Fachbereich Mathematik
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06.07.2006

12. Exercise sheet Analysis II for MCS
Summer Term 2006

(G12.1)

Let V and W be Banach spaces. We say that a function $A : V \to W$ is *affine* if there exist $c \in W$ and linear $T : V \to W$ such that

$$A(x) = T(x) + c$$

for all $x \in V$. Prove that any continuous affine function $A : V \to W$ is differentiable. Show also that if $A(x) = T(x) + c$ for $x \in V$, then $A'(x) = T$ for any $x \in V$.

(G12.2)

Let

$$U = \{(r, \phi) : r > 0,\ \ 0 < \phi < 2\pi\} \subseteq \mathbb{R}^2,$$

and let

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : U \to \mathbb{R}^2$$

be defined by

$$f_1(r, \phi) = r \cos \phi,$$
$$f_2(r, \phi) = r \sin \phi.$$

We write $x = f_1(r, \phi)$ and $y = f_2(r, \phi)$.

(i) Notice that f is injective with range

$$f(U) = \mathbb{R}^2 \backslash \{(x, 0) : x \geq 0\}.$$

Show that f has a continuous inverse $f^{-1} : f(U) \to \mathbb{R}^2$.

(ii) Prove that f is differentiable, and compute the Jacobi matrix of f.

(iii) Prove that f^{-1} is differentiable, and compute the Jacobi matrix of f^{-1}.

(G12.3)

Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x^2 y \sin(\frac{y}{x}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(i) Compute the partial derivatives of f where they exist and show that $\partial_1 f$ is not continuous in any point $(0, y)$ with $y \neq 0$.

Hint for the latter: For $y \neq 0$, consider the points $(x_n, y_n) = (y/(n\pi), y)$ for $n \in \mathbb{N}$.

(ii) Decide whether f is differentiable in $(0, y)$, for $y \in \mathbb{R}$.