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5. Tutorial Analysis II for MCS
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Let $f : [a,b] \to \mathbb{R}$ be a bounded function. For any partition $P = (a = x_0 < x_1 < \ldots < x_n = b)$ of $[a,b]$ define

$$\|P\| := \max\{x_i - x_{i-1} : 1 \leq i \leq n\},$$
$$\sigma(P,f,\xi) := \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}), \qquad \text{where } \xi_i \in [x_{i-1}, x_i], \quad 1 \leq i \leq n.$$

The sums $\sigma(P,f,\xi)$ are called the *Riemann sums* associated with the function f, the partition P, and the system of intermediate points $\xi = (\xi_i)_{i=1}^n = (\xi_1, \xi_2, \ldots, \xi_n)$.

(T5.1)

(i) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Prove that the following are equivalent.

(1) f is integrable.

(2) There is an $I \in \mathbb{R}$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - I \right| < \varepsilon$$

for any partition $P = (a = x_0 < x_1 < \ldots x_n = b)$ of $[a,b]$ with $\|P\| < \delta$ and for every choice of points $\xi_1, \ldots, \xi_n$ with $\xi_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$.

In case such an $I \in \mathbb{R}$ as specified above exists, it is the integral of f. Riemann defined the integral of $f : [a,b] \to \mathbb{R}$ as outlined in this exercise, rather than the way it is done in Hofmann's book.

Hint: For (1) $\Rightarrow$ (2), argue that there exist step functions s, t with $s \leq f \leq t$ and $\int t - \int s < \varepsilon/2$ and associated partition $P' = (a = y_0 < y_1 < \ldots < y_m = b)$. Now consider a partition $P = (a = x_0 < x_1 < \ldots < x_n = b)$ and let $\xi = (\xi)_{i=1}^n$ be a choice of points with $\xi_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$. Consider the union Π of those intervals $]x_{i-1}, x_i[$ such that there exists a $1 \leq j \leq m$ with $[x_{i-1}, x_i] \subseteq]y_{j-1}, y_j[$. Notice that there can be at most $2m$ intervals $]x_{i-1}, x_i[$ such that $]x_{i-1}, x_i[\not\subseteq \Pi$.

(ii) Let $f : [a,b] \to \mathbb{R}$ be integrable. Prove that for every sequence $(P_n)_{n\in\mathbb{N}}$ of partitions $P_n = (a = x_0^{(n)} < x_1^{(n)} < \ldots < x_{p_n}^{(n)} = b)$ with $\lim_{n\to\infty} \|P_n\| = 0$, and for all systems $\xi^{(n)} = (\xi_i^{(n)})_{i=1}^{p_n}$ of intermediate points with $\xi_i^{(n)} \in [x_{i-1}^{(n)}, x_i^{(n)}]\,(n \in \mathbb{N}, 1 \le i \le p_n)$,

$$\lim_{n\to\infty} \sigma(P_n, f, \xi^{(n)}) = \int_a^b f dx.$$

(T5.2) Let $a < b \in \mathbb{R}$, and let $\exp : [a,b] \to \mathbb{R}$ be the exponential function. Use (T5.1)(ii) to compute its integral.