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DARMSTADT

1. Juni 2006

Linear Algebra II (MCS), SS 2006, Exercise 7

Groupwork

G 28 (i) Let $u = (x_1, x_2)^t$ and $v = (y_1, y_2)^t$. Which of the following expressions are bilinear forms on $\mathbb{R}^2$?

$$\begin{array}{llll} (a) & f(u,v) = 2x_1y_2 - 3x_2y_1, & (d) & f(u,v) = x_1x_2 + y_1y_2, \\ (b) & f(u,v) = x_1 + y_2, & (e) & f(u,v) = 1, \\ (c) & f(u,v) = 3x_2y_2, & (f) & f(u,v) = 0. \end{array}$$

(ii) Let $u = (x_1, x_2, x_3)^t$ and $v = (y_1, y_2, y_3)^t$. Determine the matrix A associated with the map:

$$f(u,v) := 3x_1y_1 - 2x_1y_2 + 5x_2y_1 + 7x_2y_2 - 8x_2y_3 + 4x_3y_2 - x_3y_3.$$

(iii) Let A be a $n \times n$ matrix over K. Show that the map $f(u,v) = u^t \cdot A \cdot v$ is a bilinear form on K^n.

G 29 On $\mathbb{R}^2$ consider the quadratic form $f(x,y) = \lambda x^2 + \mu y^2$ with parameters $\lambda > \mu \in \mathbb{R}$.

(i) Let $S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$ denote the unit circle. What are the extremal values of $f|_{S^1}$?

(ii) For $\lambda = 1$ and $\mu = -\frac{1}{4}$ show that the isohypsis of height 0, i.e. the level set $M_0 := \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$, is the union of two intersecting lines. Draw a picture of them together with the isohypsis M_1 of height 1. What can you say about the distance of a point P on M_0 to M_1, as the distance of P to the origin goes to infinity?

G 30 Let ϕ be an endomorphism of a vector space V over K.

(i) Show that the following subspaces of V are invariant under ϕ:

$$(a) \quad \{0\}, \qquad (b) \quad V, \qquad (c) \quad \ker(\phi), \qquad (d) \quad \mathrm{im}(\phi).$$

(ii) Let $W_i, i \in I$ be a collection of ϕ-invariant subspaces of V. Show that $\bigcap_{i \in I} W_i$ is also ϕ-invariant.

(iii) Let $p \in K[t]$ be an arbitrary polynomial over K. Show that $\ker(p(\phi))$ is invariant under ϕ.

G 31 Let $\phi : V \to V$ be an endomorphism and $V = U \oplus W$. Show that U and W are both invariant under ϕ if and only if $\phi \circ \pi = \pi \circ \phi$, where $\pi : V \to V$ denotes the projection of V along W onto U.

G 32 Let V be a vector space and $W_1, \dots, W_r$ linear subspaces of V.

(i) Is it true that $V = W_1 \oplus \dots \oplus W_r$ if and only if $V = W_1 + \dots + W_r$ and $W_i \cap W_j = \{0\}$ for $i \neq j$? Give a proof or a counterexample!

(ii) Suppose that $\{w_{i1}, \dots, w_{in_i}\}$ is a basis of W_i for $i = 1, \dots, r$. Show that $V = W_1 \oplus \dots \oplus W_r$ if and only if $\beta = \{w_{11}, \dots, w_{1n_1}, \dots, w_{r1}, \dots, w_{rn_r}\}$ is a basis of V.

(iii) Let $U, W \subset V$ be linear subspaces with $V = U \oplus W$. Suppose that $U = U_1 \oplus U_2$ and $W = W_1 \oplus W_2$. Show that $V = U_1 \oplus U_2 \oplus W_1 \oplus W_2$.

Homework

H 23 Determine the symmetric matrices associated to each of the following quadratic forms:

$$\begin{array}{llll} (i) & q(x,y) = 4x^2 - 6xy - 7y^2, & (iii) & q(x,y,z) = 3x^2 + 4xy - y^2 + 8xz - 6yz + z^2, \\ (ii) & q(x,y) = xy + y^2, & (iv) & q(x,y,z) = x^2 - 2yz + xz. \end{array}$$

H 24 Let V be an n-dimensional vector space. Show that an endomorphism $\phi : V \to V$ has a triangular matrix representation, if and only if there exist ϕ-invariant subspaces $W_1 \subset W_2 \subset \dots \subset W_n = V$, such that $\dim W_i = i$, for $i = 1, \dots, n$.

H 25 Let $P = (-p, 0)^t, Q = (p, 0)^t$ and $X = (x_1, x_2)^t$ be points in $\mathbb{R}^2$ with $p > 0$ fixed and denote by r_P, resp. r_Q, the distance of X to P, resp. Q, in the Euclidean distance. Show that the set E of all X satisfying $r_P + r_Q = 2c$ for some constant $c > p$ is an ellipse. I.e. $E = \{X \in \mathbb{R}^2 \mid \frac{x_1^2}{\lambda^2} + \frac{x_2^2}{\mu^2} = 1\}$ with $\lambda, \mu \in \mathbb{R}^{>0}$. Remark: Gardeners use this principle to create elliptically shaped flower beds.

H 26 Let ϕ and ψ be diagonalizable endomorphisms of an n-dimensional vector space V. Show that ϕ and ψ commute if and only if they can be simultaneously diagonalized. I.e. $\phi \circ \psi = \psi \circ \phi$ if and only if there is a basis $\{e_1, \dots, e_n\}$ of V, such that the matrices of ϕ, resp. ψ, w.r.t. this basis are diagonal.

Due to the holiday 'Pfingstmontag' on 5.6.2006, lectures will instead take place on Thu. 8.6.2006 8:00 am - 9:40 am in room S1 03/123.

Linear Algebra II (MCS), SS 2006, Exercise 7, Solution

Groupwork

G 28 (i) Let $u = (x_1, x_2)^t$ and $v = (y_1, y_2)^t$. Which of the following expressions are bilinear forms on $\mathbb{R}^2$?

$$\begin{array}{llll} (a) & f(u,v) = 2x_1y_2 - 3x_2y_1, & (d) & f(u,v) = x_1x_2 + y_1y_2, \\ (b) & f(u,v) = x_1 + y_2, & (e) & f(u,v) = 1, \\ (c) & f(u,v) = 3x_2y_2, & (f) & f(u,v) = 0. \end{array}$$

(ii) Let $u = (x_1, x_2, x_3)^t$ and $v = (y_1, y_2, y_3)^t$. Determine the matrix A associated with the map:

$$f(u,v) := 3x_1y_1 - 2x_1y_2 + 5x_2y_1 + 7x_2y_2 - 8x_2y_3 + 4x_3y_2 - x_3y_3.$$

(iii) Let A be a $n \times n$ matrix over K. Show that the map $f(u,v) = u^t \cdot A \cdot v$ is a bilinear form on K^n.

To (i): *The maps in (a), (c) and (f) are bilinear forms, since they can be represented by matrices* $\begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *respectively. The other forms fail to be bilinear because* $f(0, (1,1)^t) = 1 \neq 0$ *in each case.*

To (ii): $A = \begin{pmatrix} 3 & -2 & 0 \\ 5 & 7 & -8 \\ 0 & 4 & -1 \end{pmatrix}$.

To (iii): *We have*

$$\begin{aligned} f(\lambda \cdot u + \tilde{u}, v) &= (\lambda \cdot u + \tilde{u})^t \cdot A \cdot v = (\lambda \cdot u^t + \tilde{u}^t) \cdot A \cdot y = \lambda \cdot (u^t \cdot A \cdot v) + \tilde{u}^t \cdot A \cdot v \\ &= \lambda \cdot f(u,v) + f(\tilde{u}, v), \end{aligned}$$

proving linearity in the first argument. Linearity in the second argument is proven analogously.

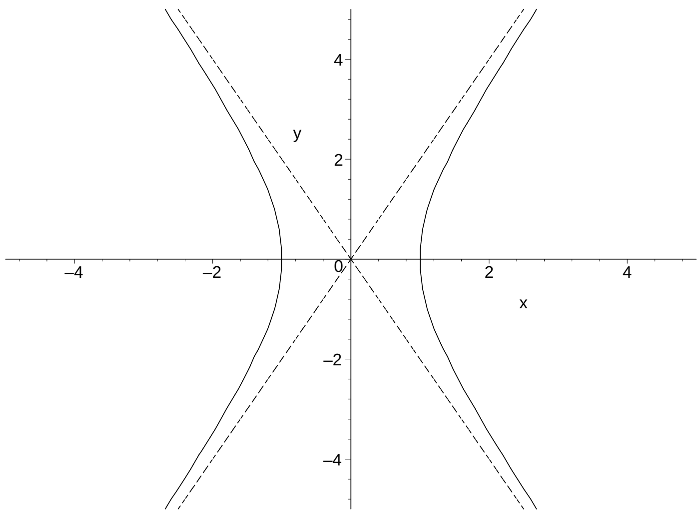
G 29 On $\mathbb{R}^2$ consider the quadratic form $f(x,y) = \lambda x^2 + \mu y^2$ with parameters $\lambda > \mu \in \mathbb{R}$.

(i) Let $S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$ denote the unit circle. What are the extremal values of $f|_{S^1}$?

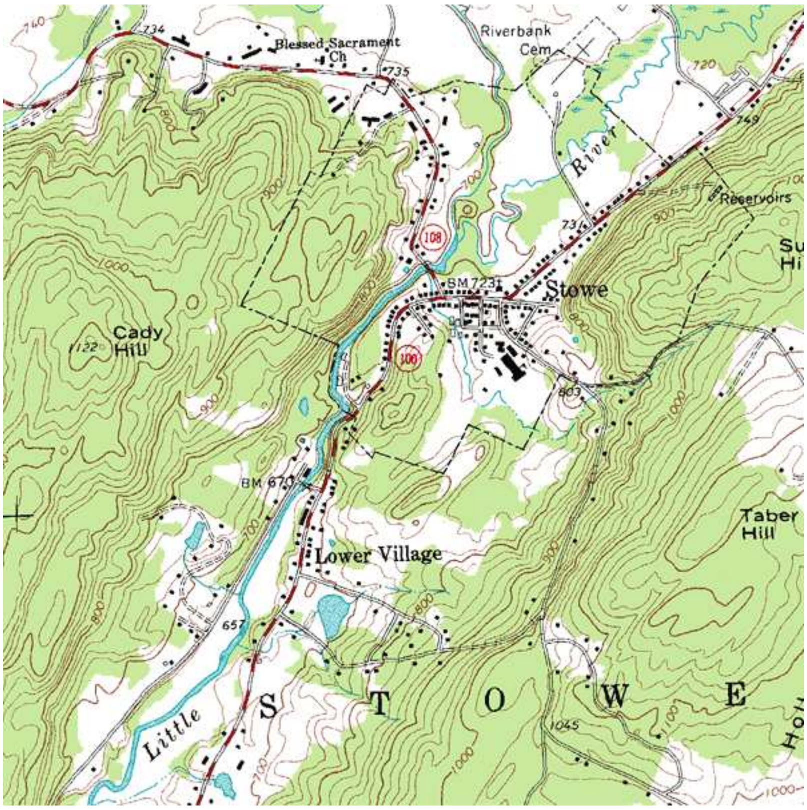
(ii) For $\lambda = 1$ and $\mu = -\frac{1}{4}$ show that the isohypsis of height 0, i.e. the level set $M_0 := \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$, is the union of two intersecting lines. Draw a picture of them together with the isohypsis M_1 of height 1. What can you say about the distance of a point P on M_0 to M_1, as the distance of P to the origin goes to infinity?

To (i): *For* $(x,y) \in S^1$, *we have* $f(x,y) = \lambda x^2 + \mu y^2 = \lambda x^2 + \mu(1 - x^2) = (\lambda - \mu)x^2 + \mu$. *Since* x *ranges through the interval* $[-1, 1]$ *we see that the extremal values are assumed for* $(0, \pm 1)$ *and* $(\pm 1, 0)$ *and the corresponding values are* μ *and* λ, *respectively.*

To (ii): *We have* $f(x,y) = 0 \Leftrightarrow x^2 = \frac{1}{4}y^2$. *Hence,* $M_0 = l_1 \cup l_2$, *where* $l_1 = \{(x,y) \in \mathbb{R}^2 \mid x = \frac{1}{2}y\}$ *and* $l_2 = \{(x,y) \in \mathbb{R}^2 \mid x = -\frac{1}{2}y\}$.



The distance of P *to* M_1 *goes to zero as the distance of* P *to the origin goes to infinity.* l_1 *and* l_2 *are the asymptotes of* M_1. *The following graphic shows an example of a topographic map with drawn isohypses:*



G 30 Let ϕ be an endomorphism of a vector space V over K.

(i) Show that the following subspaces of V are invariant under ϕ:

$$(a)\quad \{0\}, \qquad (b)\quad V, \qquad (c)\quad \ker(\phi), \qquad (d)\quad \operatorname{im}(\phi).$$

(ii) Let $W_i, i \in I$ be a collection of ϕ-invariant subspaces of V. Show that $\bigcap_{i\in I} W_i$ is also ϕ-invariant.

(iii) Let $p \in K[t]$ be an arbitrary polynomial over K. Show that $\ker(p(\phi))$ is invariant under ϕ.

To (i): *(a) and (b) are trivial. Since $0 \in \ker(\phi)$, (c) is clear as well. Furthermore, $\operatorname{im}(\phi) = \phi(V) \subset V$. Therefore, $\phi(\operatorname{im}(\phi)) \subset \operatorname{im}(\phi)$.*

To (ii): *Let $w \in \bigcap_{i\in I} W_i$ be arbitrary. Then $w \in W_i$ for all $i \in I$. Hence, $\phi(w) \in W_i$ for all $i \in I$, whence $\phi(w) \in \bigcap_{i\in I} W_i$.*

To (iii): *Let $v \in \ker(p(\phi))$ be arbitrary. Since $p(\phi)\circ\phi = \phi\circ p(\phi)$, it follows that $p(\phi)(\phi(v)) = (\phi\circ p(\phi))(v) = 0$. Hence, $\ker(\phi)$ is ϕ-invariant.*

G 31 Let $\phi : V \to V$ be an endomorphism and $V = U \oplus W$. Show that U and W are both invariant under ϕ if and only if $\phi \circ \pi = \pi \circ \phi$, where $\pi : V \to V$ denotes the projection of V along W onto U.

First suppose that U and W are ϕ-invariant. Let $v \in V$ be arbitrary. Since $V = U \oplus W$, we have a unique decomposition $v = u + w$ with $u \in U$ and $w \in W$ and hence $\pi(v) = u$. Then

$$(\phi \circ \pi)(v) = \underbrace{\phi(u)}_{\in U} = (\pi \circ \phi)(u) + \underbrace{0}_{=(\pi\circ\phi)(w)} = (\pi \circ \phi)(u + w) = (\pi \circ \phi)(v).$$

Next, suppose that ϕ and π commute. Then $\phi(U) = \phi(\pi(U)) = (\phi \circ \pi)(U) = (\pi \circ \phi)(U) \subset U$ showing that U is ϕ-invariant. If now $w \in W$ is arbitrary, then $\pi(\phi(w)) = \phi(\pi(w)) = \phi(0) = 0$ and $\phi(w) \in \ker(\pi) = W$. Therefore $\phi(W) \subset W$.

G 32 Let V be a vector space and $W_1, \ldots, W_r$ linear subspaces of V.

(i) Is it true that $V = W_1 \oplus \cdots \oplus W_r$ if and only if $V = W_1 + \cdots + W_r$ and $W_i \cap W_j = \{0\}$ for $i \neq j$? Give a proof or a counterexample!

(ii) Suppose that $\{w_{i1}, \ldots, w_{in_i}\}$ is a basis of W_i for $i = 1, \ldots, r$. Show that $V = W_1 \oplus \cdots \oplus W_r$ if and only if $\beta = \{w_{11}, \ldots, w_{1n_1}, \ldots, w_{r1}, \ldots, w_{rn_r}\}$ is a basis of V.

(iii) Let $U, W \subset V$ be linear subspaces with $V = U \oplus W$. Suppose that $U = U_1 \oplus U_2$ and $W = W_1 \oplus W_2$. Show that $V = U_1 \oplus U_2 \oplus W_1 \oplus W_2$.

To (i): *No. Take for instance $V = \mathbb{R}^2$ and for W_1, W_2, W_3 any three lines which pairwise only intersect in the origin. Then $V = W_1 + W_2 + W_3$ but the sum is not direct for dimension reasons.*

To (ii): *This is part (4) of Theorem 34.1 in the finite dimensional case.*

To (iii): *Let $\{u_{11},\dots,u_{1n_1}\},\{u_{21},\dots,u_{2n_2}\},\{w_{11},\dots,w_{1n_3}\}$ and $\{w_{21},\dots,w_{2n_4}\}$ be bases of U_1,U_2,W_1 and W_2, respectively. By (ii), or part (4) of Theorem 34.1, $\{u_{11},\dots,u_{1n_1},u_{21},\dots,u_{2n_2}\}$ is a basis of U and $\{w_{11},\dots,w_{1n_3},w_{21},\dots,w_{2n_4}\}$ is a basis of W. Since $V=U\oplus W$ and (ii) again, we have that $\{u_{11},\dots,u_{1n_1},u_{21},\dots,u_{2n_2},w_{11},\dots,w_{1n_3},w_{21},\dots,w_{2n_4}\}$ is a basis of V. Invoking (ii) for a third time, we conclude that $V=U_1\oplus U_2\oplus W_1\oplus W_2$.*

Homework

H 23 Determine the symmetric matrices associated to each of the following quadratic forms:

$$(i)\quad q(x,y)=4x^2-6xy-7y^2,\qquad (iii)\quad q(x,y,z)=3x^2+4xy-y^2+8xz-6yz+z^2,$$
$$(ii)\quad q(x,y)=xy+y^2,\qquad (iv)\quad q(x,y,z)=x^2-2yz+xz.$$

$$(i)\quad \begin{pmatrix}4&-3\\-3&-7\end{pmatrix},\quad (iii)\quad \begin{pmatrix}3&2&4\\2&-1&-3\\4&-3&1\end{pmatrix},$$
$$(ii)\quad \begin{pmatrix}0&\frac12\\\frac12&1\end{pmatrix},\quad (iv)\quad \begin{pmatrix}1&0&\frac12\\0&0&-1\\\frac12&-1&0\end{pmatrix}.$$

H 24 Let V be an n-dimensional vector space. Show that an endomorphism $\phi: V\to V$ has a triangular matrix representation, if and only if there exist ϕ-invariant subspaces $W_1\subset W_2\subset\dots\subset W_n=V$, such that $\dim W_i=i$, for $i=1,\dots,n$.

Let ϕ have a triangular matrix representation. That is, there exists some basis $\beta=\{e_1,\dots,e_n\}$ of V such that the matrix $A=(a_{ij})$ of ϕ w.r.t. β is in lower triangular form. By the latter one we mean that $a_{ij}=0$ for $i<j$. The proof works slightly modified for the upper triangular form, too. Now A being triangular implies that $\phi(e_i)=\sum_{j=1}^i a_{ij}e_j$. If we therefore put $W_i=\operatorname{span}\{e_1,\dots,e_i\}$ then $W_1\subset W_2\subset\dots\subset W_n=V$, $\dim W_i=i$ for $i=1,\dots,n$ and furthermore $\phi(W_i)\subset W_i$.
Conversely, given $W_1\subset W_2\subset\dots\subset W_n=V$ as above, we construct a basis β_i of W_i inductively as follows. For $i=1$ put $\beta_1=\{e_1\}$, where $e_1\in W_1\setminus\{0\}$ is an arbitrary element. For $i>1$ suppose that we have already constructed β_{i-1}. Since β_{i-1} is a system of linear independent vectors in W_i and $\dim W_i=i$, we may complete β_{i-1} by an element of W_i to a basis β_i of W_i. By ϕ-invariance of the W_i we conclude that $\phi(e_i)=\sum_{j=1}^i a_{ij}e_j$ for some coefficients a_{ij} in the ground field. If we put $a_{ij}:=0$ for $i<j$, then the matrix $A:=(a_{ij})$ is obviously a triangular representation matrix of ϕ.

H 25 Let $P=(-p,0)^t, Q=(p,0)^t$ and $X=(x_1,x_2)^t$ be points in $\mathbb{R}^2$ with $p>0$ fixed and denote by r_P, resp. r_Q, the distance of X to P, resp. Q, in the Euclidean distance. Show that the set E of all X satisfying $r_P+r_Q=2c$ for some constant $c>p$ is an ellipse. I.e. $E=\{X\in\mathbb{R}^2\mid \frac{x_1^2}{\lambda^2}+\frac{x_2^2}{\mu^2}=1\}$ with $\lambda,\mu\in\mathbb{R}^{>0}$. Remark: Gardeners use this principle to create elliptically shaped flower beds.

We have $r_P=\sqrt{(x_1+p)^2+x_2^2}$ and $r_Q=\sqrt{(x_1-p)^2+x_2^2}$. To get rid of the square roots in the equation $r_P+r_Q=2c$, we square it first and obtain $r_P^2+r_Q^2+2r_Pr_Q=4c^2$, which we rearrange to $2r_Pr_Q=4c^2-r_P^2-r_Q^2$ and square again to finally arrive at

$$4r_P^2r_Q^2=(4c^2-r_P^2-r_Q^2)^2. \qquad (*)$$

Now we have $r_P^2=x_1^2+x_2^2+p^2+2x_1p$ and $r_Q^2=x_1^2+x_2^2+p^2-2x_1p$, such that $r_P^2r_Q^2=(x_1^2+x_2^2+p^2)^2-4x_1^2p^2$ and $r_P^2+r_Q^2=2(x_1^2+x_2^2+p^2)$. Thus equation $(*)$ becomes

$$\begin{aligned}4(x_1^2+x_2^2+p^2)^2-16x_1^2p^2 &= (4c^2-2(x_1^2+x_2^2+p^2))^2\\ &= 16c^4-16c^2(x_1^2+x_2^2+p^2)+4(x_1^2+x_2^2+p^2)^2.\end{aligned}$$

The fourth powers cancel out and we end up with an equation of degree two, which we rewrite in the final form

$$\frac{x_1^2}{c^2}+\frac{x_2^2}{\sqrt{c^2-p^2}^2}=1.$$

Note how the condition $c>p$ enters in the calculation. Of course, there is a geometric reason for this condition. Can you see it?

If we put $\lambda:=c$ and $\mu:=\sqrt{c^2-p^2}$ then we have shown that all points of E lie on the ellipse described by $\frac{x_1^2}{\lambda^2}+\frac{x_2^2}{\mu^2}=1$.

Conversely, we have to show that every point X on the ellipse above is also a point of E. In fact, all transformations we have made were equivalence transformations. This is clear except for the two times we squared the equations. The first time we did, both sides of the equation $r_P + r_Q = 2c$ were positive and squaring the equation is an equivalence transformation. The second time, we squared the equation $2r_P r_Q = 4c^2 - r_P^2 - r_Q^2$ and the left hand side is clearly nonnegative as both r_P and r_Q are nonnegative by definition. The right hand side involves a little estimate: Note that the defining relation of the ellipse $\frac{x_1^2}{\lambda^2} + \frac{x_2^2}{\mu^2} = 1$ implies $x_1^2 \leq \lambda^2 = c^2$ and $x_2^2 \leq \mu^2 = c^2 - p^2$. Then

$$4c^2 - r_P^2 - r_Q^2 = 4c^2 - 2x_1^2 - 2x_2^2 - 2p^2 \geq 4c^2 - 2c^2 - 2c^2 + 2p^2 - 2p^2 \geq 0.$$

Thus, the second time we squared, we also did an equivalence transformation. Since all transformations can be performed in both directions, we have shown that $E = \{X \in \mathbb{R}^2 \mid \frac{x_1^2}{\lambda^2} + \frac{x_2^2}{\mu^2} = 1\}$ with $\lambda = c$ and $\mu = \sqrt{c^2 - p^2}$.

H 26 Let ϕ and ψ be diagonalizable endomorphisms of an n-dimensional vector space V. Show that ϕ and ψ commute if and only if they can be simultaneously diagonalized. I.e. $\phi \circ \psi = \psi \circ \phi$ if and only if there is a basis $\{e_1, \ldots, e_n\}$ of V, such that the matrices of ϕ, resp. ψ, w.r.t. this basis are diagonal.

Let $\beta\{e_1, \ldots, e_n\}$ be a basis as above and let $A = (a_{ij})$, resp. $B = (b_{ij})$ be the matrices of ϕ, resp. ψ w.r.t. this base. We have $a_{ij} = b_{ij} = 0$ for $i \neq 0$, by assumption. Now let $v \in V$ be arbitrary and let $v = \sum_{i=1}^n v_i e_i$ be its representation w.r.t. β. Then

$$\begin{aligned}(\phi \circ \psi)(v) &= (\phi \circ \psi)(\sum_{i=1}^n v_i e_i) = \phi(\sum_{i=1}^n v_i \psi(e_i)) = \phi(\sum_{i=1}^n v_i b_{ii} e_i) = \sum_{i=1}^n v_i a_{ii} b_{ii} e_i \\ &= (\psi \circ \phi)(v).\end{aligned}$$

It follows that ϕ and ψ commute.

*Conversely, suppose that ϕ and ψ commute. Let $E_1 \oplus \cdots \oplus E_r$ be the decomposition of V into eigenspaces of ϕ. We first claim that E_k is ψ-invariant. Let $v \in E_k$ be arbitrary. Then $\phi(v) = \lambda_k v$, where λ_k denotes the eigenvalue of ϕ corresponding to E_k. We have $\phi(\psi(v)) = \psi(\phi(v)) = \lambda_k \psi(v)$, wherefore $\psi(v) \in E_k$. Since each E_k is ψ-invariant, we have a ψ-invariant decomposition $V = E_1 \oplus \cdots \oplus E_r$ and if $\pi_k : V \to V$ denotes the projection of V onto E_k along $\bigoplus_{j=1, j\neq k}^r E_j$, then by exercise **G 31**, $\pi_k \circ \psi = \psi \circ \pi_k$. If now v is an arbitrary eigenvector of ψ to some eigenvalue, say μ, then $\psi(v) = \mu v$ implies $\mu \pi_k(v) = \pi_k(\psi(v)) = \psi(\pi_k(v))$. It follows that $\pi_k(v)$ is either zero or an eigenvector of ψ to the eigenvalue μ again. In any case, the unique decomposition $v = \sum_{k=1}^r \pi_k(v)$ with respect to the distinct eigenspaces of ϕ is also a decomposition of v into eigenvectors of ψ (or zero vectors) to a given eigenvalue. If we denote by $F_1, \ldots, F_s$ the distinct eigenspaces of ψ, it follows that $V_{kl} := \pi_k(F_l) = E_k \cap F_l$. Hence, $F_l = \bigoplus_{k=1}^r \pi_k(F_l)$ and $V = \bigoplus_{l=1}^s F_l = \bigoplus_{k=1}^r \bigoplus_{l=1}^s V_{kl}$ is a direct decomposition of V into ϕ- and ψ-invariant subspaces consisting of eigenvectors of both ϕ and ψ. Taking a basis of each V_{kl} and concatenating them yields a basis of V which diagonalizes both ϕ and ψ.*

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