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Linear Algebra II (MCS), SS 2006, Exercise 1

Groupwork

G 1 Which of the following matrices can be completed to an orthogonal matrix? Give an admissible completion, if possible.

$$\text{(i)} \begin{pmatrix} * & 5 & * \\ * & * & * \\ * & * & * \end{pmatrix}, \qquad \text{(ii)} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & * \\ 1 & 0 & * \\ 0 & -1 & * \end{pmatrix}, \qquad \text{(iii)} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & * \\ 0 & 0 & * \\ 1 & -1 & * \end{pmatrix}.$$

G 2 Let V be a finite dimensional Euclidean space and $U, W \subset V$ subspaces. Show the following identities:

$$\text{(i) } (U+W)^\perp = U^\perp \cap W^\perp, \qquad \text{(ii) } (U \cap W)^\perp = U^\perp + W^\perp, \qquad \text{(iii) } V = U \oplus U^\perp.$$

G 3 Let $(V, \langle \cdot \mid \cdot \rangle)$ be a finite dimensional Euclidean space. Show that for every $\vec{x}, \vec{y} \in V$ the following identities hold:

(i) $\langle \vec{x} + \vec{y} \mid \vec{x} - \vec{y} \rangle = \|\vec{x}\|^2 - \|\vec{y}\|^2$.

(ii) $\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\langle \vec{x} \mid \vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos\theta$,
where θ denotes the angle between $\vec{x}$ and $\vec{y}$. (Generalized theorem of Pythagoras)

(iii) $\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 = 4\langle \vec{x} \mid \vec{y} \rangle$.

(iv) $\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2$. (Parallelogram identity)

G 4 Let $(\mathbb{P}, V)$ denote affine 3-space and let $\alpha : O, \vec{e_1}, \vec{e_2}, \vec{e_3}$ be a coordinate system. Let furthermore

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Describe the affine self-map $\phi : (\mathbb{P}, V) \to (\mathbb{P}, V)$ with homogeneous matrix ${}_{\tilde{\alpha}}\tilde{\phi}_{\tilde{\alpha}} = A$ in terms of translations, rotations, reflections, etc. Does ϕ has any fixed points? Draw a picture of the situation.

Homework

H 1 Let V be a vector space and V^* its dual. Show that the map $V^* \setminus \{0\} \to H,\ f \mapsto f^{-1}(1)$ is a bijection, where H denotes the set of affine hyperplanes of V which do not pass through the origin.

H 2 Let $(\mathbb{P}, V)$ be an affine space and let $\vec{x_i}$, $i \in I$ be a basis of V. Show that any affine self-map $\phi : (\mathbb{P}, V) \to (\mathbb{P}, V)$ which commutes with each of the translations τ_{x_i}, $i \in I$ is itself a translation.

H 3 Let $(V, \langle \cdot \mid \cdot \rangle)$ be a finite dimensional Euclidean space and $U \subset V$ a subspace. Suppose that $\varphi : V \to V$ is an endomorphism with the following properties:

$$\varphi \circ \varphi = \varphi,\ \operatorname{im}(\varphi) = U,\ \ker(\varphi) = U^\perp.$$

Show that φ is the orthogonal projection of V onto U and give an explicit formula for φ in terms of a suitable basis of V and the scalar product.

Linear Algebra II (MCS), SS 2006, Exercise 1, Solution

Groupwork

G 1 Which of the following matrices can be completed to an orthogonal matrix? Give an admissible completion, if possible.

$$\text{(i)} \begin{pmatrix} * & 5 & * \\ * & * & * \\ * & * & * \end{pmatrix}, \qquad \text{(ii)} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & * \\ 1 & 0 & * \\ 0 & -1 & * \end{pmatrix}, \qquad \text{(iii)} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & * \\ 0 & 0 & * \\ 1 & -1 & * \end{pmatrix}.$$

Let a_1, a_2, a_3 denote the columns of the matrix A. If A is orthogonal, we then have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A \cdot A^t = \begin{pmatrix} \langle a_1 \mid a_1 \rangle & \langle a_1 \mid a_2 \rangle & \langle a_1 \mid a_3 \rangle \\ \langle a_2 \mid a_1 \rangle & \langle a_2 \mid a_2 \rangle & \langle a_2 \mid a_3 \rangle \\ \langle a_3 \mid a_1 \rangle & \langle a_3 \mid a_2 \rangle & \langle a_3 \mid a_3 \rangle \end{pmatrix}.$$

In particular, each column vector $a_i = (a_{1,i}, a_{2,i}, a_{3,i})^t$, $i = 1, 2, 3$ satisfies $a_{1,i}^2 + a_{2,i}^2 + a_{3,i}^2 = 1$ and thus each matrix entry has absolute value less than or equal to one. Furthermore, the a_i are mutually orthogonal. Hence, (i) and (ii) cannot be completed to an orthogonal matrix. For (iii) the only possible solutions are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \pm\sqrt{2} \\ 1 & -1 & 0 \end{pmatrix}.$$

G 2 Let V be a finite dimensional Euclidean space and $U, W \subset V$ subspaces. Show the following identities:

$$\text{(i) } (U + W)^\perp = U^\perp \cap W^\perp, \qquad \text{(ii) } (U \cap W)^\perp = U^\perp + W^\perp, \qquad \text{(iii) } V = U \oplus U^\perp.$$

There are several ways to solve this exercise.
To (i): Let $u + w \in U + W$ and $x \in U^\perp \cap W^\perp$ be arbitrary elements. Then

$$\langle u + w \mid x \rangle = \underbrace{\langle u \mid x \rangle}_{=0} + \underbrace{\langle w \mid x \rangle}_{=0} = 0,$$

which proves $U^\perp \cap W^\perp \subset (U + W)^\perp$. Conversely, let $x \in (U + W)^\perp$ and $u \in U$, $w \in W$ be arbitrary elements. Then $\langle x \mid u \rangle = 0$ and $\langle x \mid w \rangle = 0$, whence $x \in U^\perp$ and $x \in W^\perp$. This implies $x \in U^\perp \cap W^\perp$.
To (ii): Applying (i) to $U^\perp$ and $W^\perp$ gives

$$(U^\perp + W^\perp)^\perp = U^{\perp\perp} \cap W^{\perp\perp}.$$

Applying now $(\cdot)^\perp$ to both sides and using that $U^{\perp\perp} = U$, yields the desired formula.

To (iii): This is Corollary 25.3.

G 3 Let $(V, \langle \cdot \mid \cdot \rangle)$ be a finite dimensional Euclidean space. Show that for every $\vec{x}, \vec{y} \in V$ the following identities hold:

(i) $\langle \vec{x} + \vec{y} \mid \vec{x} - \vec{y} \rangle = \|\vec{x}\|^2 - \|\vec{y}\|^2$.

(ii) $\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\langle \vec{x} \mid \vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos\theta$,
where θ denotes the angle between $\vec{x}$ and $\vec{y}$. (Generalized theorem of Pythagoras)

(iii) $\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 = 4\langle \vec{x} \mid \vec{y} \rangle$.

(iv) $\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2$. (Parallelogram identity)

Using the bilinearity and symmetry of the scalar product, we obtain:

(i): $\langle \vec{x} + \vec{y} \mid \vec{x} - \vec{y} \rangle = \langle \vec{x} \mid \vec{x} \rangle - \langle \vec{x} \mid \vec{y} \rangle + \langle \vec{y} \mid \vec{x} \rangle + \langle \vec{y} \mid \vec{y} \rangle = \|\vec{x}\|^2 - \|\vec{y}\|^2$.

(ii): $\|\vec{x} - \vec{y}\|^2 = \langle \vec{x} - \vec{y} \mid \vec{x} - \vec{y} \rangle = \langle \vec{x} \mid \vec{x} \rangle - \langle \vec{x} \mid \vec{y} \rangle - \langle \vec{y} \mid \vec{x} \rangle + \langle \vec{y} \mid \vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2 \cdot \langle \vec{x} \mid \vec{y} \rangle$.
By definition of the angle between $\vec{x}$ and $\vec{y}$, we have $\langle \vec{x} \mid \vec{y} \rangle = \|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos\theta$.

(iii)& (iv): Applying (ii) to $\vec{x}, \vec{y}$, as well as $\vec{x}, -\vec{y}$ and subtracting (resp. adding) the resulting equations, yields the particular formula.

G 4 Let $(\mathbb{P}, V)$ denote affine 3-space and let $\alpha : O, \vec{e_1}, \vec{e_2}, \vec{e_3}$ be a coordinate system. Let furthermore

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Describe the affine self-map $\phi : (\mathbb{P}, V) \to (\mathbb{P}, V)$ with homogeneous matrix ${}_{\tilde{\alpha}}\tilde{\phi}_{\tilde{\alpha}} = A$ in terms of translations, rotations, reflections, etc. Does ϕ has any fixed points? Draw a picture of the situation.

From the first column we read of that the translational part of ϕ is $\tau_{\vec{v}}$ with $\vec{v} = \vec{e_1} + \vec{e_2} + \vec{e_3}$. The lower right 3×3-block of A tells us, that the linear part of ϕ is represented by the matrix

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which is an orthogonal reflection ϕ_0 on the hyperplane $\epsilon = \{\vec{x} + O \mid \langle \vec{e_1} - \vec{e_3} \mid \vec{x} \rangle = 0\}$. That is: $\phi_0(\vec{x}) = \vec{x} - 2\langle \vec{e_1} - \vec{e_2} \mid \vec{x} \rangle$ and ϕ decomposes as $\phi = \tau_{\vec{v}} \circ \phi_0$. From this description, we read of that ϕ is a proper glide reflection on ϵ. I.e. it has no fixed points, one fixed line (given by $l = \{r(\vec{e_1} + \vec{e_2} + \vec{e_3}) + O \mid r \in \mathbb{R}\}$) and one fixed plane (given by ϵ).

Homework

H 1 Let V be a vector space and V^* its dual. Show that the map $V^* \setminus \{0\} \to H,\ f \mapsto f^{-1}(1)$ is a bijection, where H denotes the set of affine hyperplanes of V which do not pass through the origin.

We first have to verify, that the map, say $\Psi : V^* \setminus \{0\} \to H$, is indeed well defined. That is, we have to verify that for any $f \in V^* \setminus \{0\}$, the point set $f^{-1}(1)$ is indeed an affine hyperplane. Since $f \neq 0$, we obtain that f has to be surjective. In particular, $f^{-1}(1)$ is not empty and we may choose some element $v \in f^{-1}(1)$. By the dimension formula for linear maps, it follows that $f^{-1}(0)$ is a hyperplane through the origin and then $f^{-1}(1) = v + f^{-1}(0)$ is an affine hyperplane, not containing the origin.

Next, we deal with the injectivity of Ψ. For this purpose suppose that f_1, f_2 have the same image under Ψ. That is $f_1^{-1}(1) = f_2^{-1}(1) =: E$ and f_1, f_2 coincide on said hypersurface. Now, for an arbitrary $v \in E$, the hyperplane $E' = -v + E$ contains the origin and is therefore a linear subspace. Since $v \notin E'$ and the codimension of E' is one, we have a direct sum decomposition of $V = E' \oplus \operatorname{span}(v)$. Thus, every element w of V can be uniquely written as a sum of an element of E' and a scalar multiple of v. Say, $w = e + r \cdot v$. Then

$$f_1(w) = f_1(e + r \cdot v) = f_1(e) + r \cdot f_1(v) = 0 + r = r = f_2(w)$$

holds for every $w \in W$ and hence $f_1 = f_2$, proving the injectivity of Ψ.

Finally, in order to proof the surjectivity of Ψ, let E be an arbitrary hypersurface in V, not containing the origin. Then for some $v \in E$ we again have a direct sum decomposition of $V = E' \oplus \operatorname{span}(v)$, where $E' = -v + E$. We define a linearform f on V by setting $f|_{E'} = 0$ and $f(v) = 1$. Obviously, $f \neq 0$ and $f^{-1}(1) = v + E' + E$.

Did you observe that we did not require V to be finite dimensional?

H 2 Let $(\mathbb{P}, V)$ be an affine space and let $\vec{x_i}$, $i \in I$ be a basis of V. Show that any affine self-map $\phi : (\mathbb{P}, V) \to (\mathbb{P}, V)$ which commutes with each of the translations τ_{x_i}, $i \in I$ is itself a translation.

Let us first assume that $\phi(0) = 0$, that is ϕ is linear. By assumption we then have $\phi \circ \tau_{x_i} = \tau_{x_i} \circ \phi$. Applying this equation to the zero vector, we obtain

$$\phi(x_i) = \tau_{x_i}(\phi(0)) = \tau_{x_i}(0) = x_i.$$

Since we may write every $v \in V$ as a unique, finite linear combination of the x_i, say $v = \sum_{i \in I} v_i \cdot x_i$ (for all but finitely many $i \in I$ we have $v_i = 0$), we have by linearity of ϕ:

$$\phi(v) = \phi(\sum_{i \in I} v_i \cdot x_i) = \sum_{i \in I} v_i \cdot \phi(x_i) = \sum_{i \in I} v_i \cdot x_i = v.$$

Thus, ϕ is the translation by the zero vector.

If ϕ is not linear, we may pass to $\phi' = \tau_{-v} \circ \phi$, where $v = \phi(0)$. Since all translations commute, we have that ϕ' satisfies our assumption and furthermore, it is a linear map. By the arguments above we thus have $\phi' = \mathrm{id}$ and hence $\phi = \tau_v \circ \phi' = \tau_v$.

H 3 Let $(V, \langle \cdot \mid \cdot \rangle)$ be a finite dimensional Euclidean space and $U \subset V$ a subspace. Suppose that $\varphi : V \to V$ is an endomorphism with the following properties:

$$\varphi \circ \varphi = \varphi, \ \mathrm{im}(\varphi) = U, \ \ker(\varphi) = U^{\perp}.$$

Show that φ is the orthogonal projection of V onto U and give an explicit formula for φ in terms of a suitable basis of V and the scalar product.

Actually, all this exercise requires of you is to locate and review the definition of an orthogonal projection in the script. For any $x \in U$ you have some $v \in V$ with $\varphi(v) = u$. By the given property, you have $\varphi(u) = \varphi \circ \varphi(v) = \varphi(v) = u$. The rest follows from theorem 25.2.