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Linear Algebra II (MCS), SS 2006, Exercise 5

Groupwork

G 19 (i) Let ϕ be an endomorphism of the Euclidean plane which has, with respect to some fixed orthogonal basis $\beta = \{e_1, e_2\}$, the matrix representation

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \text{ for some } \alpha \in [0, 2\pi).$$

Determine the (real) eigenvalues and eigenvectors of ϕ. What is their geometrical interpretation?

(ii) If in (i) we replace A by

$$\tilde{A} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

does ϕ has real eigenvalues and eigenvectors?

(iii) Suppose now that ψ is a rotation in Euclidean 3-space. That means, ψ rotates space around a certain axis through a certain angle. Are there any eigenvectors? To what eigenvalues?

G 20 Let $V_n = \mathrm{Pol}_n(K)$ be the vector space of polynomials of degree $\leq n$ over **(Corr.:)** $K = \mathbb{R}$ or $K = \mathbb{C}$. And let $D : V_n \to V_n$ be the differentiation, i.e. $D(p) = p'$.
Show that D does not have any eigenvectors with any *nonzero* eigenvalue λ. Furthermore, there is, up to a scalar factor, only one eigenvector of D to eigenvalue 0. Which polynomials satisfy $Dp = 0 \cdot p$?

G 21 Let ϕ be an endomorphism of some vector space V. Show:

(i) If λ is an eigenvalue of ϕ, then λ^2 is an eigenvalue of ϕ^2. How did you find the associated eigenvector? What about higher powers of ϕ?

(ii) If u_1 and u_2 are eigenvectors of ϕ to two *different* eigenvalues λ_1 and λ_2, then $u_1 + u_2$ is *not* an eigenvector of ϕ.

(iii) What is the relation between $\ker\phi$ and the eigenvalues and eigenvectors of ϕ?

G 22 Let A be a square matrix.

(i) If A is a triangular matrix, can you immediately say what the eigenvalues are?

(ii) Show that A and A^t have the same set of eigenvalues.

(iii) Suppose that A is invertible. Show: If λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}. What can you say about the eigenvectors?

Homework

H 13 For each of the following matrices determine the eigenvalues, eigenvectors and a basis of each eigenspace.

$$(i) \quad \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}, \qquad (ii) \quad \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

Which matrix can be diagonalized?

H 14 Let ϕ be an endomorphism of some vector space V over K. Show: If all vectors of V are eigenvectors for ϕ (to a priori different eigenvalues), then ϕ is a homothety, i.e. $\phi = r \cdot \mathrm{id}$ for some $r \in K$.

H 15 Let ϕ be an endomorphism. Show: If $\phi^2 + \phi$ has an eigenvalue -1, then ϕ^3 has an eigenvalue 1.

H 16 Let V be a vector space and let ϕ and ψ be endomorphisms of V.

(i) If $v \in V$ is an eigenvector of $\phi \circ \psi$ to the eigenvalue λ, and if $\psi(v) \neq 0$, then $\psi(v)$ is an eigenvector of $\psi \circ \phi$ to the eigenvalue λ.

(ii) If V has finite dimension, then $\psi \circ \phi$ and $\phi \circ \psi$ have the same eigenvalues.

Note that neither ϕ nor ψ themselves are required to have any eigenvalues!

Please note that the next exercise groups will, instead of Thursday, take place on the coming Monday 22.5.2006 3:20 pm-5:00 pm in room S2 15/201. Homework solutions may also be submitted on Thursday, 1.6.2006.

Linear Algebra II (MCS), SS 2006, Exercise 5, Solution

Groupwork

G 19 (i) Let ϕ be an endomorphism of the Euclidean plane which has, with respect to some fixed orthogonal basis $\beta = \{e_1, e_2\}$, the matrix representation

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \text{ for some } \alpha \in [0, 2\pi).$$

Determine the (real) eigenvalues and eigenvectors of ϕ. What is their geometrical interpretation?

(ii) If in (i) we replace A by

$$\tilde{A} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

does ϕ has real eigenvalues and eigenvectors?

(iii) Suppose now that ψ is a rotation in Euclidean 3-space. That means, ψ rotates space around a certain axis through a certain angle. Are there any eigenvectors? To what eigenvalues?

(i) *We first compute that the characteristic polynomial of A is $\chi_A(x) = det(A - x \cdot I) = x^2 - 1$. From this we read of that A has two different eigenvalues $x_1 = 1$ and $x_2 = -1$. The eigenvectors in representation w.r.t. the given basis can now be computed as the nontrivial solutions of the homogenous system $(A - x_i \cdot I) \cdot v = 0$. If $\alpha \neq 0$, then $v = (-\sin\alpha, \cos\alpha - 1)^t$ and $w = (\cos\alpha - 1, \sin\alpha)^t$ are eigenvectors to x_1, resp. x_2. If $\alpha = 0$, then $v = (1, 0)^t$ and $w = (0, 1)^t$ do. Geometrically, ϕ is an (orthogonal) reflection of the plane in the line through $v_1e_1 + v_2e_2$ alongside the line through $w_1e_1 + w_2e_2$.*

(ii) *Proceeding as before, we obtain that ϕ has real eigenvalues if and only if $\alpha = 0$ or $\alpha = \pi$. That is, ϕ is either the identity or the multiplication by minus one. In both cases, the eigenvectors are $v = (1, 0)^t$ and $w = (0, 1)^t$. Geometrically, ϕ is a rotation around the origin through the angle α.*

(iii) *ϕ now being a rotation means, that it leaves its axis of rotation pointwise fixed and the orthogonal complement to the axis invariant. That means, the elements of the axis, except for the origin, are each eigenvectors for the eigenvalue 1 and the complement of the axis is an Euclidean plane E, on which $\phi|_E$ acts as a rotation around the origin. Hence, we are in the situation of (ii) concerning the remaining eigenvalues and eigenvectors of ϕ (if any).*

G 20 Let $V_n = \mathrm{Pol}_n(K)$ be the vector space of polynomials of degree $\leq n$ over **(Corr.:)** $K = \mathbb{R}$ or $K = \mathbb{C}$. And let $D : V_n \to V_n$ be the differentiation, i.e. $D(p) = p'$.
Show that D does not have any eigenvectors with any *nonzero* eigenvalue λ. Furthermore, there is, up to a scalar factor, only one eigenvector of D to eigenvalue 0. Which polynomials satisfy $Dp = 0 \cdot p$?

If p is not constant, then D reduces the degree of p by one. Hence, there can only be constant polynomials which satisfy $Dp = \lambda \cdot p$. However, $Dp = 0$ in that case and since we assume $\lambda \neq 0$, we conclude that $p = 0$ in contradiction to $p \neq 0$, as it is assumed an eigenvector.

From this we also conclude, that the only eigenvectors of D which correspond to the eigenvalue 0 are the constant polynomials, which differ from each other by a scalar multiple.

G 21 Let ϕ be an endomorphism of some vector space V. Show:

(i) If λ is an eigenvalue of ϕ, then λ^2 is an eigenvalue of ϕ^2. How did you find the associated eigenvector? What about higher powers of ϕ?

(ii) If u_1 and u_2 are eigenvectors of ϕ to two *different* eigenvalues λ_1 and λ_2, then $u_1 + u_2$ is *not* an eigenvector of ϕ.

(iii) What is the relation between $\ker\phi$ and the eigenvalues and eigenvectors of ϕ?

(i) *Let $v \in V$ be a an eigenvector corresponding to λ. Then*

$$\phi^2(v) = \phi(\phi(v)) = \phi(\lambda \cdot v) = \lambda \cdot \phi(v) = \lambda^2 \cdot v.$$

Certainly, we may replace 2 by any positive integer k and then, by following the same line of arguments inductively, obtain that $\phi^k(v) = \lambda^k \cdot v$. So λ^k is an eigenvalue of ϕ^k with eigenvector v.

(ii) *We claim that u_1 and u_2 are linearly independent. In fact, suppose that $\alpha \cdot u_1 = u_2$ for some $\alpha \neq 0$. Now $A \cdot u_2 = \lambda_2 \cdot u_2$ implies*

$$\lambda_2 k u_1 = \lambda_2 u_2 = A \cdot u_2 = kA \cdot u_1 = k\lambda_1 u_1.$$

Therefore, $\lambda_2 k u_1 = k\lambda_1 u_1$ or equivalently $(\lambda_2 - \lambda_1) \cdot u_1 = 0$, which is a contradiction since neither $u_1 = 0$, nor $\lambda_2 = \lambda_1$.

Now suppose that $u_1 + u_2$ is an eigenvector again, say for the eigenvalue ν. Then $\nu \cdot (u_1 + u_2) = A \cdot (u_1 + u_2) = A \cdot u_1 + A \cdot u_2 = \lambda_1 \cdot u_1 + \lambda_2 \cdot u_2$. Then $\lambda_1 - \nu = 0$ and $\lambda_2 - \nu = 0$ by linear independence of u_1 and u_2. But $\lambda_1 \neq \lambda_2$, by assumption, which is a contradiction.

(iii) *$\ker \phi$ consists by definition of those $v \in V$, such that $\phi(v) = 0 = 0 \cdot v$. Hence, $\ker \phi$ is the eigenspace corresponding to the eigenvalue 0, in case this is an eigenvalue. Otherwise its just the trivial vector space. In particular, we have that an endomorphism which has not the eigenvalue 0 must be injective.*
In the same spirit, the set of fixed points of a given endomorphism is nothing but the set of eigenvectors corresponding to the eigenvalue 1, if this is an eigenvalue of the endomorphism.

G 22 Let A be a square matrix.

(i) If A is a triangular matrix, can you immediately say what the eigenvalues are?
(ii) Show that A and A^t have the same set of eigenvalues.
(iii) Suppose that A is invertible. Show: If λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}. What can you say about the eigenvectors?

(i) *The diagonal entries are the eigenvalues and the number of times each entry appears on the diagonal corresponds to its multiplicity as an eigenvalue. The reason for this is the fact that the determinant of a triangular matrix is the product of its diagonal entries. So this gives you the characteristic polynomial complete and already in factorized form. You can even read of an eigenvector corresponding to the first diagonal entry! It's $(1, 0, \dots, 0)^t$.*
Please, do not make the foolish mistake some of your fellow students do in their exams, who actually try to compute the eigenvalues of a triangular or even diagonal matrix by expanding an already factored polynomial!

(ii) *We have*

$$\chi_A(x) = \det(A - x \cdot I) = \det((A - x \cdot I)^t) = .\det(A^t - x \cdot I) = \chi_{A^t}(x).$$

Hence not only the eigenvalues of A and A^t coincide, but also their characteristic polynomials.

(iii) *Let v be an eigenvector of A corresponding to the eigenvalue $\lambda \neq 0$. By definition, $A \cdot v = \lambda \cdot v$. Multiplying both sides by A^{-1} and dividing by λ, we obtain $A^{-1} \cdot v = \frac{1}{\lambda} \cdot v$. Hence, λ^{-1} is an eigenvalue for A^{-1} with eigenvector v.*
*Also note that in **G 21** (i), we can now say that λ^k is an eigenvalue of A^k for any integer k.*

Homework

H 13 For each of the following matrices determine the eigenvalues, eigenvectors and a basis of each eigenspace.

$$(i) \quad \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}, \qquad (ii) \quad \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

Which matrix can be diagonalized?

To (i): For $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$ we have the eigenvalues $\lambda_1 = 4$, which has algebraic multiplicity one, and $\lambda_2 = -2$, which has algebraic multiplicity 2. An eigenvector for λ_1 is $v_1 = (1, 1, 2)^t$ and for λ_2, two linear independent eigenvectors are $v_2 = (1, 1, 0)^t$ and $v_3 = (-1, 0, 1)^t$. Hence, the eigenspace corresponding to λ_1 is $U_{\lambda_1} = \{t \cdot (1, 1, 2)^t \mid t \in \mathbb{R}\}$ and the one corresponding to λ_2 is $U_{\lambda_2} = \{t \cdot (1, 1, 0)^t + s \cdot (-1, 0, 1)^t \mid t, s \in \mathbb{R}\}$. It follows that A is diagonalizable.

To (ii): For $B = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$ we have the eigenvalues $\mu_1 = 4$, which has algebraic multiplicity one, and $\mu_2 = -2$, which has algebraic multiplicity 2. An eigenvector for μ_1 is $w_1 = (0, 1, 1)^t$ and for μ_2 an eigenvector is $w_2 = (1, 1, 0)^t$. There is no second eigenvector to μ_2 which is linearly independent from w_2. Hence, the eigenspace corresponding to μ_1 is $U_{\mu_1} = \{t \cdot (0, 1, 1)^t \mid t \in \mathbb{R}\}$ and the one corresponding to μ_2 is $U_{\mu_2} = \{t \cdot (1, 1, 0)^t \mid t \in \mathbb{R}\}$. It follows that A is not diagonalizable, since the geometric multiplicity of μ_2 is strictly less than its algebraic multiplicity.

H 14 Let ϕ be an endomorphism of some vector space V over K. Show: If all vectors of V are eigenvectors for ϕ (to a priori different eigenvalues), then ϕ is a homothety, i.e. $\phi = r \cdot \mathrm{id}$ for some $r \in K$.

Let $v_i, i \in I$ be some basis of V. By assumption, there is for every $i \in I$ some $\lambda_i \in K$ such that $\phi(v_i) = \lambda_i \cdot v_i$. On the other hand, for any finite subset $I' \subset I$ there is some $\mu \in K$ with $\phi(\sum_{i \in I'} v_i) = \mu \cdot \sum_{i \in I'} v_i$. Comparing this with $\phi(\sum_{i \in I'} v_i) = \sum_{i \in I'} \lambda_i \cdot v_i$, we obtain that $\sum_{i \in I'}(\lambda_i - \mu) \cdot v_i = 0$. Since the v_i are linearly independent, every coefficient must be zero and thus, $\lambda_i = \mu$ for every $i \in I'$. We can reason like above for any finite subset I' of I and therefore conclude, that $\lambda_i = \lambda_j =: r$ for all $i, j \in I$. It follows that $\phi = r \cdot \mathrm{id}$.

H 15 Let ϕ be an endomorphism. Show: If $\phi^2 + \phi$ has an eigenvalue -1, then ϕ^3 has an eigenvalue 1.

By assumption, there is some nonzero $v \in V$ such that $\phi^2(v) + \phi(v) = -v$. Applying ϕ on both sides of this equation yields $\phi^3(v) + \phi^2(v) = -\phi(v)$. Solving for $\phi^3(v)$ and using the above relation again, we obtain $\phi^3(v) = -(\phi^2(v) + \phi(v)) = -(-v) = v$. Hence ϕ^3 has the eigenvalue $+1$.

H 16 Let V be a vector space and let ϕ and ψ be endomorphisms of V.

(i) If $v \in V$ is an eigenvector of $\phi \circ \psi$ to the eigenvalue λ, and if $\psi(v) \neq 0$, then $\psi(v)$ is an eigenvector of $\psi \circ \phi$ to the eigenvalue λ.

(ii) If V has finite dimension, then $\psi \circ \phi$ and $\phi \circ \psi$ have the same eigenvalues.

Note that neither ϕ nor ψ themselves are required to have any eigenvalues!

To (i): By assumption we have $\phi \circ \psi(v) = \lambda \cdot v$. From this it follows

$$\psi \circ \phi(\psi(v)) = \psi(\phi \circ \psi(v)) = \psi(\lambda \cdot v) = \lambda \cdot \psi(v).$$

Because $\psi(v) \neq 0$, we conclude that $\psi(v)$ is an eigenvector of $\psi \circ \phi$ to the eigenvalue λ.

To (ii): Let $\Lambda := \{\lambda_i \mid i = 1, \dots, r\}$ be the set of distinct eigenvalues of $\phi \circ \psi$ and let $M := \{\mu_i \mid i = 1, \dots, s\}$ be the set of distinct eigenvalues of $\psi \circ \phi$. We claim that $\Lambda = M$. It obviously suffices to show the inclusion $\Lambda \subset M$, because if we swich the roles of ϕ and ψ, we get the other inclusion $M \subset \Lambda$.

So let $\lambda \in \Lambda$ be arbitrary. Accordingly, there exists some nonzero $v \neq 0$ satisfying $\phi \circ \psi(v) = \lambda v$. If $\psi(v) \neq 0$ we know by (i) that $\lambda \in M$. So it remains to deal with the case that $\psi(v) = 0$. In that case, ψ has non-trivial kernel $\ker(\psi) \neq \{0\}$ and the eigenvalue-equation above becomes $0 = \lambda \cdot v$. Since $v \neq 0$, it follows that $\lambda = 0$ and we have to show that $\psi \circ \phi$ has non-trivial kernel. If there is some $w \in V \setminus \{0\}$ such that $\phi(w) = v$, then $\psi \circ \phi(w) = \psi(v) = 0$. This is always the case, if ϕ is surjective. If ϕ is not surjective, then the finite dimensionality of V implies that ϕ is also not injective. Therefore, there is a $w \in V \setminus \{0\}$ such that $\phi(w) = 0$. But then also $\psi \circ \phi(w) = 0$.

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