Fachbereich Mathematik
Dr. L. Leuştean
K. Altmann, E. Briseid, S. Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

09.05.2006

4. Tutorial Analysis II for MCS
Summer Term 2006

(1) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We write

$$\lim_{n\to\infty} x_n = \infty,$$

if

$$(\forall M \in \mathbb{R})(\exists m \in \mathbb{N})(\forall n \geq m)(x_n > M).$$

(2) Let X be a subset of a metric space, and let a be an accumulation point of X. Let furthermore $f : X \to \mathbb{R}$. If

$$(\forall M \in \mathbb{R})(\exists \delta > 0)(\forall x \in X)(d(x,a) < \delta \Rightarrow f(x) > M),$$

then we say that $f(x)$ *goes to infinity as x approaches a*, and we write

$$\lim_{x\to a} f(x) = \infty.$$

(3) Let $X \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$ be an accumulation point of $X\cap\,]a,\infty[$. Let furthermore $f : X \to \mathbb{R}$. If

$$(\forall M \in \mathbb{R})(\exists \delta > 0)(\forall x \in X\cap\,]a,\infty[)(d(x,a) < \delta \Rightarrow f(x) > M),$$

then we say that $f(x)$ *goes to infinity as x approaches a from above*, and we write

$$\lim_{\substack{x\to a\\ x>a}} f(x) = \infty.$$

(4) Let now $a \in \mathbb{R}$ and $f :]a,\infty[\to \mathbb{R}$. For $l \in \mathbb{R}$ we say that $f(x)$ *converges to l as x goes to infinity*, and we write

$$\lim_{x\to\infty} f(x) = l,$$

if

$$\lim_{\substack{y\to 0\\ y>0}} f(\frac{1}{y}) = l.$$

If

$$\lim_{\substack{y\to 0\\ y>0}} f(\frac{1}{y}) = \infty,$$

then we write

$$\lim_{x \to \infty} f(x) = \infty,$$

and we say that $f(x)$ *goes to infinity as x goes to infinity.*

When saying that a certain limit *exists*, without specifying that the sequence (of reals) or the function (taking values in the reals) goes to infinity, we will take it to imply that the limit is a real number.

(T4.1)

Define in a similar way:

(i) $\lim_{n \to \infty} x_n = -\infty$, for a sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers.

(ii) $\lim_{x \to a} f(x) = -\infty$, for an accumulation point a of a subset X of a metric space, and for $f : X \to \mathbb{R}$.

(iii)
$$\lim_{\substack{x \to a \\ x < a}} f(x) = \infty,$$
for $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and an a which is an accumulation point of $X \cap]-\infty, a[$.

(iv) $\lim_{x \to -\infty} f(x) = l$ and $\lim_{x \to -\infty} f(x) = \infty$ for $f :]-\infty, a[\to \mathbb{R}$, $a \in \mathbb{R}$.

(T4.2)

(i) Prove the following version of the rule of Bernoulli and de l'Hôpital:

Let $a < b \in \mathbb{R}$ and $f, g :]a, b[\to \mathbb{R}$ be differentiable and such that

$$\lim_{\substack{x \to a \\ x > a}} f(x) = \lim_{\substack{x \to a \\ x > a}} g(x) = \infty.$$

Assume

$$\lim_{\substack{x \to a \\ x > a}} g'(x) \neq 0,$$

that is, the limit exists and is not 0. Assume further that there exists $l \in \mathbb{R}$ such that

$$\lim_{\substack{x \to a \\ x > a}} \frac{f'(x)}{g'(x)} = l.$$

Then

$$\lim_{\substack{x \to a \\ x > a}} \frac{f(x)}{g(x)} = l.$$

Hint:

Argue first that there exists a $\delta > 0$ such that for $x \in]a, a+\delta[$ we have $g(x) > 1$ and $g'(x) \neq 0$, and such that for $x \in]a, a+\delta[$ we have that

$$u(x) := \inf\{y > a : f(y) \leq \sqrt{f(x)} \wedge g(y) \leq \sqrt{g(x)}\}$$

exists.

Prove then that

$$\lim_{\substack{x \to a \\ x > a}} u(x) = a.$$

Use this together with the generalized mean value theorem to finish the proof.

(ii) Prove the following version of the rule of Bernoulli and de l'Hôpital:

Let $a \in \mathbb{R}$ and $f, g : [a, \infty[\to \mathbb{R}$ be differentiable and such that

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty.$$

Assume that

$$\lim_{x \to \infty} g'(x) \neq 0.$$

Assume further that there exists $l \in \mathbb{R}$ such that

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = l.$$

Then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = l.$$

(T4.3)

(i) Prove that for all $n \in \mathbb{N}$ we have

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0.$$

(ii) Let $n \in \mathbb{N}$ and let $P : \mathbb{R} \to \mathbb{R}$ be defined by $P(x) := a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ with $a_0, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$. Prove that

$$\lim_{x \to \infty} \frac{P(x)}{e^x} = 0.$$