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11. Tutorial Analysis II for MCS
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Space-filling curves

It seems paradoxical, but it is nevertheless true that there are curves which completely fill up higher dimensional spaces such as squares or cubes. The first example was constructed by G. Peano in 1890. Curves with this property are now called *space-filling curves* or *Peano curves*. Further examples by D. Hilbert (1891), E.H. Moore (1900), H. Lebesgue (1904), W. Sierpiński (1912), G. Pólya (1913), and others followed. The basic reference for this subject is the book

H. Sagan, *Space-filling curves*, Springer-Verlag, 1994.

In the sequel, we present a modification of Lebesgue's space-filling curve, due to I.J. Schoenberg (1938).

Let $f : \mathbb{R} \to \mathbb{R}$ be the even, two-periodic (i.e. periodic with period two) function defined by

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{3} \\ 3t-1 & \text{for } \frac{1}{3} \le t \le \frac{2}{3} \\ 1 & \text{for } \frac{2}{3} \le t \le 1 \end{cases}, \quad f(-t) = f(t),\ f(t+2) = f(t).$$

Let

$$x(t) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{f(3^{2k}t)}{2^k}, \quad y(t) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{f(3^{2k+1}t)}{2^k}.$$

The Schoenberg curve is defined by $\gamma_{sc} : [0,1] \to [0,1]^2$, $\gamma_{sc}(t) = (x(t), y(t))$.

(T11.1)

Prove the following:

(a) γ_{sc} is well-defined, that is, for all $t \in [0,1]$, $\gamma_{sc}(t) \in [0,1]^2$.

(b) γ_{sc} is a continuous function.

Hint: For (a) and (b), use the criterion of Weierstraß (Theorem 7.10).

(c) γ_{sc} is surjective.

Hint: Let $(x_0, y_0) \in [0,1]^2$ and consider the binary representations of x_0, y_0:

$$x_0 = \sum_{k=0}^{\infty} \frac{a_k}{2^{k+1}}, \;\; y_0 = \sum_{k=0}^{\infty} \frac{b_k}{2^{k+1}}, \;\; a_k, b_k \in \{0,1\}.$$

(See pages 110-113 in Hofmann, and in particular Theorem 2.37.)

Define $t_0 = \dfrac{2a_0}{3} + \dfrac{2b_0}{3^2} + \dfrac{2a_1}{3^3} + \dfrac{2b_1}{3^4} + \ldots$, and prove that $f(3^{2k} t_0) = a_k$, and $f(3^{2k+1} t_0) = b_k$.

(T11.2)

Read the proof of the following theorem.

Let X and Y be metric spaces, and let $f : X \to Y$ be continuous and bijective. Assume that X is compact. Then the inverse function $f^{-1} : Y \to X$ is continuous.

Proof: By (T12.2) in Analysis I we have that each closed subset A of X is compact. So $f(A) \subseteq Y$ is compact, by Theorem 3.51. Hence $f(A)$ is closed, by Proposition 3.45. Thus f^{-1} is continuous by Proposition 3.13 (iii).

(This result will be used in the exercise below.)

(T11.3) (Supplementary exercise)

(Netto's Theorem)

Prove the following result:

Any bijective map $g : [0,1] \to [0,1]^2$ is necessarily discontinuous.

Conclude that γ_{sc} is not injective.

Hint: Remark that if we remove a point from [0,1], we get a disconnected set, while if we remove a point from $[0,1]^2$, the set obtained is pathwise connected.