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Linear Algebra II (MCS), SS 2006, Exercise 10

Mini-Quiz

(1) The principal axes transformation for a real symmetric matrix A means...?
- ☐ to find a symmetric matrix P, such that $P^{-1}AP$ is diagonal.
- √ to find an orthogonal matrix P, such that $P^{-1}AP$ is diagonal.
- ☐ to find an invertible matrix P, such that $P^{-1}AP$ is diagonal.
- ☐ to find an invertible matrix P, such that P^tAP is diagonal.

(2) If a real symmetric $n \times n$-matrix A has only one eigenvalue λ, then...?
- √ A is already diagonal.
- ☐ $a_{ij} = \lambda$ for all $i, j = 1, \dots, n$
- ☐ $n = 1$.

(3) If W is a linear subspace of a vector space V with scalar product $\langle \cdot \mid \cdot \rangle$, then $W^\perp$ is...?
- ☐ $\{w \in W \mid w \perp W\}$.
- √ $\{v \in V \mid v \perp W\}$.
- √ $\{v \in V \mid \langle v|w\rangle = 0 \text{ for all } w \in W\}$.
- √ $\{v \in V \mid \langle w|v\rangle = 0 \text{ for all } w \in W\}$.
- ☐ $\{v \in V \mid v \perp W \text{ and } \|v\| = 1\}$.

Groupwork

G 43 Determine the eigenvalues and the principal axes transformation of the following matrix using Jacobi-rotations:

$$A = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}.$$

G 44 Determine the singular value decomposition $U^*AV = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$ of the following matrix:

$$A = \begin{pmatrix} 1 & i \\ 0 & 0 \\ -1 & -i \end{pmatrix}.$$

G 45 Let $\{u_1, \dots, u_r\}$ be an orthonormal subset of a n-dimensional vector space V with scalar product. Show that for every $v \in V$ we have:

$$\sum_{j=1}^{r} |\langle v|u_j\rangle|^2 \leq \|v\|^2. \quad \text{(Bessel's Inequality)}$$

G 46 Let V be a n-dimensional vector space with scalar product $\langle \cdot \mid \cdot \rangle$ and let $\{u_1, \dots, u_r\}$ be a basis of a subspace W of V. Suppose that $\alpha = \{v_1, \dots, v_{n-r}\}$ is a linear independent system of $n - r$ vectors, such that $\langle u_j|v_k\rangle = 0$ for all $j = 1, \dots, r$, $k = 1, \dots, n - r$. Show that α is a basis of $W^\perp$.

G 47 Verify the following polar forms for a scalar product $\langle \cdot \mid \cdot \rangle$:

(i) $\langle u|v\rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2)$ (euclidean case)

(ii) $\langle u|v\rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2) + \frac{i}{4}(\|u + iv\|^2 - \|u - iv\|^2)$ (unitary case)

G 48 Let V be a finite dimensional vector space with scalar product $\langle \cdot \mid \cdot \rangle$.

(i) Show that for every fixed $u \in V$, the map $\psi_u : V \to K$, $v \mapsto \langle u|v\rangle$ defines a linear form on V.

(ii) Show that for every element $\phi \in V^*$, i.e. every linear form $\phi : V \to K$, there is a unique vector $u \in V$ such that $\phi(v) = \psi_u(v) = \langle u, v\rangle$.

Remark: (i) and (ii) show that the map $V \to V^*, u \mapsto \psi_u$ is a bijection. However, in the unitary case, this assignment is not linear!

Homework

H 35 Determine the QR-decomposition of $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and use it to solve $Ax = b_i$ for $b_1 = (1, 1, 1)^t$ and $b_2 = (1, 0, -1)^t$.

H 36 (i) Prove: For any positively semi-definite matrix $A \in K^{n\times n}$ there is a unique positively semi-definite matrix $B \in K^{n\times n}$ such that $B^2 = A$.

(ii) Determine for each of the following matrices A, a positively semi-definite matrix B with $B^2 = A$.

$$(a) \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad (b) \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

H 37 Let A be a real symmetric $n \times n$-matrix with smallest eigenvalue $\lambda_{\min}$ and largest eigenvalue $\lambda_{\max}$.

(i) Show the following estimate for the *Rayleigh-Quotient*:

$$\lambda_{\min} \leq \frac{x^t Ax}{|x|^2} \leq \lambda_{\max} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

(ii) Let $Q(x) = x^t Ax$ and let V_+ as in Theorem 39.7. Show that $Q|_{V_+}$ has a unique minimum in $x = 0$.

H 38 Consider the two parallel circles

$$S_1 = \{(x, y, -1)^t \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$
$$S_2 = \{(x, y, 1)^t \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

in $\mathbb{R}^3$ centered at the z-axis. Now let g be the line which passes through S_1 in the point $(1, 0, -1)^t$ and through S_2 in the point $(\cos\alpha, \sin\alpha, 1)^t$ for some fixed $\alpha \in [0, 2\pi)$. Let $H_g = \bigcup_{A\in \mathrm{SO}(2)} A(g)$ where $\mathrm{SO}(2)$ denotes the set of all rotations of $\mathbb{R}^3$ around the z-axis. Show that

(i) $H_g = \{(x, y, z)^t \in \mathbb{R}^3 \mid \lambda_1 x^2 + \lambda_2 y^2 - \lambda_3 z^2 = c\}$ for some $\lambda_1, \lambda_2, \lambda_3, c \in \mathbb{R}$, i.e. H_g is a quadric.

(ii) Through every point of H_g run two distinct lines which lie completely in H_g (only for $\alpha \neq 0$).

Hints: Every element of $\mathrm{SO}(2)$ has the form $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $\theta \in [0, 2\pi)$. Use the addition theorems for trigonometric functions.

H 39 (**Optional task**)

Let V be an euclidean vector space with on-basis $\{e_1, e_2, e_3\}$ and Q a quadratic form, on V with matrix A w.r.t. this basis. Suppose that

$$1 \leq Q(v) \leq 3 \text{ for } |v| = 1,$$
$$\det(A) = 6,\ Q(e_1 - 2e_2 + 2e_3) = 27 \text{ and}$$
$$Q(2e_1 + 2e_2 + e_3) = 18.$$

(i) Determine all eigenvalues and an on-basis of eigenvectors of Q.

(ii) Determine A.

Hint: Use Corollary 39.4: If $|v| = 1$ and $Q(v) = \max\{Q(w) \mid |w| = 1\}$, then v is an eigenvector of Q.

Linear Algebra II (MCS), SS 2006, Exercise 10, Solution

Mini-Quiz

(1) The principal axes transformation for a real symmetric matrix A means...?
- ☐ to find a symmetric matrix P, such that $P^{-1}AP$ is diagonal.
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- √ A is already diagonal.
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- ☐ $\{w \in W \mid w \perp W\}$.
- √ $\{v \in V \mid v \perp W\}$.
- √ $\{v \in V \mid \langle v|w\rangle = 0 \text{ for all } w \in W\}$.
- √ $\{v \in V \mid \langle w|v\rangle = 0 \text{ for all } w \in W\}$.
- ☐ $\{v \in V \mid v \perp W \text{ and } \|v\| = 1\}$.

Groupwork

G 43 Determine the eigenvalues and the principal axes transformation of the following matrix using Jacobi-rotations:

$$A = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}.$$

The Ansatz $(1,t)A(t,-1)^t = 0$ yields $t = \pm i$. Hence, $v_1 = \frac{1}{\sqrt{2}}(1,-i)^t$ and $v_2 = \frac{1}{\sqrt{2}}(i,-1)^t$ are the principal axes thus $T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$ is the transformation matrix, which gives $T^*AT = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$.

G 44 Determine the singular value decomposition $U^*AV = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$ of the following matrix:

$$A = \begin{pmatrix} 1 & i \\ 0 & 0 \\ -1 & -i \end{pmatrix}.$$

*We have $A^*A = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}$ which is the matrix from exercise **G 43**. Thus, by the algorithm of the script, we have $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$ and $\Sigma = (2)$. Furthermore, $AV = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}$ and after division by $\Sigma_{11} = 2$, we obtain the first column vector of U, which we complete arbitrarily to an on-basis of $\mathbb{R}^3$: $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$.*

G 45 Let $\{u_1, \dots, u_r\}$ be an orthonormal subset of a n-dimensional vector space V with scalar product. Show that for every $v \in V$ we have:

$$\sum_{j=1}^{r} |\langle v|u_j\rangle|^2 \leq \|v\|^2. \quad \text{(Bessel's Inequality)}$$

This is an implication of Plancherel's formula. In fact, we may complete $\{u_1, \dots, u_r\}$ to an on-basis $\{u_1, \dots, u_n\}$ of V. Then Plancherel's formula yields $\sum_{i=j}^{n} |\langle v|u_j\rangle|^2 = \|v\|^2$. Dropping the last $n-r$ terms on the left side yields the desired inequality, since all summands are non-negative.

G 46 Let V be a n-dimensional vector space with scalar product $\langle \cdot \mid \cdot \rangle$ and let $\{u_1, \dots, u_r\}$ be a basis of a subspace W of V. Suppose that $\alpha = \{v_1, \dots, v_{n-r}\}$ is a linear independent system of $n-r$ vectors, such that $\langle u_j|v_k\rangle = 0$ for all $j = 1, \dots, r$, $k = 1, \dots, n-r$. Show that α is a basis of $W^\perp$.

We clearly have that $\operatorname{span}(\alpha) \subset W^\perp$, since $\langle u_j|v_k\rangle = 0$ for all $j = 1, \dots, r$, $k = 1, \dots, n-r$. Since $W \cap W^\perp = \{0\}$ and $V = W + W^\perp$, the dimension formula for linear subspaces yields $n = \dim V =$

$\dim W + \dim W^{\perp} = r + \dim W^{\perp}$. *Thus* $\dim W^{\perp} = n - r$, *which shows that* α *generates the whole of* $W^{\perp}$. *So* α *is indeed a basis of* $W^{\perp}$.

G 47 Verify the following polar forms for a scalar product $\langle \cdot \mid \cdot \rangle$:

(i) $\langle u|v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2)$ (euclidean case)

(ii) $\langle u|v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2) + \frac{i}{4}(\|u+iv\|^2 - \|u-iv\|^2)$ (unitary case)

To (i):

$$\begin{aligned}\frac{1}{4}(\|u+v\|^2 - \|u-v\|^2) &= \frac{1}{4}(\langle u+v|u+v\rangle - \langle u-v|u-v\rangle) \\ &= \frac{1}{4}(\langle u|u\rangle + 2\langle u|v\rangle + \langle v|v\rangle - \langle u|u\rangle + 2\langle u|v\rangle - \langle v|v\rangle) \\ &= \frac{1}{4}(4\langle u|v\rangle) = \langle u|v\rangle.\end{aligned}$$

To (ii): *This is just done as in (i).*

G 48 Let V be a finite dimensional vector space with scalar product $\langle \cdot \mid \cdot \rangle$.

(i) Show that for every fixed $u \in V$, the map $\psi_u : V \to K$, $v \mapsto \langle u|v\rangle$ defines a linear form on V.

(ii) Show that for every element $\phi \in V^*$, i.e. every linear form $\phi : V \to K$, there is a unique vector $u \in V$ such that $\phi(v) = \psi_u(v) = \langle u, v\rangle$.

Remark: (i) and (ii) show that the map $V \to V^*, u \mapsto \psi_u$ is a bijection. However, in the unitary case, this assignment is not linear!

To (i): *By definition, a scalar product is linear in the second argument and its image lies in the ground field. Hence* ψ_u *is a well defined linear form on* V.

To (ii): *Let* ϕ *be an arbitrary linear form on* V. *If* $\phi = 0$, *then clearly* $\psi_0 = \phi$. *If* $\phi \neq 0$, *then by the dimension formula for linear maps,* $W := \ker \phi$ *is a hyperplane in* V, *i.e.* $\dim W = \dim V - 1$. *Let* $u \in V$ *be such that* $u \perp W$ *and* $\phi(u) = 1$. *We claim that* $\psi_u = \phi$. *In fact,* $V = W \oplus (K \cdot u)$ *and on both summands,* ψ_u *and* ϕ *coincide. By linearity, both linear forms coincide on* V. *Uniqueness of* u *can be seen as follows, if arguable: Suppose there is some* u' *such that* $\psi_u = \phi = \psi_{u'}$. *Then for all* $v \in V$: $0 = \psi_u(v) - \psi_{u'}(v) = \langle u - u'|v\rangle$. *Hence* $u = u'$.

Homework

H 35 Determine the QR-decomposition of $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and use it to solve $Ax = b_i$ for $b_1 = (1,1,1)^t$ and $b_2 = (1,0,-1)^t$.

Applying Gram-Schmidt orthonormaliztion to the columns b_1, b_2, b_3 of A we obtain

$$\begin{aligned}
u_1 &= b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
v_1 &= \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
u_2 &= b_2 - \langle v_1 | b_2 \rangle v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \\
v_2 &= \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \\
u_3 &= b_3 - \langle v_1 | b_3 \rangle v_1 - \langle v_2 | b_3 \rangle v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \\
v_3 &= \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.
\end{aligned}$$

The vectors v_1, v_2, v_3 form the columns of the matrix Q in the QR-decomposition. Hence,

$$R = \begin{pmatrix} \langle v_1|b_1\rangle & \langle v_1|b_2\rangle & \langle v_1|b_3\rangle \\ 0 & \langle v_2|b_2\rangle & \langle v_2|b_3\rangle \\ 0 & 0 & \langle v_3|b_3\rangle \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

so A decomposes as

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Using this decomposition, we solve $Ax_i = b_i$ by first forming $Q^t b_i$ and then solving $Rx_i = Q^t b_i$ by substituting backwards. We then have:

$$Q^t b_1 = (\sqrt{3}, 0, 0)^t \implies x_1 = (1,0,0)^t.$$

$$Q^t b_2 = (0, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{2}})^t \implies x_2 = (1,-1,-1)^t.$$

H 36 (i) Prove: For any positively semi-definite matrix $A \in K^{n\times n}$ there is a unique positively semi-definite matrix $B \in K^{n\times n}$ such that $B^2 = A$.

(ii) Determine for each of the following matrices A, a positively semi-definite matrix B with $B^2 = A$.

$$(a) \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad (b) \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

To (i): *Since A is positively semi-definite, there is a unitary, resp. orthogonal, matrix S and a diagonal matrix D with nonnegative entries $\lambda_1, \ldots, \lambda_n$ such that $A = SDS^*$, resp. $A = SDS^t$. Now put $B = S\sqrt{D}S^*$, where $\sqrt{D}$ denotes the diagonal matrix whose diagonal entries are just $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$. Then $B^2 = S\sqrt{D}S^*S\sqrt{D}S^* = SDS^* = A$. This proves the existence of the square root.*

For the uniqueness, suppose that B satisfies $B^2 = A$. Let v be an eigenvector of B to the eigenvalue μ. Then v is an eigenvector of A to the eigenvalue μ^2. Hence $E_\mu(B) \subset E_{\mu^2}(A)$. Since *the eigenvalues of A are non-negative, they have a unique square root. This implies that the eigenspaces of B for distinct μ lie in the eigenspaces of A for distinct μ^2. By the pigeon hole principle, $E_\mu(B) = E_{\mu^2}(A)$. Since B is completely determined by its eigenspaces and eigenvalues, we have just proved the uniqueness of B.*

To (ii): *For $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ the eigenvalues are 2 and 0. Corresponding eigenvectors are $(1,1)^t$ and $(1,-1)^t$. Hence the transition matrix of A is $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. By (i), B is given by $S\sqrt{D}S^t = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} A$.*

For $A' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ the eigenvalues are 3 and 1. Corresponding eigenvectors are $(1,1)^t$ and $(1,-1)^t$. Hence the transition matrix of A' is the same as that of A and by (i), B' is given by $S\sqrt{D}'S^t = \frac{1}{2} \begin{pmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{pmatrix}$.

H 37 Let A be a real symmetric $n \times n$-matrix with smallest eigenvalue $\lambda_{\min}$ and largest eigenvalue $\lambda_{\max}$.

(i) Show the following estimate for the *Rayleigh-Quotient*:

$$\lambda_{\min} \leq \frac{x^t Ax}{|x|^2} \leq \lambda_{\max} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

(ii) Let $Q(x) = x^t Ax$ and let V_+ as in Theorem 39.7. Show that $Q|_{V_+}$ has a unique minimum in $x = 0$.

To (i): *Let $Q(x) := x^t Ax$. Then Q is a quadratic form on $\mathbb{R}^n$ and by Corollary 39.4 we have for all $\|x\| = 1 : \lambda_{\min} \leq Q(x) \leq \lambda_{\max}$. If now $x \in \mathbb{R}^n \setminus \{0\}$ is arbitrary, put $r := \|x\|$ and then by the above: $\lambda_{\min} \leq Q(\frac{x}{r}) \leq \lambda_{\max}$. However, $Q(\frac{x}{r}) = \frac{1}{r^2}Q(x) = \frac{Q(x)}{\|x\|^2} = \frac{x^tAx}{\|x\|^2}$ and our claim follows.*

To (ii): *Suppose that Q has a (local) minimum in $x_0 \in \mathbb{R}^n \setminus \{0\}$. Then there is some open ball $B_\epsilon(x_0)$ around x_0 such that for all $y \in B_\epsilon(x_0)$ we have $Q(y) \geq Q(x_0) =: m$. However, $Q(rx_0) = r^2 Q(x_0) = r^2 m$ for all $r \in \mathbb{R}$. In particular for all $r \in (1-\epsilon, 1+\epsilon)$ we have $r^2 m \geq m$. But for $r < 1$, clearly $r^2 m < m$, a contradiction. Hence, the unique minimum of Q is 0.*

H 38 Consider the two parallel circles

$$S_1 = \{(x, y, -1)^t \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

$$S_2 = \{(x, y, 1)^t \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

in $\mathbb{R}^3$ centered at the z-axis. Now let g be the line which passes through S_1 in the point $(1, 0, -1)^t$ and through S_2 in the point $(\cos\alpha, \sin\alpha, 1)^t$ for some fixed $\alpha \in [0, 2\pi)$. Let $H_g = \bigcup_{A \in \mathrm{SO}(2)} A(g)$ where $\mathrm{SO}(2)$ denotes the set of all rotations of $\mathbb{R}^3$ around the z-axis. Show that

(i) $H_g = \{(x, y, z)^t \in \mathbb{R}^3 \mid \lambda_1 x^2 + \lambda_2 y^2 - \lambda_3 z^2 = c\}$ for some $\lambda_1, \lambda_2, \lambda_3, c \in \mathbb{R}$, i.e. H_g is a quadric.

(ii) Through every point of H_g run two distinct lines which lie completely in H_g (only for $\alpha \neq 0$).

Hints: Every element of $\mathrm{SO}(2)$ has the form $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $\theta \in [0, 2\pi)$. Use the addition theorems for trigonometric functions.

To (i): *The line g is the set of all $(x, y, z)^t = (1, 0, -1)^t + t \cdot (\cos\alpha - 1, \sin\alpha, 2)^t$ with arbitrary $t \in \mathbb{R}$. Hence, every element of H_g has the form $A_\theta \cdot (x, y, z)^t$ for some $\theta \in [0, 2\pi)$ and $(x, y, z)^t \in g$. This* *yields:*

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} \cos\theta \\ \sin\theta \\ -1 \end{pmatrix} + t \cdot \begin{pmatrix} (\cos\alpha - 1)\cos\theta - \sin\alpha\sin\theta \\ (\cos\alpha - 1)\sin\theta + \sin\alpha\cos\theta \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta + t\cos(\alpha+\theta) - t\cos\theta \\ \sin\theta + t\sin(\alpha+\theta) - t\sin\theta \\ -1 + 2t \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta(1-t) + t\cos(\alpha+\theta) \\ \sin\theta(1-t) + t\sin(\alpha+\theta) \\ 2t - 1 \end{pmatrix}.
\end{aligned}$$

From the last equation we obtain $t = \frac{1+z}{2}$ *and substituting this into the equations for* x *and* y *gives:*

$$\begin{aligned}
x &= \cos\theta\left(\frac{1-z}{2}\right) + \left(\frac{1+z}{2}\right)\cos(\alpha+\theta) \\
y &= \sin\theta\left(\frac{1-z}{2}\right) + \left(\frac{1+z}{2}\right)\sin(\alpha+\theta).
\end{aligned}$$

Now we square both equations, add them up and use $\cos^2 + \sin^2 = 1$ *to obtain*

$$x^2 + y^2 = \left(\frac{1-z}{2}\right)^2 + \left(\frac{1+z}{2}\right)^2 + \left(\frac{1-z^2}{2}\right)(\cos\theta\cos(\alpha+\theta) + \sin\theta\sin(\alpha+\theta)).$$

We use the addition theorem for trigonometric functions one last time and arrive at

$$x^2 + y^2 - \frac{(1-\cos\alpha)}{2}z^2 = \frac{(1+\cos\alpha)}{2}$$

Choosing $\lambda_1 = \lambda_2 = 1, \lambda_3 = -\frac{(1-\cos\alpha)}{2}$ *and* $c = \frac{(1+\cos\alpha)}{2}$, *we have shown that* H_g *is included in a quadric. This quadric is for* $\alpha = 0$ *a cylinder, for* $\alpha = \pi$ *a cone and in all other cases a one-sheeted hyperboloid. We still have to verify that the quadric does not contain more points than* H_g*. In fact, it is not difficult to see that the intersection of every plane parallel to the* x, y*-plane intersects with* H_g *in a circle centered at the* z*-axis. The same holds true for the quadric. As a circle contained in another circle must coincide with the latter, we see that both sets* H_g *and the quadric must coincide.*

To (ii): *Since* H_g *is invariant under all rotations around the* z*-axis, it is also invariant under all reflections on planes which contain the* z*-axis. Now let* p *be an arbitrary point on* H_g*. By definition, there is some* $A \in \mathrm{SO}(2)$ *such that the line* $A(g)$ *passes through* p*. Under the reflection* τ *on the plane which contains* p *and the* z*-axis,* $A(g)$ *gets mapped to the line* $\tau(A(g))$ *which is different from* $A(g)$*, because we excluded the case* $\alpha = 0$ *for which* H_g *is a cylinder.*

H 39 (**Optional task**)

Let V be an euclidean vector space with on-basis $\{e_1, e_2, e_3\}$ and Q a quadratic form, on V with matrix A w.r.t. this basis. Suppose that

$$1 \le Q(v) \le 3 \text{ for } |v| = 1,$$
$$\det(A) = 6,\ Q(e_1 - 2e_2 + 2e_3) = 27 \text{ and}$$
$$Q(2e_1 + 2e_2 + e_3) = 18.$$

(i) Determine all eigenvalues and an on-basis of eigenvectors of Q.

(ii) Determine A.

Hint: Use Corollary 39.4: If $|v| = 1$ and $Q(v) = \max\{Q(w) \mid |w| = 1\}$, then v is an eigenvector of Q.

To (i): *From* $Q(e_1 - 2e_2 + 2e_3) = 27$ *we obtain* $3 = \frac{1}{9}Q(e_1 - 2e_2 + 2e_3) = Q(\frac{1}{3}e_1 - \frac{2}{3}e_2 + \frac{2}{3}e_3)$*. Thus the corollary says that* $v_1 = (\frac{1}{3}e_1 - \frac{2}{3}e_2 + \frac{2}{3}e_3)^t$ *is an eigenvector to the maximal eigenvalue* $\lambda_1 = 3$*. We can furthermore choose a normed basis* v_1, v_2, v_3 *of eigenvectors corresponding to the eigenvalues* $3 = \lambda_1 \ge \lambda_2 \ge \lambda_3 \ge 1$*. Since the determinant of a diagonalizable matrix is equal to the product of its eigenvalues, we obtain from* $\det A = 6$ *that* $2 = \lambda_2 \cdot \lambda_3$*. Now look at the restriction* $Q|_W$ *of* Q *to the subspace* $W = v_1^\perp = \mathrm{span}\{v_2, v_3\}$*. Then* $Q|_W$ *is also a quadratic form, with eigenvalues* λ_2, λ_3*. Furthermore, an eigenvector of* $Q|_W$ *is also one of* Q*. By assumption,*

$Q(2e_1+2e_2+e_3) = 18$ and as above, this implies that $2 = \frac{1}{9}Q(2e_1+2e_2+e_3) = Q(\frac{2}{3}e_1+\frac{2}{3}e_2+\frac{1}{3}e_3)$. Since $(\frac{2}{3}e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3)^t$ is orthogonal to v_1 and has norm equal to one, we see that $\lambda_2 = 2$ and may take $v_2 = (\frac{2}{3}e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3)^t$, due to the corollary again. By the determinant condition above, we also see that $\lambda_3 = 1$. Now V_1 and v_2 already determine v_3 uniquely up to a sign. For instance, we can take $v_3 = (-\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{2}{3}e_3)^t$.

To (ii): *From the data obtained in (i) we know that the transformation matrix S which diagonalizes A is given by $S = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ and the corresponding diagonal matrix is $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Accordingly, we get for $A = SDS^t = \frac{1}{3}\begin{pmatrix} 5 & 0 & 2 \\ 0 & 7 & -2 \\ 2 & -2 & 6 \end{pmatrix}$.*