(V)$ on V. So on the left hand side of the equation is a $*$-hermitian sesquilinear form and on the right hand side is a $*$-skew hermitian sesquilinear form. However, $\Lambda = \Lambda^* = -\Lambda^*$ shows that $\Lambda = 0$. Hence Ψ_i and Φ_i coincide for $i = 1, 2$ and we have shown that the decomposition is unique.

Please note that the next exercise groups will, instead of Thursday, take place on the coming Monday 12.6.2006 3:20 pm-5:00 pm in room S2 15/201. Homework solutions may also be submitted on Thursday, 22.6.2006.