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Linear Algebra II (MCS), SS 2006, Exercise 13

Mini-Quiz

(1) Let ϕ be an endomorphism. The algebraic multiplicity of an eigenvalue λ of ϕ is...?
- ☐ The number of eigenvectors equal to zero.
- ☐ The multiplicity of $(x-\lambda)$ in the characteristic polynomial of ϕ.
- ☐ The dimension of $\ker(\phi - \lambda \cdot \mathrm{id})$.

(2) The eigenvalue 2 of the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has geometric multiplicity...?

☐ 0 ☐ 1 ☐ 2 ☐ 3

(3) Which of the following matrices is in Jordan normal form?

☐ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(4) Is the Jordan normal form of a real symmetric $n \times n$-matrix A always a diagonal matrix?
- ☐ Yes, because the principal axis transformation transforms A into diagonal form.
- ☐ No, because even a symmetric matrix can have less than n distinct eigenvalues. Since the set of all orthogonal matrices is different from the set of all invertible complex matrices, the above argumentation is not correct.
- ☐ The question makes no sense and therefore deserves no answer, because the theorem on the Jordan normal form deals with complex and not real matrices.

Groupwork

G 59 Determine the Jordan normal form and Jordan bases for the following matrices:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 1 \end{pmatrix}.$$

Hint: $B^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & -1 \end{pmatrix}$ and $B^3 = 0$.

G 60 Let A be the real 4×4-matrix which has 1 as its only real eigenvalue, of algebraic multiplicity 2, and suppose that i is a complex eigenvalue of A and $A^4 \neq I$. Determine the JNF of A.

G 61 What is the geometric interpretation of an endomorphism whose JNF is given by a $2 \times 2 - \lambda$-Jordan block, resp. a $3 \times 3 - \lambda$-Jordan block. Distinguish the cases $\lambda = 0$ and $\lambda \neq 0$.

G 62 Let ϕ be an endomorphism of a finite dimensional vector space V and let $\lambda \in K$. Show:

(i) There is a maximal nontrivial, ϕ-invariant subspace W of V on which $\phi - \lambda \cdot \mathrm{id}$ is nilpotent, if λ is an eigenvalue of ϕ.

(ii) $\phi - \lambda \cdot \mathrm{id}$ is invertible if λ is not an eigenvalue of ϕ.

Homework

H 50 Determine the Jordan normal form and Jordan bases for the following matrices:

$$A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 2 & 2 \end{pmatrix}.$$

H 51 We call two matrices A and B *similar*, if there is an invertible matrix S such that $S^{-1}AS = B$. Show:

(i) The Jordan block J_k is similar to its transpose $(J_k)^t$, for every $k \in \mathbb{N}$.

(ii) Every complex (or real and trigonalizable) matrix is similar to its transpose.

Remark: A matrix is called trigonalizable, if it is similar to an upper triangular matrix.

H 52 Let V be a five dimensional real vector space and $\phi : V \to V$ an endomorphism which satisfies the following properties: 1 is an eigenvalue of algebraic multiplicity three and 2 is an eigenvalue of algebraic multiplicity two.

(i) Determine all possible Jordan normal forms of ϕ, up to permutation of the Jordan blocks.

(ii) If the eigenvalue 1 has geometric multiplicity two, which candidates remain?

H 53 Show that a trigonalizable endomorphism ϕ of a finite dimensional vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, which satisfies $\phi^k = \mathrm{id}$ for some $k \in \mathbb{N}^{>0}$ is in fact diagonalizable.

Remark: An endomorphism is called trigonalizable if it is represented by an upper triangular matrix w.r.t. a suitable basis.

H 54 Show that for every complex normal matrix A there is a polynomial p such that $A^* = p(A)$.

Hint: Compare the normal forms of A and A^* and use exercises **H12, H26**.

Linear Algebra II (MCS), SS 2006, Exercise 13, Solution

Mini-Quiz

(1) Let ϕ be an endomorphism. The algebraic multiplicity of an eigenvalue λ of ϕ is...?
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- ☐ The dimension of $\ker(\phi - \lambda \cdot \mathrm{id})$.

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- ☐ Yes, because the principal axis transformation transforms A into diagonal form.
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Groupwork

G 59 Determine the Jordan normal form and Jordan bases for the following matrices:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 1 \end{pmatrix}.$$

Hint: $$B^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & -1 \end{pmatrix} \text{ and } B^3 = 0.$$

The eigenvalues of A are obviously $\lambda_1 = 0$, with algebraic multiplicity one, and $\lambda_2 = 1$ with algebraic multiplicity 3. The generalized eigenspace V_0 to $\lambda_1 = 0$ is thus equal to the corresponding eigenspace, which is $\mathrm{Span}\{(1,0,0,0)^t\}$. For λ_2 we compute the powers of $A - \lambda_2 \cdot E$:

$$(A - E) = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, (A - E)^2 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathrm{rank}\,(A - E) = 2$ and $\mathrm{rank}\,(A - E)^2 = 1$ and $\mathrm{rank}\,(A - E)^k = 1$ for $k \geq 2$. A basis of the kernel of $(A - E)^2$ and thus the generalized eigenspace V_1 for $\lambda_2 = 1$, is given by $e_2, e_1 + e_3, e_4$. It is easy to verify, that e_2 and $e_1 + e_3$ are already annihilated by $A - E$, whereas e_4 gives rise to a Jordan chain of length two: $(A - E)e_4 = e_1 + e_2 + e_3$. A Jordan basis of V_1 is therefore $e_2, e_1 + e_2 + e_3, e_4$.

The JNF of A is thus $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *and the transition matrix is given by* $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

B is nilpotent with index 3. Looking at B^2, we choose e_3 as the head of a Jordan chain of length three: $e_1 - e_7, e_2, e_3$. There is no other chain of length three. Calculating $J(h)$ for $h < 3$, we see that there are two chains of length two. Next, we determine a basis of $\ker B^2$:

$$\ker B^2 = \mathrm{Span}\{e_2, e_6, e_1 - e_3, e_1 - e_4, e_1 - e_5, e_1 - e_7\}.$$

From this basis, only the elements $X = \{e_6, e_1 - e_3, e_1 - e_4, e_1 - e_5\}$ are independent from the above Jordan chain. From these, we choose two elements, which produce chains of length two. For instance $B(e_1 - e_3) = -e_1 - e_2 - e_3 + e_4 + e_6 + e_7$ and $B(e_1 - e_5) = e_6$. Thus, w.r.t. the basis $\{e_1 - e_7, e_2, e_3, e_6, e_1 - e_5, -e_1 - e_2 - e_3 + e_4 + e_6 + e_7, e_1 - e_3\}$ the JNF of B is $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

G 60 Let A be the real 4×4-matrix which has 1 as its only real eigenvalue, of algebraic multiplicity 2, and suppose that i is a complex eigenvalue of A and $A^4 \neq I$. Determine the JNF of A.

Since a complex eigenvalue comes always with its complex conjugate for a real matrix, the given data on the eigenvalues, leaves two possible JNF's for A: $J_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$ *and* $J_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$

The condition $A^4 \neq I$ rules out the first case, thus J_2 is the normal form of A.

G 61 What is the geometric interpretation of an endomorphism whose JNF is given by a $2 \times 2 - \lambda$-Jordan block, resp. a $3 \times 3 - \lambda$-Jordan block. Distinguish the cases $\lambda = 0$ and $\lambda \neq 0$.

In the two-dimensional case, if $\lambda \neq 0$ then $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1 & 1/\lambda \\ 0 & 1 \end{pmatrix}$ is the composition of a dilation by λ and a shearing by the factor $1/\lambda$. Just draw a picture in the plane of how the unit cube gets transformed to a stretched parallelogram. If $\lambda = 0$, then we have the projection of $\mathbb{R}^2 \to \mathbb{R}$ onto the second component, followed by the embedding of $\mathbb{R}$ into $\mathbb{R}^2$ as the line through $(1,0)^t$.

In the three-dimensional case, if $\lambda \neq 0$, then we also have a kind of shearing, even though it looks more intricate than in the two-dimensional case. However, it also helps to visualize this map by drawing a picture where the unit cube gets mapped to a stretched parallelotop. If $\lambda = 0$, then we get again a projection, this time of $\mathbb{R}^3 \to \mathbb{R}^2$ onto the x_2, x_3-plane, followed by an embedding into $\mathbb{R}^3$ on the x_1, x_2-plane.

G 62 Let ϕ be an endomorphism of a finite dimensional vector space V and let $\lambda \in K$. Show:

(i) There is a maximal nontrivial, ϕ-invariant subspace W of V on which $\phi - \lambda \cdot \mathrm{id}$ is nilpotent, if λ is an eigenvalue of ϕ.

(ii) $\phi - \lambda \cdot \mathrm{id}$ is invertible if λ is not an eigenvalue of ϕ.

To (i): *If we take for W the generalized eigenspace corresponding to λ, then the statement is obviously true.*

To (ii): *By definition of an eigenvalue, $\det(\phi - \lambda \cdot \mathrm{id})$ is zero if and only if λ is an eigenvalue of ϕ. Therefore, $\det(\phi - \lambda \cdot \mathrm{id}) \neq 0$ if λ is not an eigenvalue of ϕ and $\phi - \lambda \cdot \mathrm{id}$ is invertible by the determinant criterion.*

Homework

H 50 Determine the Jordan normal form and Jordan bases for the following matrices:

$$A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 2 & 2 \end{pmatrix}.$$

The characteristic polynomial of A is (up to a sign) $\chi_A(x) = x^3 + 3x^2 + 3x + 1 = (x+1)^3$. Therefore, A has only -1 as single eigenvalue of algebraic multiplicity three. We compute the powers of $A + E$:

$$A + E = \begin{pmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{pmatrix}, (A+E)^2 = 0.$$

Hence, there must be exactly one Jordan chain of length two and one of length one. For the two chain we choose for instance $7e_1 + 14e_2 + 7e_3, e_1$. The kernel of $A + E$ is spanned by $6e_1 + 7e_2, 5e_1 - 7e_3$. Any of these vectors complete our two chain to a Jordan basis. We thus have as transition matrix: $S = \begin{pmatrix} 7 & 1 & 6 \\ 14 & 0 & 7 \\ 7 & 0 & 0 \end{pmatrix}$.

The eigenvalues of B are 3 with algebraic multiplicity three and 2 with algebraic multiplicity one (this can be read off from the matrix without computations). The vector e_4 forms a basis of V_2. It remains to determine the generalized eigenspace for $\lambda = 3$. We have $B - 3E = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & -1 \end{pmatrix}$, $(B - 3E)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}$ *and the rank becomes stationary from there on. A basis of V_3 is given by a basis of $\ker(B - 3E)^2$, which is $e_1 + e_2, e_2 - e_3, e_1 + e_4$. We transform A with respect to the transition matrix* $S = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ *and obtain* $S^{-1}AS = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. *Since our eyes are used to these pictures now, we read off that in our new basis, e_3, e_1 is a two chain, which is complemented to Jordan basis of the 3-Jordan block by the element of the appropriate kernel: $e_1 + e_2$. We thus get another transition matrix* $T = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. *We then have achieved that*

$$T^{-1}S^{-1}AST = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

H 51 We call two matrices A and B *similar*, if there is an invertible matrix S such that $S^{-1}AS = B$. Show:

(i) The Jordan block J_k is similar to its transpose $(J_k)^t$, for every $k \in \mathbb{N}$.

(ii) Every complex (or real and trigonalizable) matrix is similar to its transpose.

Remark: A matrix is called trigonalizable, if it is similar to an upper triangular matrix.

To (i): *Since $(J_k^t)^m = (J_k^m)^t$ it follows that J_k is nilpotent of index k. Hence, there is precisely one Jordan chain of length k and by the JNF-theorem we have that $(J_k)^t$ is similar to J_k. We can even determine explicitly a transition matrix S as follows. The $k-1$-st power of $(J_k)^t$ is the matrix whose only non-vanishing column is the first one. This column is equal to e_k. Hence, the maximum Jordan chain is $e_k \leftarrow \cdots \leftarrow e_1$. The transition matrix S is therefore the identity matrix mirrored at the horizontal middle axis:* $S = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$.

To (ii): *Let A denote a trigonalizable matrix. Thus, the JNF of A exists and we have an invertible matrix S such that $S^{-1}AS = J$, where J is a Jordan matrix consisting of λ_i-Jordan blocks, where λ_i ranges through the eigenvalues of A. Each block can be written as $\lambda_i \cdot I + K_k$ for a suitable k. Now apply a transition matrix as above to the block and we obtain its transpose. If we arrange these transition matrices into a big block diagonal transition matrix, then it is easy to see that A is similar to its transpose.*

H 52 Let V be a five dimensional real vector space and $\phi : V \to V$ an endomorphism which satisfies the following properties: 1 is an eigenvalue of algebraic multiplicity three and 2 is an eigenvalue of algebraic multiplicity two.

(i) Determine all possible Jordan normal forms of ϕ, up to permutation of the Jordan blocks.

(ii) If the eigenvalue 1 has geometric multiplicity two, which candidates remain?

To (i): *by assumption, the only possible JNF's of ϕ up to permutations of the blocks are*

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&2&0\\0&0&0&0&2\end{pmatrix}, \begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&2&0\\0&0&0&0&2\end{pmatrix}, \begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&2&0\\0&0&0&0&2\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&2&1\\0&0&0&0&2\end{pmatrix}, \begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&2&1\\0&0&0&0&2\end{pmatrix}, \begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&2&1\\0&0&0&0&2\end{pmatrix}.$$

To (ii): *Under the additional requirement, there must be two 1-Jordan blocks and we only have the two possibilities:*

$$\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&2&0\\0&0&0&0&2\end{pmatrix}, \begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&2&1\\0&0&0&0&2\end{pmatrix}.$$

H 53 Show that a trigonalizable endomorphism ϕ of a finite dimensional vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, which satisfies $\phi^k = \mathrm{id}$ for some $k \in \mathbb{N}^{>0}$ is in fact diagonalizable.
Remark: An endomorphism is called trigonalizable if it is represented by an upper triangular matrix w.r.t. a suitable basis.

Being trigonalizable means that the characteristic polynomial decomposes into linear factors over the ground field. Hence the theorem of the Jordan normal form can be applied to ϕ and there is a Jordan matrix J representing ϕ w.r.t. a Jordan basis. Suppose now that there is at least one Jordan block appearing in J of size greater than one. Then the assumption that $\phi^k = \mathrm{id}$ implies that all eigenvalues have absolute value equal to one (they are in fact k-th roots of unity) and we have $J^k = E$. From this we conclude that all the off diagonal entries of J^k must necessarily vanish, however this is not the case if there is a Jordan block whose size is greater than one (if arguable, you may explicitly compute this for a $k \times k$-λ-Jordan block). Hence, we have arrived at a contradiction and the maximal size of a Jordan block must be equal to one.

H 54 Show that for every complex normal matrix A there is a polynomial p such that $A^* = p(A)$.
Hint: Compare the normal forms of A and A^* and use exercises **H12, H26**.

According to the spectral theorem for normal matrices, A and A^* are both unitary diagonalizable. Since A commutes with A^* by the very definition of normality, we can diagonalize both matrices simultaneously. Let D be the diagonal matrix obtained from A in that fashion. Then $\bar{D}$ is the corresponding diagonal matrix for A^*. If we let p denote the unique complex polynomial of least degree $\leq n$ which maps each diagonal entry λ_i onto $\bar{\lambda}_i$ (the existence follows from exercise **H12***), then obviously $p(D) = \bar{D}$. Now note that $p(SAS^{-1}) = Sp(A)S^{-1}$ holds for all invertible matrices S and all polynomials p. Hence, $p(A) = A^*$.*