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Linear Algebra II (MCS), SS 2006, Exercise 6

Mini-Quiz

(1) If $\phi : V \to V$ is an endomorphism and λ is an eigenvalue of ϕ, then the eigenspace U_λ of ϕ to the eigenvalue λ is...?
- ☐ The set of all eigenvectors to the eigenvalue λ.
- √ The set consisting of all eigenvectors to the eigenvalue λ and the zero vector.
- ☐ $\ker(\lambda \cdot \mathrm{id})$.

(2) An endomorphism ϕ of an n-dimensional vector space is diagonalizable if and only if...?
- ☐ ϕ has n distinct eigenvalues.
- ☐ ϕ has only one eigenvalue whose geometric multiplicity is equal to n.
- √ n is equal to the sum of the geometric multiplicities of the distinct eigenvalues of ϕ.

(3) Suppose that A and B are $n \times n$-matrices and that there exists an invertible $n \times n$-matrix P such that $A = PBP^{-1}$. Then A and B have...?
- √ The same eigenvalues.
- ☐ The same eigenvectors.
- ☐ The same eigenspaces.

Groupwork

G 23 Compute the eigenvalues and eigenvectors of the following matrices over $\mathbb{R}$, resp. over $\mathbb{C}$:

$$(i) \quad A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \qquad (ii) \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Also determine transformation matrices P, Q and their inverses P^{-1}, Q^{-1}, such that $P^{-1}AP$, resp. $Q^{-1}BQ$ are diagonal matrices.

G 24 Let

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

Determine the eigenspaces. Can A be diagonalized?
Hint: Use that the characteristic polynomial of A is $\chi_A(x) = -(x+2)^2(x-4)$.

G 25 Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Determine the eigenvalues and eigenspaces of A and show that A can be diagonalized over $\mathbb{C}$, but not over $\mathbb{R}$. Find a *real* transformation matrix P, such that $P^{-1}AP$ is of the form $\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$, where $\lambda \in \mathbb{R}$ and B is a real 2×2 square matrix.

G 26 Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -2 & -1 & -1 \end{pmatrix}.$$

(i) Show that for $p(x) = x^3 - x$ we have $p(A) = 0$.
(ii) Compute A^{2006}.

Remark: $-p(x)$ is the characteristic polynomial of A.

G 27 Let V be an n-dimensional vector space and $U, W \subset V$ complementary subspaces. I.e. $V = U + W$ and $U \cap W = \{0\}$. Give a geometric interpretation of the real eigenvalues and eigenvectors of the following endomorphisms: Reflection of V on U along W, Projection of V onto U along W, Central scaling of V, Scaling of V in n linearly independent directions.
Hint: The first step to the solution of this exercise is to write down the correct definition of each endomorphism. Thinking of $n = 2$ or $n = 3$ may help your imagination.

Homework

H 17 In each of the following cases compute the eigenvalues and eigenvectors of the given matrix. Diagonalize the matrix over $\mathbb{R}$ or $\mathbb{C}$, if possible:

$$(i) \quad \begin{pmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix}, \qquad (ii) \quad \begin{pmatrix} -4 & 4 & 4 & -1 \\ -3 & 4 & 3 & -1 \\ -5 & 4 & 5 & -1 \\ -5 & 4 & 5 & -1 \end{pmatrix}$$

H 18 The elements of the sequence $u_1, u_2, \dots$ given by the initial values $u_1 = u_2 = 1$ and the recurrence relation

$$u_{n+1} = u_{n-1} + u_n$$

are called *Fibonacci numbers*. The first 14 Fibonacci numbers were produced for the first time in 1228 in the manuscripts of Leonardo of Pisa (Fibonacci).

(i) Compute the first 8 Fibonacci numbers.

(ii) We write the Fibonacci numbers as entries of a vector in the following way:

$$x_n := \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$$

for $n = 2, 3, \dots$. Find a matrix A such that

$$Ax_n = x_{n+1}.$$

Express x_n in terms of x_2 and A.

(iii) Derive $u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ (*Binet's formula*) by computing x_n with help of the above.

H 19 In exercise **G 25** the basis of the eigenspaces, and hence the matrix P, can be chosen such that B is a spiral $\begin{pmatrix} r \cdot \cos\alpha & -r \cdot \sin\alpha \\ r \cdot \sin\alpha & r \cdot \cos\alpha \end{pmatrix}$, with $r > 0$ and $\alpha \in [0, 2\pi)$. Determine such a P as well as r and α.

H 20 Let T be the endomorphism $D^2 + D + \mathrm{id}$ on the vector space of polynomials of degree $\leq n$ (c.f. exercise **G 20**). I.e. $Tp(x) = p''(x) + p'(x) + p(x)$. Can T be diagonalized? (**Corr.:** We assume that $K = \mathbb{R}$ or $K = \mathbb{C}$!)

H 21 Compute the characteristic polynomial of

$$A := \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix},$$

with $a_i \in K$ for $i = 0, 1, \dots, n-1$.

H 22 (i) Given a polynomial of degree n of the form **Corr.:** $p(x) = (-1)^n(x^n + a_{n-1}x^{n-1} + \dots + a_0) \in K[x]$, is there always an endomorphism ϕ of an n-dimensional vector space V over K, such that p is the characteristic polynomial of ϕ?

(ii) For each of the following polynomials p, q, give a matrix which has p, resp. q as its characteristic polynomial.

(i) **(Corr.:)** $p(x) = -(x^3 - 5x^2 + 6x + 8)$, $\qquad (ii)$ $q(x) = x^4 - 5x^3 + 7x + 4$.

Hint: Exercise **H 21** may be helpful for the solution.

Linear Algebra II (MCS), SS 2006, Exercise 6, Solution

Mini-Quiz

(1) If $\phi : V \to V$ is an endomorphism and λ is an eigenvalue of ϕ, then the eigenspace U_λ of ϕ to the eigenvalue λ is...?
- ☐ The set of all eigenvectors to the eigenvalue λ.
- √ The set consisting of all eigenvectors to the eigenvalue λ and the zero vector.
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(2) An endomorphism ϕ of an n-dimensional vector space is diagonalizable if and only if...?
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(3) Suppose that A and B are $n \times n$-matrices and that there exists an invertible $n \times n$-matrix P such that $A = PBP^{-1}$. Then A and B have...?
- √ The same eigenvalues.
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Groupwork

G 23 Compute the eigenvalues and eigenvectors of the following matrices over $\mathbb{R}$, resp. over $\mathbb{C}$:

$$(i) \quad A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \qquad (ii) \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Also determine transformation matrices P, Q and their inverses P^{-1}, Q^{-1}, such that $P^{-1}AP$, resp. $Q^{-1}BQ$ are diagonal matrices.

To (i): The real (and thus also complex) eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -1$. The corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Hence, $P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ and $P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$.

To (ii): The (complex) eigenvalues are $\lambda_1 = 1 - i$ and $\lambda_2 = 1 + i$. The corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Hence, $Q = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ and $Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$.

G 24 Let

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

Determine the eigenspaces. Can A be diagonalized?
Hint: Use that the characteristic polynomial of A is $\chi_A(x) = -(x+2)^2(x-4)$.

The eigenspace to the eigenvalue $\lambda_1 = -2$ is $\mathbb{R} \cdot (1,1,0)^t$. The eigenspace to the eigenvalue $\mu = 4$ is $\mathbb{R} \cdot (0,1,1)^t$. The geometric multiplicity of λ is therefore less than its algebraic multiplicity and A is not diagonalizable.

G 25 Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Determine the eigenvalues and eigenspaces of A and show that A can be diagonalized over $\mathbb{C}$, but not over $\mathbb{R}$. Find a *real* transformation matrix P, such that $P^{-1}AP$ is of the form $\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$, where $\lambda \in \mathbb{R}$ and B is a real 2×2 square matrix.

The characteristic polynomial of A is $\chi_A(x) = -x(x^2+3) = -x(x - i\sqrt{3})(x + i\sqrt{3})$. Therefore A has one real eigenvalue $\lambda_1 = 0$ and two imaginary eigenvalues $\lambda_2 = i\sqrt{3}$ and $\lambda_3 = -i\sqrt{3}$, which are complex conjugated to each other. The corresponding complex eigenvectors are $v_1 = (1,1,-1)^t$, $v_2 = (1 - i\sqrt{3}, 1 + i\sqrt{3}, 2)^t$ and $v_3 = (1 + i\sqrt{3}, 1 - i\sqrt{3}, 2)^t$. Note that $\overline{v_2} = v_3$.

Since there are three different eigenvalues over $\mathbb{C}$, we conclude that A is diagonalizable over $\mathbb{C}$. However, A is not diagonalizable over $\mathbb{R}$. From chapter 32 we know that with respect to the basis $w_1 = v_1, w_2 = \Im(v_2)$ and $w_3 = \Re(v_2)$ our matrix A gets transformed to a block-matrix as claimed. In fact, put $P = \begin{pmatrix} 1 & -\sqrt{3} & 1 \\ 1 & \sqrt{3} & 1 \\ -1 & 0 & 2 \end{pmatrix}$. Then $P^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & -2 \\ -\sqrt{3} & \sqrt{3} & 0 \\ 1 & 1 & 2 \end{pmatrix}$ and $P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} \\ 0 & \sqrt{3} & 0 \end{pmatrix}$.

G 26 Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -2 & -1 & -1 \end{pmatrix}.$$

(i) Show that for $p(x) = x^3 - x$ we have $p(A) = 0$.
(ii) Compute A^{2006}.
Remark: $-p(x)$ is the characteristic polynomial of A.

We have $A^2 = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ *and* $A^3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -2 & -1 & -1 \end{pmatrix} = A$. *Hence,* $p(A) = A^3 - A = 0$ *and it follows that* $A^{2k-1} = A$ *and* $A^{2k} = A^2$ *for all* $k \in \mathbb{N} \setminus \{0\}$. *Thus,* $A^{2006} = A^2$.
A more systematic approach would be to compute the eigenvalues of A*, which are* $0, -1$ *and* 1 *and then determine a transformation matrix* P *such that* $P^{-1}AP$ *is the diagonal matrix* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $A^{2006} = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}$.

G 27 Let V be an n-dimensional vector space and $U, W \subset V$ complementary subspaces. I.e. $V = U + W$ and $U \cap W = \{0\}$. Give a geometric interpretation of the real eigenvalues and eigenvectors of the following endomorphisms: Reflection of V on U along W, Projection of V onto U along W, Central scaling of V, Scaling of V in n linearly independent directions.
Hint: The first step to the solution of this exercise is to write down the correct definition of each endomorphism. Thinking of $n = 2$ or $n = 3$ may help your imagination.

The reflection of V *on* U *along* W *is the endomorphism* $\phi : V \to V$ *which restricted to* U *is the identity and which restricted to* W *is minus the identity. I.e.* $\phi|_U = \mathrm{id}_U$, *resp.* $\phi|_W = -\mathrm{id}_W$. *It follows that* ϕ *satisfies the identity* $\phi^2 = \mathrm{id}$, *so if* λ *is an eigenvalue of* ϕ *with eigenvector* $v \neq 0$, *then* $v = \phi^2 v = \lambda^2 \cdot v$. *Hence, the only possible eigenvalues are* $\lambda_1 = 1$ *and* $\lambda_2 = -1$. *Obviously,* U*, the set of fixed points of* ϕ*, is the eigenspace to* $\lambda_1 = 1$ *and* W *is the eigenspace to* $\lambda_2 = -1$. *In geometric terms,* U *is the mirror plane and* W *is the direction of reflection. Since the geometric multiplicities, i.e.* $\dim U$ *and* $\dim W$ *add up to* $n = \dim V$, *it follows that the reflection* ϕ *is diagonalizable.*

The projection of V *onto* U *along* W *is the map, which restricted to* U *is the identity and which restricted to* W *is the zero map. I.e.* $\phi|_U = \mathrm{id}_U$, *resp.* $\phi|_W = 0$. *It follows that* ϕ *satisfies the identity* $\phi^2 = \phi$, *so if* λ *is an eigenvalue of* ϕ *with eigenvector* $v \neq 0$, *then* $\lambda \cdot v = \phi v = \phi^2 v = \lambda^2 \cdot v$. *Hence, the only possible eigenvalues are* $\lambda_1 = 1$ *and* $\lambda_2 = 0$. *Obviously,* U*, the set of fixed points of* ϕ*, is the eigenspace to* $\lambda_1 = 1$ *and* W *is the eigenspace to* $\lambda_2 = 0$. *In geometric terms,* U *is the projection plane and* W *is the direction of projection. Since the geometric multiplicities, i.e.* $\dim U$ *and* $\dim W$ *add up to* $n = \dim V$, *it follows that the reflection* ϕ *is diagonalizable.*

The central scaling of V *by some scaling factor* λ *is the map* $\phi(v) = \lambda \cdot v$ *for all* $v \in V$. *Clearly,* ϕ *is diagonalizable, since every nonzero vector* $v \in V$ *is an eigenvector. In fact,* ϕ *is diagonal in the sense, that w.r.t to any basis* $e_1, \ldots, e_n$, ϕ *is represented by the matrix* $A = \lambda \cdot \mathrm{id}$. *It is not difficult to see, that this property characterizes a central scaling.*

Scaling of V *in* n *directions* $v_1, \ldots, v_n$, *with scaling factors* $\lambda_1, \ldots, \lambda_n$ *is the endomorphism* ϕ, *which is characterized by* $\phi v_i = \lambda_i v_i$, *for* $i = 1, \ldots, n$. *Hence the* λ_i *are the eigenvalues of* ϕ *and each* v_i *is an eigenvector to* λ_i. *Since the eigenvectors form a basis of* V, *we have that* ϕ *is diagonalizable.*

Homework

H 17 In each of the following cases compute the eigenvalues and eigenvectors of the given matrix. Diagonalize the matrix over $\mathbb{R}$ or $\mathbb{C}$, if possible:

$$(i) \quad \begin{pmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix}, \qquad (ii) \quad \begin{pmatrix} -4 & 4 & 4 & -1 \\ -3 & 4 & 3 & -1 \\ -5 & 4 & 5 & -1 \\ -5 & 4 & 5 & -1 \end{pmatrix}$$

To (i): The characteristic polynomial of $A := \begin{pmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix}$ *is* $\chi_A(x) = -(x-2)(x-1)^2$. *Therefore the eigenvalues are* $\lambda_1 = 1$ *with algebraic multiplicity* 2 *and* $\lambda_1 = 2$ *with algebraic multiplicity* 1. *An eigenvector to* λ_1 *is* $v_1 = (-1,0,1)^t$ *and there is no second eigenvector to* λ_1 *linearly independent to* v_1. *Hence, the geometric multiplicity of* λ_1 *is* 1 *and we already know, that* A *is not diagonalizable. An eigenvector to* λ_2 *is* $v_2 = (-2,1,0)^t$.

To (ii): The characteristic polynomial of $B := \begin{pmatrix} -4 & 4 & 4 & -1 \\ -3 & 4 & 3 & -1 \\ -5 & 4 & 5 & -1 \\ -5 & 4 & 5 & -1 \end{pmatrix}$ *is* $\chi_B(x) = x^2(x-1)(x-3)$. *The eigenvalues are thus* $\mu_1 = 0$ *with alg. multiplicity* 2, $\mu_2 = 1$ *with alg. multiplicity* 1 *and* $\mu_3 = 3$ *with alg. multiplicity* 1. *Two linearly independent eigenvectors to* μ_1 *are* $w_1 = (1,0,1,0)^t$ *and* $w_2 = (0,1,0,4)^t$. *For* μ_3 *an eigenvector is given by* $w_3 = (-1,1,-3,-3)^t$ *and for* μ_4 *an eigenvector is given by* $w_4 = (1,1,1,1)^t$. *It follows that* B *is diagonlizable. A corresponding transformation matrix* P *with* $P^{-1}BP$ *diagonal is given by* $P = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -3 & 1 \\ 0 & 4 & -3 & 1 \end{pmatrix}$.

H 18 The elements of the sequence $u_1, u_2, \ldots$ given by the initial values $u_1 = u_2 = 1$ and the recurrence relation

$$u_{n+1} = u_{n-1} + u_n$$

are called *Fibonacci numbers*. The first 14 Fibonacci numbers were produced for the first time in 1228 in the manuscripts of Leonardo of Pisa (Fibonacci).

(i) Compute the first 8 Fibonacci numbers.

(ii) We write the Fibonacci numbers as entries of a vector in the following way:

$$x_n := \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$$

for $n = 2, 3, \ldots$. Find a matrix A such that

$$Ax_n = x_{n+1}.$$

Express x_n in terms of x_2 and A.

(iii) Derive $u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ (*Binet's formula*) by computing x_n with help of the above.

(i) *The first Fibonacci numbers are* 1, 1, 2, 3, 5, 8, 13, 21.

(ii) *Put* $A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. *Then* $x_{n+1} = A \cdot x_n$ *and further* $x_{n+1} = A^{n-1} \cdot x_2$, *as can be verified by induction.*

(iii) *The eigenvalues of* A *are* $\lambda_1 = \frac{1+\sqrt{5}}{2}$ *and* $\lambda_2 = \frac{1-\sqrt{5}}{2}$. *The correspnding eigenvectors are* $v_1 = (\lambda_1, 1)^t$ *and* $v_2 = (\lambda_2, 1)^t$. *Put* $P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$, *then* $P^{-1}AP$ *is diagonal with* λ_1, λ_2 *as the diagonal entries. More precisely,* $P^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$ *and then*

$$A^{n-1} = P\begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix}P^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} \lambda_1^n - \lambda_2^n & -\lambda_2\lambda_1^n + \lambda_1\lambda_2^n \\ \lambda_1^{n-1} - \lambda_2^{n-1} & -\lambda_2\lambda_1^{n-1} + \lambda_1\lambda_2^{n-1} \end{pmatrix}$$

Multiplying this with x_2, *we obtain for* u_n, *as the second component of the resulting vector,*

$$u_n = \frac{1}{\sqrt{5}}\left(\lambda_1^{n-1} - \lambda_2^{n-1} - \lambda_2\lambda_1^{n-1} + \lambda_1\lambda_2^{n-1}\right) = \frac{1}{\sqrt{5}}\left(\lambda_1^{n-1} + \lambda_1^{n-2} - (\lambda_2^{n-1} + \lambda_2^{n-2})\right).$$

Finally, using $\lambda_i + 1 = \lambda_i^2$ *for* $i = 1, 2$ *we obtain Binet's formula from the last equation.*

H 19 In exercise **G 25** the basis of the eigenspaces, and hence the matrix P, can be chosen such that B is a spiral $\begin{pmatrix} r \cdot \cos\alpha & -r \cdot \sin\alpha \\ r \cdot \sin\alpha & r \cdot \cos\alpha \end{pmatrix}$, with $r > 0$ and $\alpha \in [0, 2\pi)$. Determine such a P as well as r and α.

*If we take P as in the solution of exercise **G 25**, we obtain* $B = \begin{pmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$, *wherefore* $r = \sqrt{3}$ *and* $\alpha = \pi/2$.

H 20 Let T be the endomorphism $D^2 + D + \mathrm{id}$ on the vector space of polynomials of degree $\leq n$ (c.f. exercise **G 20**). I.e. $Tp(x) = p''(x) + p'(x) + p(x)$. Can T be diagonalized? (**Corr.:** We assume that $K = \mathbb{R}$ or $K = \mathbb{C}$!)

Suppose that $p \neq 0$ is an eigenvector of T for the eigenvalue λ and further suppose that $\deg(p) = k$. Then $\lambda \cdot p = Tp = D^2p + Dp + p$ and substracting p on both sides yields $D^2p + Dp = (\lambda - 1) \cdot p$. On the left hand side is a polynomial of degree $< k$, whereas on the right hand side we have a polynomial of degree k, unless $\lambda = 1$. Thus, necessarily $\lambda = 1$ and $D^2p + Dp = 0$. Another comparison of the degrees of the summands yields that p has degree zero and is hence a constant. In fact, $p = c$ for some $c \in K \setminus \{0\}$ is an eigenvector and every eigenvector is of this form. Hence, the geometric multiplicity of $\lambda = 1$ is 1 and T is not diagonalizable.

H 21 Compute the characteristic polynomial of

$$A := \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix},$$

with $a_i \in K$ for $i = 0, 1, \dots, n-1$.

We claim that

$$\chi_A(x) = \det(A - x \cdot \mathrm{id}) = (-1)^n \left(x^n + \sum_{k=0}^{n-1} a_k x^k \right) = (-1)^n (x^n + a_{n-1}x^{n-1} + \dots + a_0).$$

The proof goes by induction over n. For this purpose we index A by n, for reasons of bookeeping i.e. $A = A_n$. For $n = 1$ we have $A_1 = (-a_0)$ and $\chi_{A_1}(x) = -x - a_0$, as in the formula. For $n \to n+1$, we develop the matrix by the last row and obtain

$$\begin{aligned} \det(A_{n+1} - x \cdot \mathrm{id}) &= (-1)^{n+n-1} \det(A_n - x \cdot \mathrm{id}) + (-a_n - x)(-x)^n \\ &= (-1)^{n+1}(x^{n+1} + a_n x^n) - \det(A_n - x \cdot \mathrm{id}) \\ &= (-1)^{n+1}(x^{n+1} + a_n x^n) - (-1)^n \left(x^n + \sum_{k=0}^{n-1} a_k x^k \right) \\ &= (-1)^{n+1} \left(x^{n+1} + a_n x^n + x^n + \sum_{k=0}^{n-1} a_k x^k \right) \\ &= (-1)^{n+1} \left(x^{n+1} + \sum_{k=0}^{n} a_k x^k \right). \end{aligned}$$

We applyied the induction hypothese in the third equation.

H 22 (i) Given a polynomial of degree n of the form **Corr.:** $p(x) = (-1)^n(x^n + a_{n-1}x^{n-1} + \dots + a_0) \in K[x]$, is there always an endomorphism ϕ of an n-dimensional vector space V over K, such that p is the characteristic polynomial of ϕ?

(ii) For each of the following polynomials p, q, give a matrix which has p, resp. q as its characteristic polynomial.

(i) **(Corr.:)** $p(x) = -(x^3 - 5x^2 + 6x + 8)$, $\qquad (ii)$ $q(x) = x^4 - 5x^3 + 7x + 4$.

Hint: Exercise **H 21** may be helpful for the solution.

*To (i): Take some basis ov V and define ϕ with respect to this basis by a matrix A as in exercise **H 21**. Then obviously, the answer is positive.*

To(ii): Using (i), we put $P := \begin{pmatrix} 0 & 0 & -8 \\ 1 & 0 & -6 \\ 0 & 1 & -5 \end{pmatrix}$ *and* $Q := \begin{pmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}$. *Then* $\chi_P(x) = p(x)$ *and* $\chi_Q(x) = q(x)$.