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29. Juni 2006

Linear Algebra II (MCS), SS 2006, Exercise 11

Mini-Quiz

(1) The adjoint ϕ^* of an endomorphism ϕ of an euclidean vector space V is defined by...?
- ☐ $\langle \phi^*(v)|\phi^*(w)\rangle = \langle v|\phi(w)\rangle$
- √ $\langle v|\phi(w)\rangle = \langle \phi(v)^*|w\rangle$
- ☐ $\langle \phi(v)|w\rangle = \langle w|\phi(v)\rangle$

for all $v, w \in V$.

(2) An endomorphism ϕ of an euclidean vector space V is called self-adjoint, if...?
- ☐ $\langle \phi(v)|\phi(w)\rangle = \langle v|w\rangle$
- √ $\langle v|\phi(w)\rangle = \langle \phi(v)|w\rangle$
- ☐ $\langle \phi(v)|w\rangle = \langle w|\phi(v)\rangle$

for all $v, w \in V$.

(3) If $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues of a selfadjoint endomorphism, v_i, resp. U_i, is an eigenvector, resp. the eigenspace, for λ_i, $i = 1, \ldots, r$, then for $i \neq j$:
- ☐ $\lambda_i \perp \lambda_j$.
- √ $v_i \perp v_j$.
- √ $U_i \perp U_j$.

(4) Is there a scalar product on $\mathbb{R}^2$ such that the shearing associated with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is self adjoint?
- ☐ Yes, put $\langle x, y\rangle := x_1y_1 + x_1y_2 + x_2y_2$.
- ☐ Yes, the standard scalar product already has this property.
- √ No, because A is not symmetric.

Groupwork

G 49 Determine the diagonal form of the following matrix after a principal axes transformation:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Do so just by thought, without lengthy computations.
Hint: What is the image of A? What does A do to the elements of its image?

G 50 Let $A = \frac{1}{2}\begin{pmatrix} 1 & -1 & -\sqrt{2} \\ -1 & 1 & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$. Show that A is orthogonal and determine its axis of rotation in $\mathbb{R}^3$, as well as the angle ω of rotation around this axis. What is the normal form of A?

G 51 Let ϕ be an endomorphism of a vector space V with scalar product. Show:
- (i) A linear subspace U of V is ϕ-invariant if and only if $U^\perp$ is ϕ^*-invariant.
- (ii) $\ker\phi = (\operatorname{im}\phi^*)^\perp$ and $\operatorname{im}\phi = (\ker\phi^*)^\perp$.
- (iii) If ϕ is orthogonal (or unitary) and U is ϕ-invariant, then $U^\perp$ is also ϕ-invariant.

G 52 Let ϕ and ψ be endomorphisms of a finite dimensional vector space V with scalar product and $\lambda \in K$. Show:

(i)	$(\phi + \psi)^* = \phi^* + \psi^*$,	(ii)	$(\phi\psi)^* = \psi^*\phi^*$,
(iii)	$(\lambda \cdot \phi)^* = \bar{\lambda}\phi^*$,	(iv)	$(\phi^*)^* = \phi$,
(v)	$\mathrm{id}^* = \mathrm{id},\ 0^* = 0$	(vi)	$(\phi^{-1})^* = (\phi^*)^{-1}$, if ϕ is invertible.

G 53 (i) Show that for every invertible complex $n \times n$ matrix A there are a uniquely determined positively definite hermitean matrix H and a unitary matrix S with $A = HS$. More precisely, $H = \sqrt{(AA^*)}$ (see ex. **H 36**) and $S = H^{-1}A$. If A is real then so are H and S. The above decomposition of A into H and S is called *polar decomposition*.

(ii) Determine the polar decomposition of the matrix $\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}$

Homework

H 40 Compute the pseudoinverse of $A = \begin{pmatrix} 1 & i \\ 0 & 0 \\ -1 & -i \end{pmatrix}$ and use it to determine the best approximate solution of $Ax = b$ with **Corr.:** $b = (1, 1, 1)^t$.

H 41 Show that the following properties of an endomorphism $\phi : V \to V$ of a vector space with scalar product $\langle \cdot \mid \cdot \rangle$ are equivalent:

(i) $\phi^* = \phi^{-1}$,

(ii) $\langle \phi(u) | \phi(v) \rangle = \langle u | v \rangle$ for all $u, v \in V$,

(iii) $\|\phi(u)\| = \|u\|$ for all $u \in V$.

(iv) $\|\phi(u) - \phi(v)\| = \|u - v\|$ for all $u, v \in V$.

Remark: If V is euclidean, a ϕ with the above properties is called *orthogonal.* If V is unitary, then such a ϕ is called *unitary.*

H 42 Let V be a vector space with scalar product $\langle \cdot \mid \cdot \rangle$ and ϕ an endomorphism of V. Show that each of the following conditions imply that $\phi = 0$:

(i) $\langle \phi(u) | v \rangle = 0$ for all $u, v \in V$.

(ii) V is unitary and $\langle \phi(u) | u \rangle = 0$ for all $u \in V$.

(iii) ϕ is selfadjoint and $\langle \phi(u) | u \rangle = 0$ for all $u \in V$.

Give an example of an endomorphism $\phi \neq 0$ on an euclidean space V with $\langle \phi(u) | u \rangle = 0$ for all $u \in V$.
Hint: In some cases it may be helpful to polarize the quadratic form $\langle \phi(u) | u \rangle$ (i.e. what is $\langle \phi(u) | v \rangle$?). In (ii), what happens if you substitute u with iu?

H 43 Let V be a vector space with scalar product $\langle \cdot \mid \cdot \rangle$ and let $\alpha = \{e_1, \ldots, e_n\}$ be any basis of V. Let further A denote the Gram-matrix of $\langle \cdot \mid \cdot \rangle$ w.r.t. α and let $\phi : V \to V$ be an endomorphism. Show that ϕ is selfadjoint if and only if **Corr.:** $A \cdot \phi^\alpha$ is a hermitean matrix.

H 44 Let V be a finite dimensional vector space with scalar product $\langle \cdot \mid \cdot \rangle$ and basis α.

(i) Suppose that ϕ is an endomorphism associated to a hermitean form Φ on V. Show that ϕ^α is a hermitean matrix, if α is an on-basis.

(ii) For $V = \mathbb{R}^2$ with the standard scalar product and $\alpha = \{(1, 2)^t, (0, -1)^t\}$ consider the endomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\phi^\alpha = \begin{pmatrix} 0 & -1 \\ -9 & 2 \end{pmatrix}$. Show that ϕ is selfadjoint, i.e. $\phi = \phi^*$, and determine the symmetric bilinear form Φ associated with ϕ.

Linear Algebra II (MCS), SS 2006, Exercise 11, Solution

Mini-Quiz

(1) The adjoint ϕ^* of an endomorphism ϕ of an euclidean vector space V is defined by...?
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- ☐ $\langle \phi(v)|w\rangle = \langle w|\phi(v)\rangle$

for all $v, w \in V$.

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- √ $\langle v|\phi(w)\rangle = \langle \phi(v)|w\rangle$
- ☐ $\langle \phi(v)|w\rangle = \langle w|\phi(v)\rangle$

for all $v, w \in V$.

(3) If $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of a selfadjoint endomorphism, v_i, resp. U_i, is an eigenvector, resp. the eigenspace, for $\lambda_i, i = 1, \dots, r$, then for $i \neq j$:
- ☐ $\lambda_i \perp \lambda_j$.
- √ $v_i \perp v_j$.
- √ $U_i \perp U_j$.

(4) Is there a scalar product on $\mathbb{R}^2$ such that the shearing associated with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is self adjoint?
- ☐ Yes, put $\langle x, y\rangle := x_1y_1 + x_1y_2 + x_2y_2$.
- ☐ Yes, the standard scalar product already has this property.
- √ No, because A is not symmetric.

Groupwork

G 49 Determine the diagonal form of the following matrix after a principal axes transformation:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Do so just by thought, without lengthy computations.
Hint: What is the image of A? What does A do to the elements of its image?

We see at a glance that A has rank equal to one. The image is the span of the vector $v = (1, 1, 1, 1, 1)^t$. Accordingly, the kernel is 4-dimensional. If we apply A to its image, then $Av = (5, 5, 5, 5, 5)^t = 5 \cdot (1, 1, 1, 1, 1)^t$ wherefore v is an eigenvector to the eigenvalue 5. We conclude that the normal form of A is $\begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

G 50 Let $A = \frac{1}{2}\begin{pmatrix} 1 & -1 & -\sqrt{2} \\ -1 & 1 & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$. Show that A is orthogonal and determine its axis of rotation in $\mathbb{R}^3$, as well as the angle ω of rotation around this axis. What is the normal form of A?

It is an easy calculation that $AA^t = A^tA = I$. Hence, A is orthogonal. Since $A \neq A^t$, the axis of rotation can be calculated as the kernel of $A - A^t = \begin{pmatrix} 0 & 0 & -\sqrt{2} \\ 0 & 0 & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$. This yields $v_1 = \frac{1}{\sqrt{2}}(1, -1, 0)^t$, which is obviously an eigenvector to the eigenvalues 1. The angle of rotation is determined by $\cos\omega = \frac{1}{2}(\mathrm{tr}A - \det A) = \frac{1}{2}(1 - 1) = 0$. Hence, $\omega = \frac{\pi}{2}$. The normal form is given by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

G 51 Let ϕ be an endomorphism of a vector space V with scalar product. Show:
- (i) A linear subspace U of V is ϕ-invariant if and only if $U^\perp$ is ϕ^*-invariant.
- (ii) $\ker\phi = (\mathrm{im}\,\phi^*)^\perp$ and $\mathrm{im}\,\phi = (\ker\phi^*)^\perp$.
- (iii) If ϕ is orthogonal (or unitary) and U is ϕ-invariant, then $U^\perp$ is also ϕ-invariant.

To (i): *Suppose that U is ϕ-invariant and let $v \in U^\perp$ be arbitrary. Then for all $u \in U$ we have*

$$\langle \phi^*(v) | u \rangle = \langle v | \phi(u) \rangle = 0.$$

Therefore $U^\perp$ is ϕ^*-invariant. For the other implication we interchange the rôles of ϕ and ϕ^* and U and $U^\perp$ and use that $(\phi^*)^* = \phi$ and $(U^\perp)^\perp = U$.

To (ii): *Let $v \in \ker \phi$ be arbitrary. Then for all $\phi^*(w) \in \operatorname{im} \phi^*$ we have $\langle v | \phi^*(w) \rangle = \langle \phi(v) | w \rangle = 0$. Hence, $\ker \phi \subset (\operatorname{im} \phi^*)^\perp$. Conversely, if $v \in (\operatorname{im} \phi^*)^\perp$, then for all $w \in V$ we have $0 = \langle v | \phi^*(w) \rangle = \langle \phi(v) | w \rangle$. So we also have $\ker \phi \supset (\operatorname{im} \phi^*)^\perp$ and thus in fact equality. Again, interchanging the roles of ϕ and ϕ^* and using $(U^\perp)^\perp = U$ we obtain the second equality.*

To (iii): *By (i) we have that $U^\perp$ is ϕ^*-invariant. Since in particular, $\phi^* = \phi^{-1}$ is bijective, we have $\phi^*(U^\perp) = U^\perp$. If we apply ϕ on both sides of this equation, we end up, using $\phi \circ \phi^* = \mathrm{id}$, with $U^\perp = \phi(U^\perp)$ which shows that $U^\perp$ is ϕ-invariant.*

G 52 Let ϕ and ψ be endomorphisms of a finite dimensional vector space V with scalar product and $\lambda \in K$. Show:

(i)	$(\phi + \psi)^* = \phi^* + \psi^*$,	(ii)	$(\phi\psi)^* = \psi^*\phi^*$,
(iii)	$(\lambda \cdot \phi)^* = \bar{\lambda}\phi^*$,	(iv)	$(\phi^*)^* = \phi$,
(v)	$\mathrm{id}^* = \mathrm{id},\ 0^* = 0$	(vi)	$(\phi^{-1})^* = (\phi^*)^{-1}$, if ϕ is invertible.

To (i): $\langle (\phi + \psi)^*(v) | w \rangle = \langle v | (\phi + \psi)(w) \rangle = \langle v | \phi(w) \rangle + \langle v | \psi(w) \rangle = \langle \phi^*(v) | w \rangle + \langle \psi^*(v) | w \rangle = \langle (\phi^* + \psi^*)(v) | w \rangle$.

To (ii): $\langle (\phi \circ \psi)^*(v) | w \rangle = \langle v | (\phi \circ \psi)(w) \rangle = \langle v | \phi(\psi(w)) \rangle = \langle \phi^*(v) | \psi(w) \rangle = \langle (\psi^* \circ \phi^*)(v) | w \rangle$.

To (iii): $\langle (\lambda\phi)^*(v) | w \rangle = \langle v | (\lambda\phi)(w) \rangle = \lambda \langle v | \phi(w) \rangle = \lambda \langle \phi^*(v) | w \rangle = \langle \bar{\lambda}\phi^*(v) | w \rangle$.

To (iv): $\langle (\phi^*)^*(v) | w \rangle = \langle v | \phi^*(w) \rangle = \langle \phi(v) | w \rangle$.

To (v): $\langle (\mathrm{id}^*)(v) | w \rangle = \langle v | \mathrm{id}(w) \rangle = \langle \mathrm{id}(v) | w \rangle$ *and* $0^* = 0$ *by (iii) with* $\lambda = 0$.

To (vi): *Since $\phi \circ \phi^{-1} = \mathrm{id}$ we can apply (ii) and (v) to obtain $(\phi^{-1})^* \circ \phi^* = \mathrm{id}$, whence $(\phi^{-1})^* = (\phi^*)^{-1}$.*

G 53 (i) Show that for every invertible complex $n \times n$ matrix A there are a uniquely determined positively definite hermitean matrix H and a unitary matrix S with $A = HS$. More precisely, $H = \sqrt{(AA^*)}$ (see ex. **H 36**) and $S = H^{-1}A$. If A is real then so are H and S. The above decomposition of A into H and S is called *polar decomposition*.

(ii) Determine the polar decomposition of the matrix $\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}$

To (i): *Since A is invertible, AA^* is clearly positively definite, as it is congruent to the identity matrix. By exercise **H 36** the square root $H = \sqrt{(AA^*)}$ is a well defined, unique and positively definite hermitean matrix. If we define $S = H^{-1}A$, then according to the rules of exercise **G 52**, which clearly hold for matrices, too:*

$$S^*S = (H^{-1}A)^*H^{-1}A = A^*(H^*)^{-1}H^{-1}A = A^*(H^2)^{-1}A = A^*(AA^*)^{-1}A = I,$$

which shows that S is unitary. We furthermore have $HS = HH^{-1}A = A$, which completes the proof of existence for the decomposition. In order to prove the uniqueness of the decomposition, suppose that $A = HS$ as claimed. Then $H = AS^*$ and $H^2 = HH^* = AS^*SA^* = AA^*$. Since the root of AA^* is unique, we have $H = \sqrt{AA^*}$ and then $S = H^{-1}A$ is also uniquely determined.

To (ii): *For* $A = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}$ *we have* $AA^* = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, *then* $H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ *and* $S = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$.

Homework

H 40 Compute the pseudoinverse of $A = \begin{pmatrix} 1 & i \\ 0 & 0 \\ -1 & -i \end{pmatrix}$ and use it to determine the best approximate solution of $Ax = b$ with **Corr.:** $b = (1, 1, 1)^t$.

*In exercise **G44** the singular value decomposition of A was computed as* $U^*AV = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$, *with* $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$, $V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$ *and* $\Sigma = (2)$. *The pseudoinverse A^+ of A is then* $A^+ = V \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^* = \frac{1}{4} \begin{pmatrix} 1 & 0 & -1 \\ -i & 0 & i \end{pmatrix} = \frac{1}{4}A^*$. *Hence, the best approximate solution for $Ax = b$ is $x = A^+b = 0$.*

H 41 Show that the following properties of an endomorphism $\phi : V \to V$ of a vector space with scalar product $\langle\cdot\mid\cdot\rangle$ are equivalent:

(i) $\phi^* = \phi^{-1}$,

(ii) $\langle\phi(u)|\phi(v)\rangle = \langle u|v\rangle$ for all $u, v \in V$,

(iii) $\|\phi(u)\| = \|u\|$ for all $u \in V$.

(iv) $\|\phi(u) - \phi(v)\| = \|u - v\|$ for all $u, v \in V$.

Remark: If V is euclidean, a ϕ with the above properties is called *orthogonal.* If V is unitary, then such a ϕ is called *unitary.*

(i) $\Leftrightarrow$ (ii): *$\langle u|v\rangle = \langle\phi^* \circ \phi(u)|v\rangle = \langle\phi(u)|\phi(v)\rangle$, for all $u, v \in V$. The other implication follows from $\langle u|v\rangle = \langle\phi(u)|\phi(v)\rangle = \langle\phi^* \circ \phi(u)|v\rangle$, for all $u, v \in V$ and therefore $\phi^* \circ \phi(v) = v$ for all $v \in V$. This in turn implies $\phi^* = \phi$.*

(ii) $\Rightarrow$ (iii): *In (ii), just take $v = u$ and take the square root from $\|\phi(u)\|^2 = \langle\phi(u)|\phi(u)\rangle = \langle u|u\rangle = \|u\|^2$.*

(iii) $\Leftrightarrow$ (iv): *(iv) follows from (iii) by substituting u with $u-v$ and conversely, if we put $v = 0$, we obtain (iii) from (iv).*

(iii) $\Rightarrow$ (ii): *The key is polarization: Since $\|\phi(u)\| = \|u\|$ for all $u \in V$, we also have $\|\phi(u + v)\|^2 = \|u + v\|^2$ for all $u, v \in V$. By expansion of the expressions on both sides we obtain:*

$$\|\phi(u)\|^2 + \|\phi(v)\|^2 + 2\Re\langle\phi(u)|\phi(v)\rangle = \|u\|^2 + \|v\|^2 + 2\Re\langle u|v\rangle.$$

This yields $\Re\langle\phi(u)|\phi(v)\rangle = \Re\langle u|v\rangle$, which concludes the proof of the euclidean case. In the unitary case, we have $\Im\langle u|v\rangle = \Re\langle iu|v\rangle$. This in combination with the former result yields $\Im\langle\phi(u)|\phi(v)\rangle = \Re\langle\phi(iu)|\phi(v)\rangle = \Re\langle iu|v\rangle = \Im\langle u|v\rangle$. Thus in total we have: $\langle\phi(u)|\phi(v)\rangle = \langle u|v\rangle$ for all $u, v \in V$.

H 42 Let V be a vector space with scalar product $\langle\cdot\mid\cdot\rangle$ and ϕ an endomorphism of V. Show that each of the following conditions imply that $\phi = 0$:

(i) $\langle\phi(u)|v\rangle = 0$ for all $u, v \in V$.

(ii) V is unitary and $\langle\phi(u)|u\rangle = 0$ for all $u \in V$.

(iii) ϕ is selfadjoint and $\langle\phi(u)|u\rangle = 0$ for all $u \in V$.

Give an example of an endomorphism $\phi \neq 0$ on an euclidean space V with $\langle\phi(u)|u\rangle = 0$ for all $u \in V$.
Hint: In some cases it may be helpful to polarize the quadratic form $\langle\phi(u)|u\rangle$ (i.e. what is $\langle\phi(u)|v\rangle$?). In (ii), what happens if you substitute u with iu?

To (i): *This is clear, since a scalar product is non-degenerate. I.e. the only vector orthogonal to every other vector is the zero vector.*

To (ii): *We polarize $q(u) := \langle\phi(u)|u\rangle$. I.e. $0 = q(u + v) = \langle\phi(u + v)|u + v\rangle = \langle\phi(u) + \phi(v)|u + v\rangle = \langle\phi(u)|u\rangle + \langle\phi(u)|v\rangle + \langle\phi(v)|u\rangle + \langle\phi(v)|v\rangle = q(u) + q(v) + \langle\phi(v)|u\rangle + \langle\phi(u)|v\rangle = \langle\phi(v)|u\rangle + \langle\phi(u)|v\rangle$. If we replace u by iu, we obtain $0 = \langle\phi(v)|iu\rangle + \langle\phi(iu)|v\rangle = i\langle\phi(v)|u\rangle - i\langle\phi(u)|v\rangle$ and therefore $\langle\phi(v)|u\rangle - \langle\phi(u)|v\rangle = 0$. Adding this to the first equation yields $\langle\phi(v)|u\rangle = 0$ for all $u, v \in V$. By (i), this implies $\phi = 0$.*

To (iii): *As in (ii), we polarize and obtain $0 = \langle\phi(v)|u\rangle + \langle\phi(u)|v\rangle = \langle\phi(v)|u\rangle + \langle u|\phi^*(v)\rangle = \langle\phi(v)|u\rangle + \overline{\langle\phi(v)|u\rangle} = 2\Re\langle\phi(v)|u\rangle$ for all $u, v \in V$. As in exercise* **H41** *we show that this implies $\langle\phi(v)|u\rangle = 0$ for all $u, v \in V$. By (i) again, we then conclude that $\phi = 0$.*

H 43 Let V be a vector space with scalar product $\langle\cdot\mid\cdot\rangle$ and let $\alpha = \{e_1, \ldots, e_n\}$ be any basis of V. Let further A denote the Gram-matrix of $\langle\cdot\mid\cdot\rangle$ w.r.t. α and let $\phi : V \to V$ be an endomorphism. Show that ϕ is selfadjoint if and only if **Corr.:** $A \cdot \phi^\alpha$ is a hermitean matrix.

Let $A = (a_{ij}) := (\langle e_i|e_j\rangle)$ and $(f_{ij}) := \phi^\alpha$. We have to show that $\langle\phi(u)|v\rangle = \langle u|\phi(v)\rangle \Leftrightarrow (A \cdot \phi^\alpha)^* = A \cdot \phi^\alpha$. The left hand side is clearly equivalent to

$$\langle\phi(e_i)|e_j\rangle = \langle e_i|\phi(e_j)\rangle \tag{*}$$

for all $i, j = 1, \ldots n$. Now $\phi(e_i) = \sum_{k=1}^n f_{ki}e_k$ and thus

$$\begin{aligned}
(*) \quad &\Leftrightarrow \quad \sum_{k=0}^{n} \bar{f}_{ki}\langle e_k|e_j\rangle = \sum_{k=0}^{n} f_{kj}\langle e_i|e_k\rangle \Leftrightarrow \sum_{k=0}^{n} \bar{f}_{ki}a_{kj} = \sum_{k=0}^{n} f_{kj}a_{ik} \\
&\Leftrightarrow \quad (\phi^\alpha)^* \cdot A = A \cdot \phi^\alpha \Leftrightarrow (A^* \cdot \phi^\alpha)^* = A \cdot \phi^\alpha \\
&\Leftrightarrow \quad (A \cdot \phi^\alpha)^* = A \cdot \phi^\alpha.
\end{aligned}$$

H 44 Let V be a finite dimensional vector space with scalar product $\langle\cdot\mid\cdot\rangle$ and basis α.

(i) Suppose that ϕ is an endomorphism associated to a hermitean form Φ on V. Show that ϕ^α is a hermitean matrix, if α is an on-basis.

(ii) For $V = \mathbb{R}^2$ with the standard scalar product and $\alpha = \{(1,2)^t, (0,-1)^t\}$ consider the endomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\phi^\alpha = \begin{pmatrix} 0 & -1 \\ -9 & 2 \end{pmatrix}$. Show that ϕ is selfadjoint, i.e. $\phi = \phi^*$, and determine the symmetric bilinear form Φ associated with ϕ.

To (i): *We have by definition:* $\Phi(u,v) = \langle u|\phi(v)\rangle$ *for all* $u, v \in V$. *Let* $\alpha = \{e_1, \ldots, e_n\}$ *be an on-basis. Let then* $A := (a_{ij}) = (\Phi(e_i, e_j))$ *be the Gram-matrix of* Φ *and* $(f_{ij}) := \phi^\alpha$. *Then* $a_{ij} = \Phi(e_i, e_j) = \langle e_i|\phi(e_j)\rangle = \sum_{k=1}^n f_{jk}\langle e_i|e_k\rangle = f_{ji}$. *Since* $a_ij = \overline{aji}$, *the same holds for* f_{ij}. *Thus* ϕ^α *is hermitean.*

To (ii): *Let* β *be the standard basis. Then the transition matrix from* α *to* β *is given by* $T = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$, *which is coincidentally equal to its inverse* T^{-1}. *Now* $T\phi^\alpha T = \phi^\beta = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}$, *which is a symmetric matrix. Thus* ϕ *is a selfadjoint endomorphism and the associated symmetric bilinear form is given by* $\Phi(u,v) = \langle u|\phi(v)\rangle = -2u_1v_1 + u_1v_2 + u_2v_1 + 4u_2v_2$.