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Linear Algebra II (MCS), SS 2006, Exercise 12

Mini-Quiz

(1) A Jordan-block of size k is the matrix $A = (a_{ij})$ with...?

- ☐ $a_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{else} \end{cases}$
- ☐ $a_{ij} = \begin{cases} 0 & \text{if } j = i+1 \\ 1 & \text{else} \end{cases}$
- ☐ $a_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{else} \end{cases}$

(2) An endomorphism ϕ is called nilpotent if...?

- ☐ $\phi^n = 0$ for all $n \in \mathbb{N}$.
- ☐ $\phi^n = 0$ for some $n \in \mathbb{N}$.
- ☐ $\phi(v^n) = 0$ for some $n \in \mathbb{N}$ and $v \in V$.

(3) For a nilpotent endomorphism ϕ and a vector $v \in V$, the Jordan chain $J(v)$ consists of...?

- ☐ the head $\sigma(v)$ and the tail v.
- ☐ the vectors $0 \neq \phi^{k-1}(v), \ldots, \phi(v), v$ with $\phi^k(v) = 0$.
- ☐ the number of Jordan bases of length exactly $\|v\|$.

(4) Given a nilpotent matrix A, there is an invertible matrix S, such that...?

- ☐ $S^{-1}AS$ is a nilpotent Jordan matrix whose number of Jordan blocks equals the algebraic multiplicity of the eigenvalue 0.
- ☐ S^tAS is diagonal with index of nilpotency equal to the number of Jordan blocks.
- ☐ $S^{-1}AS$ is a nilpotent Jordan matrix whose number of Jordan blocks equals the geometric multiplicity of the eigenvalue 0.

Groupwork

G 54 (i) Draw the diagram corresponding to the following nilpotent Jordan matrix:

$$\begin{pmatrix}
0 & 1 &&&&&&&& \\
& 0 & 1 &&&&&&& \\
&& 0 & 1 &&&&&& \\
&&& 0 &&&&&& \\
&&&& 0 & 1 &&&& \\
&&&&& 0 & 1 &&& \\
&&&&&& 0 &&& \\
&&&&&&& 0 & 1 & \\
&&&&&&&& 0 & 1 \\
&&&&&&&&& 0
\end{pmatrix}$$

(ii) Determine the nilpotent Jordan matrix corresponding to the following diagram:

$$\begin{array}{ccccccccccc}
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet && \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet &&&&&& \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet &&&&&&
\end{array}$$

G 55 Determine bases for which the following nilpotent matrices are in Jordan normal form:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hint: $B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B^4 = 0.$

G 56 Let ϕ and ψ be nilpotent endomorphisms of a vector space V which have nilpotency indices k, rep. l and which commute. I.e. $\phi \circ \psi = \psi \circ \phi$. Show that (i) $\phi + \psi$ and (ii) $\phi \circ \psi$ are also nilpotent. Give in each case also an estimate for the index of nilpotency in terms of k and l.

G 57 Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that A is already in Jordan-normal form. Put $v_1 := e_1 + e_3, v_3 := e_1 - e_3$.

(i) Show that $\{v_1, v_3\}$ is a basis of the eigenspace of A to the eigenvalue 0 (i.e. a basis of $\ker A$).

(ii) Show that there exists no vector v_2 such that $\{v_1, v_2, v_3\}$ is a Jordan-basis for A.

Thus, in general, if one wants to determine the Jordan normal form of a matrix, it is the wrong idea to start with a basis of the eigenspaces!

G 58 (i) Suppose you are given a nilpotent $n \times n$-matrix A with nilpotency index m and you know the rank of each $A^k, k = 1, \ldots, m$. How can you determine the Jordan normal form of A?

(ii) In case that A is a 9×9-matrix with nilpotency index 5 and $\operatorname{rank} A = 5, \operatorname{rank} A^2 = 3, \operatorname{rank} A^3 = 2$ and $\operatorname{rank} A^4 = 1$, determine the Jordan normal form of A.

Homework

H 45 Let ϕ be a nilpotent endomorphism of an n-dimensional vector space V. Suppose that there exists a $v \in V$ such that $\phi^{n-1}(v) \neq 0$. Show that for every $0 \leq k \leq n$ there is a k-dimensional ϕ-invariant subspace W_k of V. For which k does there exist a $n-k$-dimensional ϕ-invariant complement of W_k? Determine the normal form of ϕ.

H 46 Show that the elements of a Jordan chain $J(v), v \neq 0$ form a linear independent set.

H 47 Let V be a finite dimensional vector space and let ϕ be a nilpotent endomorphism. Suppose that $V = W_1 \oplus \cdots \oplus W_k$ is a direct decomposition of V into ϕ-invariant subspaces, such that each W_i contains a Jordan-chain of length $\dim W_i$. Show that $\dim(\ker \phi) = k$ and $\operatorname{rank} \phi = n - k$.

H 48 Let V be a n-dimensional vector space and $\phi : V \to V$ be nilpotent with index of nilpotency n. Show that there exist no endomorphism ψ of V satisfying $\psi^2 = \phi$.

H 49 Let N be a nilpotent matrix. Show that $I + N$ is invertible and determine its inverse.
Hint: Geometric series and telescope sums!

Linear Algebra II (MCS), SS 2006, Exercise 12, Solution

Mini-Quiz

(1) A Jordan-block of size k is the matrix $A = (a_{ij})$ with...?

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- ☐ the head $\sigma(v)$ and the tail v.
- ☐ the vectors $0 \neq \phi^{k-1}(v), \dots, \phi(v), v$ with $\phi^k(v) = 0$.
- ☐ the number of Jordan bases of length exactly $\|v\|$.

(4) Given a nilpotent matrix A, there is an invertible matrix S, such that...?

- ☐ $S^{-1}AS$ is a nilpotent Jordan matrix whose number of Jordan blocks equals the algebraic multiplicity of the eigenvalue 0.
- ☐ S^tAS is diagonal with index of nilpotency equal to the number of Jordan blocks.
- ☐ $S^{-1}AS$ is a nilpotent Jordan matrix whose number of Jordan blocks equals the geometric multiplicity of the eigenvalue 0.

Groupwork

G 54 (i) Draw the diagram corresponding to the following nilpotent Jordan matrix:

$$\begin{pmatrix}
0 & 1 & & & & & & & & \\
 & 0 & 1 & & & & & & & \\
 & & 0 & 1 & & & & & & \\
 & & & 0 & & & & & & \\
 & & & & 0 & 1 & & & & \\
 & & & & & 0 & 1 & & & \\
 & & & & & & 0 & & & \\
 & & & & & & & 0 & 1 & \\
 & & & & & & & & 0 & 1 \\
 & & & & & & & & & 0
\end{pmatrix}$$

(ii) Determine the nilpotent Jordan matrix corresponding to the following diagram:

$$\begin{array}{ccccccccccc}
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & & \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & & & & & & \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & & & & & &
\end{array}$$

To (i):

$$\begin{array}{ccccccc}
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & & \\
\bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & &
\end{array}$$

To (i): *Let J_k denote the Jordan-block of size $k \times k$. Then the nilpotent Jordan matrix corresponding to the diagram is:*

$$\begin{pmatrix}
J_6 & & & & \\
 & J_6 & & & \\
 & & J_5 & & \\
 & & & J_3 & \\
 & & & & J_3
\end{pmatrix}$$

G 55 Determine bases for which the following nilpotent matrices are in Jordan normal form:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hint: $B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B^4 = 0.$

According to the algorithm, we first determine the index of nilpotency of A:

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A^3 = 0.$$

The index of nilpotency is therefore 3. *The image of* A^2 *is spanned by* e_1, *which is the tail of the Jordan-chain* $J(e_4) = \{e_1, e_2, e_4\}$. *The next* $h < 3$ *with* $J(h) > 0$ *can be determined over the ranks of* A^l: *The rank of* A^2 *is one and the rank of* A *is two. Hence,* $J(2) = \text{rank}\,(A^3) + \text{rank}\,(A) - 2\text{rank}\,(A^2) = 0 + 2 - 2 = 0$. *Therefore* $h = 1$ *and indeed* $J(1) = \text{rank}\,(A^2) + 5 - 2\text{rank}\,(A) = 1 + 5 - 4 = 2$. *The kernel of* A *is* $\ker(A) = \text{Span}\{(0, 1, -1, 1, 0)^t, (0, 0, 1, 0, -1)^t\}$. *Thus* $\{e_1, e_2, e_4, e_2 - e_3 + e_4, e_3 - e_5\}$ *is a Jordan-basis of* A, *and the corresponding normal form is* $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

For B *the index of nilpotency is* 4. *The image of* B^3 *is spanned by* $e_1 - e_6 = (1, 0, 0, 0, 0, -1, 0)^t$, *which is the tail of* $J(e_4) = \{e_1 - e_6, e_2, e_3, e_4\}$. *We have* $\text{rank}\, B^3 = 1, \text{rank}\, B^2 = 2, \text{rank}\, B = 4$. *Thus* $J(3) = 0, J(2) = 1$ *and* $J(1) = 1$. *The next* $h < 4$ *with* $J(h) > 0$ *is therefore* $h = 2$. *The kernel of* B^2 *is spanned by* $\{e_1, e_2, e_5, e_6, e_7 - 2e_3 - e_4\}$ *and we see that* $e_1 - e_6, e_2 \in J(e_4)$ *are also elements of the kernel. Let* $X := \{e_1, e_5, \}$, *Then* X *together with* $e_1 - e_6, e_2$ *span the kernel and we are interested in the elements of* X *with J-rank* $h = 2$. *For instance,* $J(e_1) = \{e_5, e_1\}$ *fulfills this requirement and* e_4, e_1 *are J-independent. Furthermore, we do not have to look for other elements with J-rank 2, since this would violate J-independency. Next, we look for an* $h < 2$ *with* $J(h) > 1$. *This is necessarily* $h = 1$ *and* $J(1) = 1$, *as computed above. The kernel of* B *is spanned by* $\{e_5, e_1 - e_6, e_2 + 2e_3 + e_4 - 2e_1 - e_7\}$. *The elements* e_5 *and* $e_1 - e_6$ *lie also in the J-span of* e_4, e_1. *We therefore complete our J-basis to* $\{e_4, e_1, e_2 + 2e_3 + e_4 - 2e_1 - e_7\}$. *That is,* $\{e_1 - e_6, e_2, e_3, e_4, e_5, e_1, e_2 + 2e_3 + e_4 - 2e_1 - e_7\}$ *is a Jordan basis for* B *and the Jordan normal form is* $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

G 56 Let ϕ and ψ be nilpotent endomorphisms of a vector space V which have nilpotency indices k, rep. l and which commute. I.e. $\phi \circ \psi = \psi \circ \phi$. Show that (i) $\phi + \psi$ and (ii) $\phi \circ \psi$ are also nilpotent. Give in each case also an estimate for the index of nilpotency in terms of k and l.

Since ϕ and ψ commute as endomorphisms, the powers of $\phi+\psi$ satisfy the binomial formula $(\phi+\psi)^n = \sum_{j=0}^{n}\binom{n}{j}\phi^{n-j}\circ\psi^j$. If we choose $n = k+l-1$, then $\phi^{n-j} = 0$ for $j = 0,\dots,l$ and $\psi^j = 0$ for $j = l+1,\dots,n$. Hence, $\phi+\psi$ is nilpotent with index at most $\min\{k+l-1, \dim V\}$.

Concerning $\phi\circ\psi$, we have $(\phi\circ\psi)^n = \phi^n\circ\psi^n$, again due to the commutativity of ϕ with ψ. Hence, $\phi\circ\psi$ is nilpotent with index at most $\min\{k,l\}$.

G 57 Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that A is already in Jordan-normal form. Put $v_1 := e_1 + e_3, v_3 := e_1 - e_3$.

(i) Show that $\{v_1, v_3\}$ is a basis of the eigenspace of A to the eigenvalue 0 (i.e. a basis of $\ker A$).
(ii) Show that there exists no vector v_2 such that $\{v_1, v_2, v_3\}$ is a Jordan-basis for A.

Thus, in general, if one wants to determine the Jordan normal form of a matrix, it is the wrong idea to start with a basis of the eigenspaces!

To (i): *Since e_1, e_3 are obviously eigenvectors of A to EV 0, so are v_1 and v_3, since they are linear combinations of the former. Furthermore, the kernel is two dimensional, hence $\{v_1, v_3\}$ is a basis of $\ker(A)$.*

To (ii): *Suppose that $v_2 = (a,b,c)^t$ is such a vector. The transition matrix is then given by $S = \begin{pmatrix} 1 & a & 1 \\ 0 & b & 0 \\ 1 & c & -1 \end{pmatrix}$. Note that necessarily $b \neq 0$. The inverse of S is then $S^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{a+c}{2b} & \frac{1}{2} \\ 0 & \frac{1}{b} & 0 \\ \frac{1}{2} & \frac{c-a}{2b} & -\frac{1}{2} \end{pmatrix}$ and one computes $S^{-1}AS = \begin{pmatrix} 0 & \frac{b}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b}{2} & 0 \end{pmatrix}$. This is unequal to A, regardless of the choice of $b \neq 0$.*

G 58 (i) Suppose you are given a nilpotent $n\times n$-matrix A with nilpotency index m and you know the rank of each $A^k, k = 1,\dots,m$. How can you determine the Jordan normal form of A?

(ii) In case that A is a 9×9-matrix with nilpotency index 5 and $\operatorname{rank} A = 5, \operatorname{rank} A^2 = 3, \operatorname{rank} A^3 = 2$ and $\operatorname{rank} A^4 = 1$, determine the Jordan normal form of A.

To (i): *According to the script we have $J_{\geq}(h) = \operatorname{rank}(A^{k-1}) - \operatorname{rank}(A^k)$ and by definition, $J_{\geq}(h)$ is the number of J-chains of length at least h in any Jordan basis. Furthermore, $J(h)$, the number of J-chains of length at exactly h in any Jordan basis, is determined by $J_{\geq}(h)$ over $J(h) = J_{\geq}(h) - J_{\geq}(h+1)$. Together this yields $J(h) = \operatorname{rank}(A^{k-1}) + \operatorname{rank}(A^{k+1}) - 2\operatorname{rank}(A^k)$. As the knowledge of $J(h)$ for every h determines the JNF of A, we have shown that knowing the ranks of the powers of A determines the JNF of any nilpotent A.*

To (ii): *According to (i) we have $J(1) = 2, J(2) = 1, J(3) = J(4) = 0$ and $J(5) = 1$. Thus the Jordan normal form of A is $\begin{pmatrix} J_5 & & & \\ & J_2 & & \\ & & J_1 & \\ & & & J_1 \end{pmatrix}$, where J_k denotes the Jordan block of size $k\times k$.*

Homework

H 45 Let ϕ be a nilpotent endomorphism of an n-dimensional vector space V. Suppose that there exists a $v \in V$ such that $\phi^{n-1}(v) \neq 0$. Show that for every $0 \leq k \leq n$ there is a k-dimensional ϕ-invariant subspace W_k of V. For which k does there exist a $n-k$-dimensional ϕ-invariant complement of W_k? Determine the normal form of ϕ.

By assumption, ϕ is nilpotent of index n and $J(v) = \{\phi^{n-1}(v),\dots,\phi(v), v\}$ is a maximal Jordan chain of length n. Let $W_k := \operatorname{Span}\{\phi^{n-1}(v),\dots,\phi^{n-k}(v)\}$. Then $\dim W_k = k, W_k \subset W_{k+1}$ and $\phi(W_k) \subset W_{k-1} \subset W_k$. In fact, this is true because $J(v)$ forms a basis of V and $\phi(\phi^k(v)) = \phi^{k+1}(v)$ for $k = 0,\dots,n-1$ and $\phi(\phi^{n-1}(v)) = 0$. Since $J(v)$ forms a Jordan basis of V, the Jordan normal form of ϕ is obviously the $n\times n$-Jordan block. From this it follows that the only W_k's which have ϕ-invariant complements are $W_0 = \{0\}$ and $W_n = V$.

H 46 Show that the elements of a Jordan chain $J(v), v \neq 0$ form a linear independent set.

We have $J(v) = \{0 \neq \phi^{k-1}(v),\dots,\phi(v), v\}$, where $\phi : V \to V$ is nilpotent. Suppose that we have numbers $a_0,\dots,a_{k-1} \in K$ such that $a_0 v + a_1\phi(v) + \ldots + a_{k-1}\phi^{k-1}(v) = 0$. After applying ϕ^{k-1} to this equation, the only term that remains is $a_0\phi^{k-1}(v) = 0$, the other terms are zero. Since $\phi^{k-1}(v)$ and also every other element of $J(v)$ is different from zero, it follows that $a_0 = 0$. Hence our former equation *becomes $a_1\phi(v) + \ldots + a_{k-1}\phi^{k-1}(v) = 0$ and this time applying ϕ^{k-2} yields $a_1 = 0$. Continuing in this way, we end up with $a_0 = \cdots = a_{k-1} = 0$. Thus the elements of $J(v)$ are linearly independent.*

H 47 Let V be a finite dimensional vector space and let ϕ be a nilpotent endomorphism. Suppose that $V = W_1 \oplus \cdots \oplus W_k$ is a direct decomposition of V into ϕ-invariant subspaces, such that each W_i contains a Jordan-chain of length $\dim W_i$. Show that $\dim(\ker\phi) = k$ and $\operatorname{rank}\phi = n - k$.

Obviously, we obtain a Jordan basis by concatenating the maximal Jordan chains of each subspace W_i. The JNF of ϕ is then a block diagonal with k Jordan blocks, their sizes running through $\dim W_i, i = 1, \ldots, k$. The rank, resp. dimension of the kernel of ϕ is then the sum of the ranks, resp. dimensions of the kernels of the Jordan blocks. Each Jordan block has a one-dimensional kernel and rank equal to $\dim W_i - 1$. This yields the stated formulas.

H 48 Let V be a n-dimensional vector space and $\phi : V \to V$ be nilpotent with index of nilpotency n. Show that there exist no endomorphism ψ of V satisfying $\psi^2 = \phi$.

Suppose that ψ would exist. By assumption, ψ were also nilpotent (of index at most $2 \cdot k$ if k is the index of ϕ). However, it is easy to see that the ranks of the powers of a nilpotent endomorphism are strictly decreasing. Since ϕ has by definition rank equal to $n-1$, ψ would necessarily have rank equal to n. However, this would contradict the nilpotency of ψ.

H 49 Let N be a nilpotent matrix. Show that $I + N$ is invertible and determine its inverse.
Hint: Geometric series and telescope sums!

Let n denote the index of nilpotency of N. We claim that $A := \sum_{k=0}^{n-1}(-1)^k N^k$ is the inverse of $I + N$. In fact,

$$(I+N)\cdot A = \sum_{k=0}^{n-1}(-1)^k N^k + N \cdot \sum_{k=0}^{n-1}(-1)^k N^k = \sum_{k=0}^{n-1}(-1)^k N^k - \sum_{k=1}^{n-1}(-1)^k N^k + (-1)^n N^n = I.$$