Fachbereich Mathematik
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10. Home work Analysis II for MCS
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(H10.1)

Consider the vector space $(C[0,1], \|\cdot\|_\infty)$ of continuous functions $f : [0,1] \to \mathbb{R}$ equipped with the supremum norm. For $n \in \mathbb{N}$ let $P_n \subset C[0,1]$ be the finite dimensional subspace of polynomials of degree less than or equal to n. We then call $p_b \in P_n$ a polynomial of best approximation to $f \in C[0,1]$ if

$$\|f - p_b\|_\infty = \inf\{\|f-p\|_\infty : p \in P_n\}.$$

(i) Prove that any polynomial $p_b \in P_n$ of best approximation to f satisfies $\|p_b\|_\infty \leq 2\|f\|_\infty$.

(ii) Prove that each $f \in C[0,1]$ possesses a polynomial $p_b \in P_n$ of best approximation.

(H10.2)

Define $f_n : \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$ by

$$f_n(x) := \frac{\sin(nx)}{n}.$$

(i) Show that (f_n) converges uniformly on $\mathbb{R}$ and determine its limit function $f : \mathbb{R} \to \mathbb{R}$.

(ii) Show that the sequence of derivatives (f_n') does not converge (not even pointwise).

(H10.3)

Let (f_n) be a sequence of real-valued functions on a set X. We say that (f_n) is *uniformly Cauchy* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m \geq N$ and all $x \in X$.

Prove that (f_n) is uniformly convergent to some $f : X \to \mathbb{R}$ if and only if it is uniformly Cauchy.