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Linear Algebra II (MCS), SS 2006, Exercise 3

Groupwork

G 10 Compute the determinants of the following matrices:

$$\begin{pmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\end{pmatrix}, \quad \begin{pmatrix}1&0&1&0\\2&2&2&4\\1&0&0&0\\4&1&3&1\end{pmatrix}$$

G 11 Show that

$$\det\begin{pmatrix}x&1&1\\1&x&1\\1&1&x\end{pmatrix} = (x-1)^2(x+2), \quad \det\begin{pmatrix}a^2+1&ab&ac\\ab&b^2+1&bc\\ac&bc&c^2+1\end{pmatrix} = a^2+b^2+c^2+1.$$

G 12 Show that for an orthogonal matrix $A \in M(n,\mathbb{R})$, we have $\det(A) = \pm 1$.

G 13 Show that for a block matrix of the kind $M := \begin{pmatrix}A&B\\0&C\end{pmatrix}$, with A and C square matrices, we have: $\det(M) = \det(A)\cdot\det(C)$. Is the rule: $\det\begin{pmatrix}A&B\\C&D\end{pmatrix} = \det(A)\cdot\det(D) - \det(B)\cdot\det(C)$ also true?

G 14 Show that for any matrix $A = (a_{ij}) \in M(n,K)$ we have: $\det(a_{ij}) = \det((-1)^{i+j}\cdot a_{ij})$

Homework

H 6 Show that:

$$\det\begin{pmatrix}a&b&c&d\\-b&a&-d&c\\-c&d&a&-b\\-d&-c&b&a\end{pmatrix} = (a^2+b^2+c^2+d^2)^2$$

(Hint: Look at $A\cdot A^t$ and use that the determinant of A is a continuous function of a,b,c and d!)

H 7 Prove by induction over n that

$$\det\begin{pmatrix}1&x_1&\dots&x_1^{n-1}\\\vdots&\vdots&&\vdots\\1&x_n&\dots&x_n^{n-1}\end{pmatrix} = \prod_{1\le i<j\le n}(x_j - x_i)$$

(Remark: Recall that the empty product is defined as unity. I.e. $\prod_{i\in\emptyset} a_i := 1$.)

H 8 Compare the efforts of computing the determinant of a matrix with the Gaussian algorithm resp. the Leibniz formula.

(i) Determine the number of multiplications and additions required to compute the determinant of the square matrix $A = (a_{ij}) \in M(n,K)$

(a) by the Leibniz formula.

(b) by transforming A via Gaussian algorithm to echelon form and multiplying the diagonal entries.

(ii) Suppose a computer can perform addition and multiplication in 0.2 micro seconds. Give an estimate of the maximal value of n in case you want to compute the determinant of A within 48 hours of computation time using method (a), resp. method (b).

Linear Algebra II (MCS), SS 2006, Exercise 3, Solution

Groupwork

G 10 Compute the determinants of the following matrices:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 4 \\ 1 & 0 & 0 & 0 \\ 4 & 1 & 3 & 1 \end{pmatrix}$$

Using the Gaussian algorithm, resp. Laplace expansion we get:

$$\det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} = 4, \quad \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 4 \\ 1 & 0 & 0 & 0 \\ 4 & 1 & 3 & 1 \end{pmatrix} = 2.$$

G 11 Show that

$$\det \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} = (x-1)^2(x+2), \quad \det \begin{pmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{pmatrix} = a^2+b^2+c^2+1.$$

Use, for instance, Sarrus rule in both cases to verify the identities. E.g. the first identity computes as follows:

$$\det \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} = x^3+1+1-x-x-x = x^3-3x+2 = (x-1)^2(x+2)$$

G 12 Show that for an orthogonal matrix $A \in M(n, \mathbb{R})$, we have $\det(A) = \pm 1$.

Since $A \cdot A^t = I$ *and using that* $\det$ *is multiplicative as well as* $\det(A) = \det(A^t)$, *we have*

$$1 = \det I = \det(A) \cdot \det(A^t) = \det(A)^2.$$

Hence, $\det(A) = \pm 1$

G 13 Show that for a block matrix of the kind $M := \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, with A and C square matrices, we have: $\det(M) = \det(A) \cdot \det(C)$. Is the rule: $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \cdot \det(D) - \det(B) \cdot \det(C)$ also true?

First proof: Let S_i, $i = 1, 2$ *be the transformation matrix which transforms* A, *resp.* B *into echelon form, i.e.* $S_1 A S_1^{-1} = T_1$ *and* $S_2 A S_2^{-1} = T_2$, *for some trigonal matrices* T_1 *and* T_2. *Then* $S := \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$ *is a transformation matrix with inverse* $S^{-1} := \begin{pmatrix} S_1^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix}$, *which transforms* M *into echelon form:*

$$SMS^{-1} = \begin{pmatrix} S_1 A S_1^{-1} & S_1 B S_2^{-1} \\ 0 & S_2 C S_2^{-1} \end{pmatrix} = \begin{pmatrix} T_1 & S_1 B S_2^{-1} \\ 0 & T_2 \end{pmatrix}.$$

Therefore, using that $\det$ *is multiplicative, we have*

$$\det(M) = \det(SMS^{-1}) = \det(T_1) \cdot \det(T_2) = \det(A) \cdot \det(C).$$

Second proof: Let $A \in M(n\times n, K)$ and $C \in M(m\times m, K)$. Then $M = (a_{ij}) \in M((n+m)\times(n+m), K)$ and by the Leibniz formula we have

$$\det(M) = \sum_{\sigma\in S_{n+m}} \operatorname{sign}(\sigma)\cdot a_{1,\sigma(1)}\cdot\ldots\cdot a_{n,\sigma(n)}\cdot a_{n+1,\sigma(n+1)}\cdot\ldots\cdot a_{n+m,\sigma(n+m)}.$$

Since $a_{ij} = 0$ for $n+1 \le i \le n+m$, $1 \le j \le m$, we see that only those $\sigma \in S_{n+m}$ contribute to the sum which map $\{1,\ldots,n\}$ to itself and hence also $\{n+1,\ldots,n+m\}$ to itself. That is, $\sigma = \tau\circ\nu$ with $\tau \in S_n$, $\nu \in S_m$ (we assume here, that S_m permutes the elements of $\{n+1,\ldots,n+m\}$). Then

$$\begin{aligned}
\det(M) &= \sum_{\sigma\in S_{n+m}} \operatorname{sign}(\sigma)\cdot a_{1,\sigma(1)}\cdot\ldots\cdot a_{n,\sigma(n)}\cdot a_{n+1,\sigma(n+1)}\cdot\ldots\cdot a_{n+m,\sigma(n+m)}\\
&= \sum_{\tau\in S_n,\nu\in S_m} \operatorname{sign}(\tau)\cdot\operatorname{sign}(\nu)\cdot a_{1,\tau(1)}\cdot\ldots\cdot a_{n,\tau(n)}\cdot a_{n+1,\nu(n+1)}\cdot\ldots\cdot a_{n+m,\nu(n+m)}\\
&= \left(\sum_{\tau\in S_m}\operatorname{sign}(\tau)\cdot a_{1,\tau(1)}\cdot\ldots\cdot a_{n,\tau(n)}\right)\cdot\left(\sum_{\nu\in S_n}\operatorname{sign}(\nu)\cdot a_{n+1,\nu(n+1)}\cdot\ldots\cdot a_{n+m,\nu(n+m)}\right)\\
&= \det(A)\cdot\det(C)
\end{aligned}$$

Counterexamples for the formula $\det\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \det(A)\cdot\det(D) - \det(B)\cdot\det(C)$ are, for instance, the second matrix of exercise **G10** *or the matrix in* **H6**. *of course, the formula does not even make sense in case that A, B, C and D are not square matrices.*

G 14 Show that for any matrix $A = (a_{ij}) \in M(n, K)$ we have: $\det(a_{ij}) = \det((-1)^{i+j}\cdot a_{ij})$

Using the Leibniz formula we have

$$\begin{aligned}
\det((-1)^{i+j}\cdot a_{ij}) &= \sum_{\sigma\in S_n}\operatorname{sign}(\sigma)\cdot(-1)^{\sigma(1)+1}\cdot a_{\sigma(1),1}\cdot\ldots\cdot(-1)^{\sigma(n)+n}\cdot a_{\sigma(n),n}\\
&= \sum_{\sigma\in S_n}\operatorname{sign}(\sigma)\cdot(-1)^{\sum_{k=1}^n \sigma(k)+k}\cdot a_{\sigma(1),1}\cdot\ldots\cdot a_{\sigma(n),n}.
\end{aligned}$$

Now $\sum_{k=1}^n \sigma(k)+k = 2\cdot\frac{n(n+1)}{2} = n(n+1)$ which is an even number for every $n \in \mathbb{N}$. Hence, $(-1)^{\sum_{k=1}^n \sigma(k)+k} = 1$ for every $n \in \mathbb{N}$ and using the Leibniz formula again in the above equation, we have shown that $\det(a_{ij}) = \det((-1)^{i+j}\cdot a_{ij})$.

Homework

H 6 Show that:

$$\det\begin{pmatrix} a & b & c & d\\ -b & a & -d & c\\ -c & d & a & -b\\ -d & -c & b & a\end{pmatrix} = (a^2+b^2+c^2+d^2)^2$$

(Hint: Look at $A\cdot A^t$ and use that the determinant of A is a continuous function of a, b, c and d!)

Let $A := \begin{pmatrix} a & b & c & d\\ -b & a & -d & c\\ -c & d & a & -b\\ -d & -c & b & a\end{pmatrix}$. Then $A\cdot A^t = (a^2+b^2+c^2+d^2)\cdot I_4$ and we obtain $det(A)^2 = (a^2+b^2+c^2+d^2)^4$. From this we deduce that $|\det(A)| = (a^2+b^2+c^2+d^2)^2$. However, putting $a = 1, b = c = d = 0$ we see that $\det(A) = 1 > 0$, thus, by continuity, the non-negative root is the right one.

H 7 Prove by induction over n that

$$\det \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

(Remark: Recall that the empty product is defined as unity. I.e. $\prod_{i\in\emptyset} a_i := 1$.)

For $n = 1$, we have $\det(1) = 1 = \prod_{1\le i<j\le 1}(x_j - x_i)$ (compare with the remark!). Suppose now that we have already proved the statement for n and every $x_1, \dots, x_n \in K$. Note that formally the matrix we determine the determinant of is given by $A(n) := (a_{ij}) := (x_i^{j-1})$, $i, j = 1, \dots, n$ with the convention $x_i^0 = 1$. Hence, for each row, the elements satisfy the simple recursive identity $a_{i,0} = 1,\ a_{i,j+1} = a_{ij} \cdot x_i$. Motivated by this observation, we transform $\det(A(n + 1))$ in the first n steps, by adding to the k-th column the $(k - 1)$-st column multiplied by $(-x_{n+1})$. We do so from right to left, i.e. we start with the $(n + 1)$-st column and finish with the second column. This yields

$$\begin{aligned} \det(A(n+1)) &= \det \begin{pmatrix} 1 & x_1 - x_{n+1} & \dots & x_1^n - x_1^{n-1}x_{n+1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n - x_{n+1} & \dots & x_n^n - x_n^{n-1}x_{n+1} \\ 1 & 0 & \dots & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & (x_1 - x_{n+1}) & \dots & x_1^{n-1}(x_1 - x_{n+1}) \\ \vdots & \vdots & & \vdots \\ 0 & (x_n - x_{n+1}) & \dots & x_n^{n-1}(x_n - x_{n+1}) \\ 1 & 0 & \dots & 0 \end{pmatrix}. \end{aligned}$$

Where in the last equation, we have already eliminated the entries in the first column. We now move the last row to the top by n row exchanges then extract the factor $(x_i - x_{n+1})$ out of every row, which gives:

$$\det(A(n+1)) = (-1)^n \cdot (x_1 - x_{n+1}) \cdot \ldots \cdot (x_n - x_{n+1}) \cdot \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix}.$$

We may now use the induction hypothesis and conclude that $\det(A(n + 1)) = \prod_{1\le i<j\le n+1}(x_j - x_i)$.

H 8 Compare the efforts of computing the determinant of a matrix with the Gaussian algorithm resp. the Leibniz formula.

(i) Determine the number of multiplications and additions required to compute the determinant of the square matrix $A = (a_{ij}) \in M(n, K)$

(a) by the Leibniz formula.

(b) by transforming A via Gaussian algorithm to echelon form and multiplying the diagonal entries.

(ii) Suppose a computer can perform addition and multiplication in 0.2 micro seconds. Give an estimate of the maximal value of n in case you want to compute the determinant of A within 48 hours of computation time using method (a), resp. method (b).

(i) In the Leibniz formula we have $n!$ summands, each of which consists of $n + 1$ factors. This gives $(n+1)\cdot n! = (n+1)!$ multiplications and $n!$ additions. So in total, we have $(n+1)!+n!$ operations. If we do not count the signum as a separate multiplication, which is a reasonable assumption, then we would 'just'have $n \cdot n!$ multiplications and a total of $(n + 1)!$ operations.

For the Gaussian algorithm, we have $n \cdot (n - 1)$ multiplications and $n \cdot (n - 1)$ additions to clear the first column. For the second column, we have $(n - 1) \cdot (n - 2)$ multiplications and additions

each, and so on, until in the n-th step we have achieved echelon form and just have to multiply the n diagonal elements. Summarizing, we have $n + \sum_{k=1}^{n-1} k(k+1) = \frac{n(n^2+2)}{3}$ multiplications and $\sum_{k=1}^{n-1} k(k+1) = \frac{n(n^2-1)}{3}$ additions, which makes a total of $\frac{n(2n^2+1)}{3}$ operations.
For $n = 2$ the number of operations required is the same for both methods. For $n = 3$, the difference is still not very large. From then on however, the Gaussian algorithm is clearly the more efficient one.

(ii) *48 hours are $1.728 \cdot 10^{11}$ micro seconds. By our assumption, this equals $8.64 \cdot 10^{11}$ operations. Hence, the maximal possible n for the Leibniz formula is $n = 13$, whereas the Gaussian algorithm can tackle a $n \times n$-matrix with $n = 10902$ in the same time period!*