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27.06.2006

11. Tutorial Analysis II for MCS
Summer Term 2006

Space-filling curves

It seems paradoxical, but it is nevertheless true that there are curves which completely fill up higher dimensional spaces such as squares or cubes. The first example was constructed by G. Peano in 1890. Curves with this property are now called *space-filling curves* or *Peano curves*. Further examples by D. Hilbert (1891), E.H. Moore (1900), H. Lebesgue (1904), W. Sierpiński (1912), G. Pólya (1913), and others followed. The basic reference for this subject is the book

H. Sagan, *Space-filling curves*, Springer-Verlag, 1994.

In the sequel, we present a modification of Lebesgue's space-filling curve, due to I.J. Schoenberg (1938).

Let $f : \mathbb{R} \to \mathbb{R}$ be the even, two-periodic (i.e. periodic with period two) function defined by

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{3} \\ 3t-1 & \text{for } \frac{1}{3} \le t \le \frac{2}{3} \\ 1 & \text{for } \frac{2}{3} \le t \le 1 \end{cases}, \quad f(-t) = f(t),\ f(t+2) = f(t).$$

Let

$$x(t) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{f(3^{2k}t)}{2^k}, \quad y(t) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{f(3^{2k+1}t)}{2^k}.$$

The Schoenberg curve is defined by $\gamma_{sc} : [0,1] \to [0,1]^2$, $\gamma_{sc}(t) = (x(t), y(t))$.

(T11.1)

Prove the following:

(a) γ_{sc} is well-defined, that is, for all $t \in [0,1]$, $\gamma_{sc}(t) \in [0,1]^2$.

(b) γ_{sc} is a continuous function.

Hint: For (a) and (b), use the criterion of Weierstraß (Theorem 7.10).

(c) γ_{sc} is surjective.

Hint: Let $(x_0, y_0) \in [0,1]^2$ and consider the binary representations of x_0, y_0:

$$x_0 = \sum_{k=0}^{\infty} \frac{a_k}{2^{k+1}}, \quad y_0 = \sum_{k=0}^{\infty} \frac{b_k}{2^{k+1}}, \quad a_k, b_k \in \{0,1\}.$$

(See pages 110-113 in Hofmann, and in particular Theorem 2.37.)

Define $t_0 = \frac{2a_0}{3} + \frac{2b_0}{3^2} + \frac{2a_1}{3^3} + \frac{2b_1}{3^4} + \ldots$, and prove that $f(3^{2k}t_0) = a_k$, and $f(3^{2k+1}t_0) = b_k$.

Solution.

(a) For any $k \in \mathbb{N}_0$, let $\phi_k, \psi_k : \mathbb{R} \to \mathbb{R}$, $\phi_k(t) = \frac{f(3^{2k}t)}{2^k}$, $\psi_k(t) = \frac{f(3^{2k+1}t)}{2^k}$. Since $f(t) \in [0,1]$ for all $t \in \mathbb{R}$, we get that $0 \le \phi_k(t), \psi_k(t) \le \frac{1}{2^k}$ for all $t \in \mathbb{R}, k \in \mathbb{N}_0$, so $\|\phi_k\|_\infty, \|\psi_k\|_\infty \le \frac{1}{2^k}$ for all $k \in \mathbb{N}_0$. Using the fact that the geometric series $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ is convergent with $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$, we can apply the Weierstraß criterion (Theorem 7.10) to get that the series $\sum_{k=0}^{\infty} \phi_k(t), \sum_{k=0}^{\infty} \psi_k(t)$ are uniformly convergent. Thus, $0 \le x(t), y(t) \le \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{1}{2}\cdot 2 = 1$, hence $\gamma_{sc}(t) \in [0,1]^2$ for all $t \in [0,1]$.

(b) It is easy to see that f is continuous, so ϕ_k, ψ_k are continuous functions for all $k \in \mathbb{N}_0$. Applying now the fact that the sum of a uniformly convergent series of continuous functions is a continuous function, we get that x, y are continuous functions, hence γ_{sc} is continuous.

(c) Let $(x_0, y_0) \in [0,1]^2$. Consider the binary representations of x_0, y_0:

$$x_0 = \frac{a_0}{2} + \frac{a_1}{2^2} + \ldots + \frac{a_k}{2^{k+1}} + \ldots = \sum_{k=0}^{\infty} \frac{a_k}{2^{k+1}}, \quad a_k \in \{0,1\},$$

$$y_0 = \frac{b_0}{2} + \frac{b_1}{2^2} + \ldots + \frac{b_k}{2^{k+1}} + \ldots = \sum_{k=0}^{\infty} \frac{b_k}{2^{k+1}}, \quad b_k \in \{0,1\}.$$

Let $(c_k)_{k\geq 0}$ be defined by $c_{2k} = a_k$ and $c_{2k+1} = b_k$ for $k \geq 0$. Since $\frac{2c_k}{3^{k+1}} = \frac{2}{3} \cdot \frac{c_k}{3^k} \leq \frac{2}{3} \cdot \frac{1}{3^k}$ for any $k \in \mathbb{N}_0$, and the geometric series $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$ is convergent with $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{3}{2}$, it follows that the series $\sum_{k=0}^{\infty} \frac{2c_k}{3^{k+1}}$ is convergent, so we can define

$$t_0 = \sum_{k=0}^{\infty} \frac{2c_k}{3^{k+1}},$$

and $0 \leq t_0 \leq \frac{2}{3} \cdot \frac{3}{2} = 1$. We shall prove that $\gamma_{sc}(t_0) = (x_0, y_0)$. First, let us remark that

$$\begin{aligned} 3^k t_0 &= \text{even number } (= 2p) + \frac{2c_k}{3} + \frac{2c_{k+1}}{3^2} + \frac{2c_{k+2}}{3^3} + \dots \\ &= 2p + \frac{2c_k}{3} + \frac{2}{3^2}\left(c_{k+1} + \frac{c_{k+2}}{3} + \dots\right) \end{aligned}$$

Let $\alpha = \frac{2c_k}{3} + \frac{2}{3^2}\left(c_{k+1} + \frac{c_{k+2}}{3} + \dots\right)$. Then $\alpha \geq \frac{2c_k}{3}$, and

$$\alpha \leq \frac{2c_k}{3} + \frac{2}{3^2}\left(1 + \frac{1}{3} + \dots\right) = \frac{2c_k}{3} + \frac{2}{3^2}\sum_{k=0}^{\infty}\frac{1}{3^k} = \frac{2c_k}{3} + \frac{2}{3^2}\cdot\frac{3}{2} = \frac{2c_k}{3} + \frac{1}{3}.$$

Hence, $\alpha \in \left[\frac{2c_k}{3}, \frac{2c_k}{3} + \frac{1}{3}\right]$. Since f is two-periodic, it follows immediately by induction on n that $f(2n+t) = f(t)$ for all $n \in \mathbb{N}_0, t \in \mathbb{R}$. Thus, $f\left(3^k t_0\right) = f(2p+\alpha) = f(\alpha)$. If $c_k = 0$, then $\alpha \in \left[0, \frac{1}{3}\right]$, so $f\left(3^k t_0\right) = f(\alpha) = 0$; if $c_k = 1$, then $\alpha \in \left[\frac{2}{3}, 1\right]$, so $f\left(3^k t_0\right) = f(\alpha) = 1$. Thus, $f\left(3^k t_0\right) = c_k$ for all $k \in \mathbb{N}_0$. It follows that

$$\begin{aligned} x(t_0) &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{f(3^{2k}t_0)}{2^k} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{c_{2k}}{2^k} = \sum_{k=0}^{\infty}\frac{a_k}{2^{k+1}} = x_0, \\ y(t_0) &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{f(3^{2k+1}t_0)}{2^k} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{c_{2k+1}}{2^k} = \sum_{k=0}^{\infty}\frac{b_k}{2^{k+1}} = y_0. \end{aligned}$$

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(T11.2)

Read the proof of the following theorem.

Let X and Y be metric spaces, and let $f : X \to Y$ be continuous and bijective. Assume that X is compact. Then the inverse function $f^{-1} : Y \to X$ is continuous.

Proof: By (T12.2) in Analysis I we have that each closed subset A of X is compact. So $f(A) \subseteq Y$ is compact, by Theorem 3.51. Hence $f(A)$ is closed, by Proposition 3.45. Thus f^{-1} is continuous by Proposition 3.13 (iii).

(This result will be used in the exercise below.)

Solution. Not applicable.

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(T11.3) (Supplementary exercise)

(Netto's Theorem)

Prove the following result:

Any bijective map $g : [0,1] \to [0,1]^2$ is necessarily discontinuous.

Conclude that γ_{sc} is not injective.

Hint: Remark that if we remove a point from [0,1], we get a disconnected set, while if we remove a point from $[0,1]^2$, the set obtained is pathwise connected.

Solution.

Assume that g is continuous. Since $[0,1]$ is a compact subset of $\mathbb{R}$, it follows by what we proved above that the inverse function $g^{-1} : [0,1]^2 \to [0,1]$ is also continuous. Let us remove a point t_0 from the open interval $]0,1[$, and its image $g(t_0)$ from $[0,1]^2$. Then $A := [0,1] \setminus \{t_0\} = [0,t_0[\cup]t_0,1]$ is not an interval, so it is not connected (fill in the details). On the other hand, the set $B := [0,1]^2 \setminus \{g(t_0)\}$ is pathwise connected (give a detailed proof), so by Proposition 4.37 it is connected. Remark that $g^{-1}(B) = A$. This contradicts Theorem 3.15, which states that the image of a connected set under a continuous function between metric spaces is connected. ■